The Stackel matrix business PhL 8.13.11

These are the raw notes, with lots of clumsy stuff and exploratory sections all retained. See meta notes for a better presentation.

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History. See http://www.apprendre-math.info/anglais/historyDetail.htm?id=Stackel . The person here is in fact German Paul Stäckel 1862-1919 (I think it is "shtay'kel")

Die Integration der Hamilton-Jacobischen Differentialgleichung mittelst Separation der Variablen, 1891, Habilitation

Here is a footnote from somewhere

"1 By now, these potentials had come to be known as St¨ackel potentials in the astronomical literature.

This seems unwarranted. First, it is poor practice in physics to associate a name with an equation if a perfectly adequate descriptive term exists. On these grounds alone, the term 'separable potential' is preferable to 'St¨ackel potential'. And, second, there is no reason to associate the name of Paul St¨ackel with coordinate systems and potentials that he never wrote down! St¨ackel was a prominent differential geometer, latterly Professor of Mathematics at Heidelberg. In his Habilitationschrift in 1891 at Halle, St¨ackel wrote down the condition for the Hamilton-Jacobi equation to separate in a given coordinate system on a general Riemannian manifold in the form of the vanishing of a determinant (which has reasonably enough come to be called the St¨ackel determinant). St¨ackel did not derive the coordinate systems in Euclidean 3-space for which his determinant vanishes, far less the form of the separable
potentials in these coordinates. This work was left to Weinacht (1924) and Eisenhart (1948). In fact, St¨ackel's result is limited, as it does not even provide a comprehensive test for separability. St¨ackel's determinant for a separable system only vanishes if it is written down in the separable coordinate system itself. The finding of a general criterion for identifying whether a potential is separable in some coordinate system remains an outstanding research problem. "

Explanation of the Stackel Method of Separating Helmholtz

Start Off

I have been putting this off for a long time, but now the time is right. The subject is specifically how to try to find the separated equations for the Helmholtz equation in arbitrary 3D orthogonal curvilinear coordinates. I will be crudely following the presentation of M&F on page 509 of Vol I, section 5.1. This work is pretty high up in my building of bricks, a lot had to happen before. For example

- appropriate matrix theorems
- understanding of curvilinear coordinates and how differential operators look in same.
- some understanding of PDE's and ODE's
- understanding the significance of the Helmholtz equation!

Consider our known form of ∇^2 given an orthog curvilinear system: (see "tensors and curvilinear doc")

$$\nabla^2 u = (1/[h_1 h_2 h_3]) \{ \partial_1 [(h_1 h_2 h_3 / h_1^2) (\partial_1 u)] + \text{cyclic} \} \qquad (1)$$

Let's define $H = h_1 h_2 h_3$ to make things more compact, then we have

$$\nabla^2 \psi = H^{-1} \{ \partial_1 [(H/h_1^2)(\partial_1 \psi)] + \text{cyclic} \} = H^{-1} \Sigma_n \partial_n [(H/h_n^2)(\partial_n \psi)] \qquad (2)$$

If we now assume that

$$\psi(\xi_1,\xi_2,\xi_3) = X_1(\xi_1) X_2(\xi_2) X_3(\xi_3)$$

At this point, we adopt a certain **shorthand notation**. First, we show coordinate dependence of a function by replacing ξ_n with just the integer n and remove the commas. For example, the above reads

$$\psi(123) = X_1(1)\, X_2(2)\, X_3(3)$$

Second, if we have a function $f_1(1)$ which depends on the variable corresponding to its subscript, then we will shorten it and write just f_1, and this implies that it is really $f_1(\xi_1)$. Then we have

$$\psi(123) = X_1X_2X_3 \tag{3}$$

If we insert (3) into (2) we get

$$\nabla^2\psi = H^{-1}\{\partial_1[(H/h_1^2)(\partial_1\{X_1X_2X_3\})] + \text{cyclic}\}$$

$$= H^{-1}\{X_2X_3\{\partial_1[(H/h_1^2)(\partial_1X_1)] + \text{cyclic}\}$$

$$(\nabla^2 + k_1^2)\psi = H^{-1}\{X_1X_2X_3\{\partial_1[(H/h_1^2)(\partial_1X_1)]/X_1 + \text{cyclic}\} + k_1^2X_1X_2X_3$$

or

$$(\nabla^2 + k_1^2)\psi = H^{-1}\{\partial_1[(H/h_1^2)(\partial_1X_1)]/X_1 + \text{cyclic}\}\psi + k_1^2\psi = 0 \tag{4}$$

Now make a new definition

$$w_1(123) = (H/h_1^2) \qquad // \; w_1 \text{ is } \textit{determined} \text{ by curvilinear system!} \tag{5}$$

Then we have

$$(\nabla^2 + k_1^2)\psi = H^{-1}\{\partial_1[w_1(\partial_1X_1)]/X_1 + \text{cyclic}\}\psi + k_1^2\psi = 0 \tag{6}$$

Now, suppose we look for separated equations of this completely general 2nd order form

$$L_1X_1 = 0 \qquad L_1X_1 = \partial_1[p_1(\partial_1X_1)] + r_1\partial_1X_1 + q_1X_1 \tag{7}$$

Right here we are going to change a symbol name to make future equations more compatible with the literature on "separation". Stakgold and others like to use p and q for the two functions shown above in (7). We will soon be discarding r, and q will become part of something else, but we now want to change p to f. In this way, our earlier equations below will look more in line with this literature. Thus

$$L_1X_1 = 0 \qquad L_1X_1 = \partial_1[f_1(\partial_1X_1)] + r_1\partial_1X_1 + q_1X_1 \tag{7a}$$

When some subject is new to "the student", it helps provide a solid platform for learning the new stuff if different sources use the same notation. Once it is understood, then we can change to any notation we want. It is in the early phases that the symbol name choices seem to help. So this f notation is what you see in the two heavy-weight books on the subject of separation, M&F and M&S.

First required assumption

We want L_1 to appear inside (6). To make this happen, it seems that the only way would be to arrange to have

$$w_1(123) = g_1(23)f_1(1) \tag{8}$$

which them means that

$$g_1(23)f_1(1) = (H/h_1^2) \tag{8a}$$

Note that this is a *restriction* on our curvilinear system -- it will rule out some systems. If we find a system that meets this restriction, then both $g_1(23)$ and f_1 are *determined*, so we now regard these as given quantities.

If condition (8) were met, we would then have

$$(\nabla^2 + k_1^2)\psi = H^{-1} \{ g_1(23)\, \partial_1[f_1 (\partial_1 X_1)] / X_1 + \text{cyclic}\}\psi + k_1^2\psi = 0 \tag{9}$$

Notice that $\partial_1[f_1 (\partial_1..)] = L_1 - r_1\partial_1 + q_1$ so we then have

$$(\nabla^2 + k_1^2)\psi = H^{-1} \{ g_1(23)\, [(L_1 - r_1\partial_1 - q_1)X_1] / X_1 + \text{cyclic}\}\psi + k_1^2\psi = 0 \tag{10}$$

Some Side Comments

We have now achieved our goal of causing the L_n to appear in (6). If for the moment we assume that X_n are some arbitrary functions, and then $\psi(123) = X_1X_2X_3$ is then not a solution of Helmholtz, and if we think of the $L_{3D} = (\nabla^2 + k_1^2)$, then we are saying that

$$L_{3D}\,[X_1X_2X_3] = [\, H^{-1} \{ g_1(23)\, [(L_1 - r_1\partial_1 - q_1)X_1] / X_1 + \text{cyclic}\} + k_1^2 \,]\, [X_1X_2X_3]$$

If we think of $X_1X_2X_3$ as spanning a certain subspace of 3D functions, then within this subspace we have found a particular form for the action of the L_{3D} operator in terms of operators L_1, L_2 and L_3. But we are not able to express the operator L_{3D} itself directly in terms of the L_n due to the appearance of the X_n functions inside the RHS operator in front of $[X_1X_2X_3]$.

Now if we further restrict $X_1X_2X_3$ to the space of functions for which $L_{3D}\,[X_1X_2X_3] = 0$ AND for which we have $L_nX_n = 0$, AND if we also assume that $r_n = 0$, then the above becomes

$$L_{3D}\,[X_1X_2X_3] = [\, H^{-1} \{ g_1(23)\, [(- q_1) + \text{cyclic}\} + k_1^2 \,]\, [X_1X_2X_3] = 0$$

where now the X_n no longer appear "inside", and acting on this subspace we then have

$$L_{3D} = [\, H^{-1} \{ g_1(23)\, [(- q_1) + \text{cyclic}\} + k_1^2 \,]$$

which is then just a multiple of the unity operator. Our task will then be to arrange to have this multiple be 0, and then we will have $L_{3D} = 0$ on this solution subspace, and hence $L\psi = 0$.

Showing that the r_n must vanish

Now suppose we assume that X_1 is a solution of $L_1X_1 = 0$ and $\psi(123) = X_1X_2X_3$ is a solution of Helmholtz. Then we have from (10)

$$(\nabla^2 + k_1^2)\psi = H^{-1}\{ g_1(23)\,[(-r_1\partial_1 - q_1)X_1]/X_1 + \text{cyclic}\}\psi + k_1^2\psi = 0 \qquad (11)$$

What we need now is this

$$\{ g_1(23)\,[(-r_1\partial_1 - q_1)X_1]/X_1 + \text{cyclic}\} + H(123)\,k_1^2 = 0 \qquad (12)$$

Let's write this out

$$g_1(23)\,[(r_1\partial_1+q_1)X_1]/X_1 + g_2(31)\,[(r_2\partial_2+q_2)X_2]/X_2 + g_3(12)\,[(r_3\partial_3+q_3)X_3]/X_3 = H(123)\,k_1^2$$

Now define $F_1(1; X_1) \equiv [(r_1\partial_1+q_1)X_1]/X_1$, then we have

$$g_1(23)\,F_1(1; X_1)) + g_2(31)\,F_2(2; X_2)) + g_3(12)\,F_3(3; X_3)) = k_1^2H(123) \qquad (13)$$

where now all coordinate dependences are shown. Remember that the g_n are already determined by the curvilinear system. Can we now find functions r_n, q_n, X_n which make this be true for all ξ_1,ξ_2,ξ_3 ? What can we say about the X_n? If we are given r_n and q_n (the p_n are already determined, recall), then since the X_n must be solutions to $L_nX_n = 0$, the X_n are also determined, at least as a linear combination of independent solutions. We require that our equation (13) be true for ANY solutions X_n of $L_nX_n = 0$, that is what we mean by our whole separation idea. So let's write

$$X_n = a_nu_n+b_nv_n$$

where a_n and b_n are arbitrary constants and u_n and v_n are any independent solutions of $L_nX_n = 0$. Then our equation (13) becomes

$$g_1(23)\,F_1(1; a_1u_1+b_1v_1)) + g_2(31)\,F_2(2; a_2u_2+b_2v_2)) + g_3(12)\,F_3(3; a_3u_3+b_3v_3)) = k_1^2H(123)$$

where we have six constants on the LHS that are allowed to take any values whatsoever! If you found a solution to the above, if you were to slightly change one of the constants, it would no longer be a solution. So I conclude that there IS no solution of the above equation valid for all values of the 6 constants.

Now, suppose we assume that all three $r_n = 0$. Then something magic happens! We then have

$$g_1(23)\,[(r_1\partial_1+q_1)X_1]/X_1 + g_2(31)\,[(r_2\partial_2+q_2)X_2]/X_2 + g_3(12)\,[(r_3\partial_3+q_3)X_3]/X_3 = H(123)\,k_1^2$$

$$g_1(23)\,q_1(1) + g_2(31)\,q_2(2) + g_3(12)q_3(3) = H(123)\,k_1^2 \qquad (14)$$

Now we no longer have dependence on the X_n functions, and we ask: Can we find functions q_n which make the equation (14) be true? At least we have a *chance* of the answer being yes we can. Remember: if we can satisfy the equation we keep processing along in this doc, then we have found a separated solution to our PDE.

As a reminder, we had to set $r_n = 0$ to even have a chance, and then our assumed form for the separated operators is

$$L_1X_1 = \partial_1[f_1(\partial_1X_1)] + q_1X_1 = 0$$

We shall now modify this slightly by dividing by p_1

$$\mathcal{L}_1X_1 = (1/f_1)\partial_1[f_1(\partial_1X_1)] + (q_1/f_1)X_1 = 0$$

Let's now change names $(q_1/f_1) \rightarrow v_1$ so we then have

$$\mathcal{L}_1X_1 = (1/f_1)\partial_1[f_1(\partial_1X_1)] + v_1X_1 = 0 \qquad\qquad q_1 = f_1v_1$$

and (14) becomes

$$g_1(23)\, v_1(1)\, f_1(1) + g_2(31)\, v_2(2)\, f_2(2) + g_3(12) v_3(3)\, f_3(3) = H(123)\, k_1^2 \qquad (15)$$

or

$$\Sigma_n\ g_n(\neq n)\, f_n(n) v_n(n) = H(123)\, k_1^2$$

Note added: We could if we like combine these last two equations to get

$$\Sigma_n\ g_n\, f_n v_n(n) = H\, k_1^2$$

$$\Sigma_n\ g_n\, (1/X_n)\partial_n[f_n(\partial_n X_n)] = -H\, k_1^2$$

and we could then use (2.2) which will appear below, and which says $H/S = g_n(\neq n) f_n(n) / M_n(\neq n)$, to get

$$g_n = HM_n/[S\, f_n] \qquad\qquad // \; H/(g_n f_n) = S/M_n$$

to obtain this processed form of the Helmholtz equation

$$\Sigma_n\ M_n/[S\, f_n\, X_n]\, \partial_n[f_n(\partial_n X_n)] = -k_1^2$$

This form does lead us to imagine there are two free "separation constants" in the traditional sense:

$$(M_1(\Phi)/[\, S(\Phi) f_1 X_1])\, \partial_1\{f_1(\partial_1 X_1)\} = c_1^2$$
$$(M_2(\Phi)/[\, S(\Phi) f_2 X_2])\, \partial_2\{f_2(\partial_2 X_2)\} = c_2^2$$
$$(M_3(\Phi)/[\, S(\Phi) f_3 X_3])\, \partial_3\{f_3(\partial_3 X_3)\} = c_3^2$$

where we have the constraint that

$$c_1^2 + c_2^2 + c_3^2 = -k_1^2$$

giving then two free separation constants. One has to wonder how these might be related to the two free constants we see in the separated equation we will soon display below

$$(1/f_n)\partial_n[f_n(\partial_n X_n)] + [\, k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0$$

We can rewrite a few lines back as

$$(1/H)\,\Sigma_n\; g_n\,(1/X_n)\partial_n[f_n(\partial_n X_n)] = -\,k_1^2$$

So this is another way to write things

$$(1/H)\,(g_1/X_1)\,\partial_1\{f_1(\partial_1 X_1)\} = c_1^2$$
$$(1/H)\,(g_2/X_2)\,\partial_2\{f_2(\partial_2 X_2)\} = c_2^2$$
$$(1/H)\,(g_3/X_3)\,\partial_3\{f_3(\partial_3 X_3)\} = c_3^2$$

Question: is this correct? Maybe this conclusion only applies when you have two things in a sum. Like so

$$a(x_1) + b(x_2) = k$$

As you vary x_2 and hold x_1 fixed, suppose $b(x_2)$ were NOT a constant. Then the LHS would vary, but it cannot vary, so $b(x_2)$ must be a constant.

$$a(x_1) + b(x_2) + c(x_3) = k$$

Suppose $c(x_3)$ is NOT a constant and hold x_1 and x_2 fixed. Then the LHS would vary, but again it cannot vary, so $c(x_3)$ is a constant. So yes, it does work no matter how many you add.

So here we go:

$$(1/f_1X_1)\,\partial_1\{f_1(\partial_1 X_1)\} = (H/g_1f_1)c_1^2 = -\,[\, k_1^2\Phi_{11}(1) + k_2^2\Phi_{12}(1) + k_3^2\Phi_{13}(1)]$$
$$(1/f_2X_2)\,\partial_2\{f_2(\partial_2 X_2)\} = (H/g_2f_2)c_2^2 = -\,[\, k_1^2\Phi_{21}(2) + k_2^2\Phi_{22}(2) + k_3^2\Phi_{23}(2)]$$
$$(1/f_3X_3)\,\partial_3\{f_3(\partial_3 X_3)\} = (H/g_3f_3)c_3^2 = -\,[\, k_1^2\Phi_{31}(3) + k_2^2\Phi_{32}(3) + k_3^2\Phi_{33}(3)]$$

We can see that if k_2 and k_3 are specified, then c_1^2 and c_2^2 and c_3^2 will be determined, and it will turn out that they will be constants, and they will add up to $-k_1^2$. We can write as

$$(S/M_1)c_1^2 = -\,[\, k_1^2\Phi_{11}(1) + k_2^2\Phi_{12}(1) + k_3^2\Phi_{13}(1)]$$
$$(S/M_2)c_2^2 = -\,[\, k_1^2\Phi_{21}(2) + k_2^2\Phi_{22}(2) + k_3^2\Phi_{23}(2)]$$
$$(S/M_3)c_3^2 = -\,[\, k_1^2\Phi_{31}(3) + k_2^2\Phi_{32}(3) + k_3^2\Phi_{33}(3)]$$

using our usual friend above. Well, let's go ahead with the matrix form

$$-\begin{pmatrix} c_1^2/M_1 \\ c_2^2/M_2 \\ c_3^2/M_3 \end{pmatrix} = (\Phi/S) \begin{pmatrix} k_1^2 \\ k_2^2 \\ k_3^2 \end{pmatrix}$$

where (Φ/S) is a matrix of determinant 1, so I think this means length is preserved so we will have

$$k_1^4 + k_2^4 + k_3^4 = (c_1^2/M_1)^2 + (c_2^2/M_2)^2 + (c_3^2/M_3)^2$$

This does not seem possible to me because the LHS is a constant and the RHS is not. I guess I am wrong, you would need Φ to have the form of a rotation matrix, ie, Hermitian, and no doubt Φ does not have that property. Forms I have seen such as ellipsoidals certainly are not symmetric hence not Hermitian!

If we do invert, we would get

$$-\begin{pmatrix} k_1^2 \\ k_2^2 \\ k_3^2 \end{pmatrix}/S = \Phi^{-1} \begin{pmatrix} c_1^2/M_1 \\ c_2^2/M_2 \\ c_3^2/M_3 \end{pmatrix}$$

and in our notation below we will have

$$G_{pq}(\neq p) \equiv [\Phi^T]^{-1}_{pq} \qquad \Rightarrow G_{pq}^T(\neq p) = [\Phi]^{-1}_{pq} \qquad G_{pq}(\neq p) = \text{cof}(\Phi_{pq})$$

$$-\begin{pmatrix} k_1^2 \\ k_2^2 \\ k_3^2 \end{pmatrix}/S = G^T \begin{pmatrix} c_1^2/M_1 \\ c_2^2/M_2 \\ c_3^2/M_3 \end{pmatrix} \qquad M_p = G_{p1}$$

The k_i^2 will then be a function of all the cofactors of Φ and will be some mess, stop. Go back

$$\begin{aligned} c_1^2 &= -(M_1/S)\,[\,k_1^2\Phi_{11}(1) + k_2^2\Phi_{12}(1) + k_3^2\Phi_{13}(1)] \\ c_2^2 &= -(M_2/S)\,[\,k_1^2\Phi_{21}(2) + k_2^2\Phi_{22}(2) + k_3^2\Phi_{23}(2)] \\ c_3^2 &= -(M_3/S)\,[\,k_1^2\Phi_{31}(3) + k_2^2\Phi_{32}(3) + k_3^2\Phi_{33}(3)] \end{aligned}$$

All I can say is that in some specific system, once we compute Φ and S and the M_i, we can compute these c_1^2 in terms of the k_i^2 and they have to come out being constants. I need to try this in some system. It just looks wrong, doesn't it! In the first equation, suppose the three functions are different and none are constants. Well, M_1 and S are pretty complicated and could somehow offset this? Well, suppose k_2 and k_3 are both 0. Then it certainly looks wrong. How can $(M_1/S)\,\Phi_{11}(1)$ be a constant? Something must be wrong here, another day. On the other hand, in this case look what happens if we add the three rows. The $M\Phi$ sum is just S, so we get $c_1^2 + c_2^2 + c_3^2 = -k_1^2$ which is correct.

Introduction of the Nine Φ functions

Now nothing stops us from writing v_1 in this rather obscure form:

$$v_1 = A\Phi_{11}(1) + B\Phi_{12}(1) + C\Phi_{13}(1)$$

We have just replaced one unknown function with a linear combination of three unknown functions with some unknown coefficients! Furthermore, nothing stops us from selecting $A = k_1^2$ which is our Helmholtz parameter. So let's do this, and then let's rename B and C to be k_2^2 and k_3^2 where these last two are arbitrary. So we then have

$$v_1 = k_1^2\Phi_{11}(1) + k_2^2\Phi_{12}(1) + k_3^2\Phi_{13}(1)$$

Thus, we can write our separated equation as

$$\mathcal{L}_1 X_1 = (1/f_1)\partial_1[f_1(\partial_1 X_1)] +[k_1^2\Phi_{11}(1) + k_2^2\Phi_{12}(1) + k_3^2\Phi_{13}(1)]X_1 = 0$$

Suppose we then do this same thing for the other two separated equations, so we cause 6 more unknown functions of the Φ type to appear. Since these functions are all arbitrary so far (and have, for example, an arbitrary overall scale), we can select the same three constants k_1^2, k_2^2, k_3^2 to be the linear coefficients. We then have

$$\mathcal{L}_n X_n = (1/f_n)\partial_n[f_n(\partial_n X_n)] +v_n X_n = 0$$

appearing as

$$\mathcal{L}_n X_n = (1/f_n)\partial_n[f_n(\partial_n X_n)] +[k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0 \qquad (16)$$

where again, all 9 of the Φ functions are unknown, and where k_2^2 and k_3^2 are arbitrary constants. Note that each Φ function depends on a single variable corresponding to its first index. We have shown above that $\mathcal{L}_n$ must be writable in this manner in order to even have a distant chance of having equation (15) be satisfied. This equation (15) has to be satisfied for us to conclude that $X_1X_2X_3$ is a solution of $L_{3D}[X_1X_2X_3] = 0$ so we can say we have found a separated solution! We can now rewrite (15) in our new assumed notation as

$$g_1(23)\, v_1(1)\, f_1(1) + g_2(31)\, v_2(2)\, f_2(2) + g_3(12) v_3(3)\, f_3(3) = H(123)\, k_1^2$$

$$g_1(23)\, v_1(1)\, f_1(1) + \text{cyclic} = H(123)\, k_1^2$$

$$\Sigma_n\, g_n(\neq n) v_n(n) f_n(n) = H(123) k_1^2$$

$$\Sigma_n\, g_n(\neq n) [k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)] f_n(n) = H(123) k_1^2 \qquad (17)$$

where now for example $g_1(23)$ is written as $g_1(\neq 1)$. We wonder now if we can find 9 functions Φ_{ij} such that equation (17) will be true when k_2^2 and k_3^2 are arbitrary constants. Of course k_1^2 being the Helmholtz parameter is also arbitrary. Comparing coefficients of the k^2 terms, we would then have to have

$$\begin{aligned}
&\Sigma_n\, g_n(\neq n)\, \Phi_{n1}(n)\, f_n(n) = H(123)\\
&\Sigma_n\, g_n(\neq n)\, \Phi_{n2}(n)\, f_n(n) = 0\\
&\Sigma_n\, g_n(\neq n)\, \Phi_{n3}(n)\, f_n(n) = 0
\end{aligned} \qquad (18)$$

At this point we have sort of caught up with M&F. Here is the form they "assume" for the separated equation, and it is identical to my equation (16) above which I repeat here

$$\frac{1}{f_n}\frac{\partial}{\partial \xi_n}\left[f_n \frac{\partial X_n}{\partial \xi_n}\right] + [k_1^2\Phi_{n1} + k_2^2\Phi_{n2} + k_3^2\Phi_{n3}]X_n = 0 \qquad (5.1.28)$$

$$\mathcal{L}_n X_n = (1/f_n)\partial_n[f_n(\partial_n X_n)] + [\, k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0 \qquad (16)$$

Stating the processed Helmholtz equation as a matrix equation

Now let's try to cast (18) as a matrix equation. To get it into standard form, we write

$$\begin{aligned}
&\Sigma_n\, \Phi^T_{1n}(n)\, g_n(\neq n) f_n(n) = H(123)\\
&\Sigma_n\, \Phi^T_{2n}(n)\, g_n(\neq n) f_n(n) = 0\\
&\Sigma_n\, \Phi^T_{3n}(n)\, g_n(\neq n) f_n(n) = 0
\end{aligned} \qquad (19)$$

or

$$\Phi^T\, V = W$$

where

$$V = \begin{pmatrix} g_1(23)f_1(1) \\ g_2(31)f_2(2) \\ g_3(12)f_3(3) \end{pmatrix} \qquad W = \begin{pmatrix} H(123) \\ 0 \\ 0 \end{pmatrix}$$

Following tradition, we define to $S = \det(\Phi)$.

Inverting this matrix equation

If the above equation is to have a solution for V, we must have $S \neq 0$ and we know that

$$V = [\Phi^T]^{-1}W \qquad \text{where} \qquad [\Phi^T]^{-1}_{pq} = \text{cof}(\Phi_{pq})/S = (-1)^{p+q}\,\text{min}(\Phi_{pq})/S$$

where we use a standard formula for an inverse matrix. For example, we have

$$S\ [\Phi^{T}]^{-1}_{21} = (-1)^{2+1} \min(\Phi_{21}) = -\min(\Phi_{21}) = -[\ \Phi_{12}(1)\Phi_{23}(3) - \Phi_{13}(1)\Phi_{32}(3)\ ]$$

where we need this visual aid to verify this is correct

$$\Phi = \begin{pmatrix} \Phi_{11}(1) & \Phi_{12}(1) & \Phi_{13}(1) \\ \Phi_{21}(2) & \Phi_{22}(2) & \Phi_{23}(2) \\ \Phi_{31}(3) & \Phi_{32}(3) & \Phi_{23}(3) \end{pmatrix}$$

A minor determinant is formed by crossing out a row and a column. In our example above we crossed out the second row and the first column of Φ. Since the second row is gone, we have nothing that is a function of ξ_2 in our matrix element $[\Phi^{T}]^{-1}_{21}$. It will always be the case that $[\Phi^{T}]^{-1}_{pq}$ will not be a function of variable ξ_p so in our previous notation we would use ≠p to indicate this.

Introduction of G_{pb} as notation for cof(Φ_{pq}), and then M_p as G_{p1}.

So for another shorthand, lets define

$$G_{pq}(\neq p) \equiv [\Phi^{T}]^{-1}_{pq} \qquad // = \text{cof}(\Phi_{pq}) = \text{a number, not a matrix}$$

Then our solution vector's components (times S) are given by

$$SV_p = G_{pq}(\neq p)W_q$$

But $W_q = \delta_{q,1}H(123)$ so we have

$$SV_p = G_{pq}(\neq p)\ \delta_{q,1}H(123) = G_{p1}(\neq p)\ H(123)$$

But we defined V such that $V_p = g_p(\neq p)f_p(p)$. Thus we have

$$g_p(\neq p)f_p(p) = G_{p1}(\neq p)\ H(123)/S \qquad p = 1,2,3 \qquad (20)$$

This then is our processed version of equation (19) that we are trying to solve, and if we can solve, then we have found a separated solution!

Now let us recall some earlier equations:

$w_1(123) \equiv (H/h_1^2)$ // w_1 is *determined* by our of curvilinear system! *(5)*
$w_1(123) = g_1(23)f_1(1)$ // a restriction on our curvilinear system *(8)*

Therefore we know that

$$g_1(23)p_1(1) = g_1(23)f_1(1) = H(123)/h_1^2$$

or more generally

$$g_p(\neq p)f_p(p) = H(123)/h_p^2 \qquad p = 1,2,3 \qquad (21)$$

So insert this into (20) and we get

$$1/h_p^2 = G_{p1}(\neq p)/S = \text{cof}(\Phi_{p1})/S \equiv M_p(\neq p)/S \qquad (22)$$

where to save one useless index we define $M_p(\neq p)$ as shown.

Note: M&F use the word "minor" to mean what we call "cofactor". The terms cofactor or co-factor do not appear in either M&F volume! Was this a standard thing at one point in history? In any event, M&F use this notation M_p which they call a "minor" but which we call a "cofactor". Thus we have

$$1/h_p^2 = M_p(\neq p)/S \qquad (23)$$

and we can then quote (to show we are on track now with M&F)

$$1/h_n^2 = M_n/S \qquad (5.1.31)$$

Now let's look back again at (20) above where we had

$$1/h_p^2 = g_p(\neq p)f_p(p)/H(123)$$

Combining this with (22) says

$$H(123)\, M_p(\neq p)/S = g_p(\neq p)f_p(p)$$

$$H(123)/S = g_p(\neq p)f_p(p)/\, M_p(\neq p)$$

or

$$H(123)/S(123) = g_1(\neq 1)f_1(1)/\, M_p(\neq 1) = g_2(\neq 2)f_2(2)/\, M_2(\neq 2) = g_3(\neq 3)f_3(3)/\, M_3(\neq 3) \qquad (24)$$

Remember that the $g_i f_i$ factors are determined by our curvilinear system, since we said in (21) that

$$g_p(\neq p)f_p(p) = H(123)/h_p^2$$

whereas the factors M_p depend on our choice for 6 of the 9 functions of the matrix Φ.

Now stare at (24). The first equality says that, as a function of ξ_1 , $H(123)/S = K(\xi_2,\xi_3)\, f_1(\xi_1)$. In other words, H(123)/S is proportional to f(1). But it is also proportional to f(2) and f(3) as the next two equalities tell us. We must therefore be able to write

$$H(123)/S(123) = K\, f_1(1)\, f_2(2)\, f_3(3)$$

The constant K cannot depend on ξ_1 because then the first equality would not be true! Similarly for ξ_2 and ξ_3. So K is just a constant. Now look back at our separated equations

$$\mathcal{L}_n X_n = (1/f_n)\partial_n[f_n(\partial_n X_n)] + [\, k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0 \qquad (16)$$

If we multiply f_n by 3, this equation is unchanged. Thus, the scale of the f_n functions is irrelevant. If we have some scale where $H(123)/S = K\, f_1(1)\, f_2(2)\, f_3(3)$ for some K, then we can just change the scale say of f_1 to cancel K. Thus, we loose no generality by setting K = 1 to get

$$H(123)/S(123) = f_1(1)\, f_2(2)\, f_3(3) \qquad (25)$$

and again we quote M&F

the *Robertson condition*

$$h_1h_2h_3/S = f_1(\xi_1)f_2(\xi_2)f_3(\xi_3) \qquad (5.1.32)$$

Summary of key equations obtained in the previous section

We define $H = H(123) = h_1h_2h_3$. We define $S = S(123) = \det(\Phi)$.

Our first assumption was this: [from (8a)]

$$g_n(\neq n)f_n(n) = H/h_n^2 \qquad n = 1,2,3 \qquad (2.1)$$

The RHS is forced by our CL system, and we have to assume that it has the functional form shown on the LHS. This may or may not work. If it does not work, our CL system is not separable. This equation determines both $f_n(n)$ and $g_n(\neq n)$ [ignoring a relative scaling constant] from the CL system.

Our next interesting equation was this [from (20)]

$$g_n(\neq n)f_n(n) = M_n(\neq n)\, H/S \qquad n = 1,2,3$$

which can be written this way so the LHS does not depend on n,

$$H/S = g_n(\neq n)f_n(n)/\, M_n(\neq n) \qquad n = 1,2,3 \qquad (2.2)$$

Combining the above two equations gives this one [appears as (23)]

$$1/h_n^2 = M_n(\neq n)/S \qquad (2.3)$$

We note in passing that this says

$$M_n h_n^2 = S \quad \Rightarrow \quad M_i h_i^2 = M_j h_j^2 \quad \Rightarrow \quad M_i/M_j = h_j^2/h_i^2 \qquad (2.3a)$$

which appears in M&S page 7 1.27.

The next equation is the Robertson condition [from (25)]

$$H/S = f_1(1)\, f_2(2)\, f_3(3) \qquad (2.4)$$

Since we found the f_i from 2.1, this last tells us S. Again in passing, for the last two equations above we can solve each for 1/S and set these equal to get

$$f_1(1)\, f_2(2)\, f_3(3)/H = 1/M_n h_n^2 \quad \Rightarrow \quad M_n\, f_1(1)\, f_2(2)\, f_3(3) = 1/h_n^2 \qquad (2.4a)$$

which is the other equation appearing in M&S page 7 1.27.

We can combine (2.4) with the second last to get

$$H/S = f_1(1)\, f_2(2)\, f_3(3) = g_1(\neq 1) f_1(1)/ M_1(\neq 1) \quad \Rightarrow \quad f_2(2)\, f_3(3) = g_1(\neq 1) / M_1(\neq 1)$$

so we then have these results

$$\begin{aligned} M_1(\neq 1) &= g_1(\neq 1)/[\, f_2(2)\, f_3(3)] \\ M_2(\neq 2) &= g_2(\neq 2)/[\, f_3(3)\, f_1(1)] \\ M_3(\neq 3) &= g_3(\neq 3)/[\, f_1(1)\, f_2(2)] \end{aligned} \qquad (2.5)$$

We found the g_n and the f_n from (2.1), so this last result tells us the M_n.

So given some CL system, we know from these few equations how to find:

f_n g_n S M_n

More work is needed to actually determine the 9 functions of matrix Φ, in theory it can be done with the above information.

Review of what has happened to this point:

We sought operators L_n of the most general 2nd order form such that

$$L_n X_n = \partial_n[f_n(\partial_n X_n)] + r_n \partial_n X_n + q_n X_n = 0 \qquad n=1,2,3$$

In order to have any chance of success, we had to make our first assumption [(5) with (8)],

$$g_n(\neq n) f_n(n) = H(123)/h_n^2$$

This is a functional-form restriction for the RHS! Some systems won't give a RHS which has the functional form on the LHS. If our system *does* meet this condition, then, apart from a multiplicative constant, which we shall see does not matter, the $f_n(n)$ are determined, and hence also the $g_n(\neq n)$ are determined.

With this assumption, we were able to write the Helmholtz equation in various forms:

$$(\nabla^2 + k_1^2)\psi = H^{-1}\{ g_1(23)\,\partial_1[p_1(\partial_1 X_1)]/X_1 + \text{cyclic}\}\psi + k_1^2\psi = 0 \qquad (9)$$

$$(\nabla^2 + k_1^2)\psi = H^{-1}\{ g_1(23)\,[(L_1 - r_1\partial_1 - q_1)X_1]/X_1 + \text{cyclic}\}\psi + k_1^2\psi = 0 \qquad (10)$$

$$\{ g_1(23)\,[(- r_1\partial_1 - q_1)X_1]/X_1 + \text{cyclic}\} + H(123)\,k_1^2 = 0 \qquad (12)$$

We showed that if X_n are arbitrary solutions of their $L_nX_n = 0$, then it was impossible to solve this last equation unless we select all $r_n = 0$. In that case the above became

$$g_1(23)\,q_1(1) + g_2(31)\,q_2(2) + g_3(12)q_3(3) = H(123)\,k_1^2 \qquad (14)$$

At this point our candidate L_n were rescaled [$p_1 \rightarrow f_1$ and $(q_1/p_1) \rightarrow v_1$] and had this self-adjoint form

$$\mathcal{L}_nX_n = (1/f_n)\partial_n[f_n(\partial_nX_n)] + v_nX_n \qquad n = 1,2,3$$

which shows why the scaling of f_n does not matter. We do not know the v_n, we just have this self-adjoint form. Note that L_{3D} was formally adjoint in its 3D space. We WANT to have self-adjoint $\mathcal{L}_n$ operators so we can do SL work with are separated equations so we are not unhappy to have been forced this way.

Next we chose to write the functions $v_n(n)$ in a very obscure but certainly allowable manner

$$v_n = k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)$$

where k_1^2 is the Helmholtz parameter, and where we hope to find that k_2^2 and k_3^2 are arbitrary parameters, which we shall then refer to as our "separation constants". We have now suddenly involved a set of 9 as yet unknown functions which form a matrix known as the Stackel Matrix

$$\Phi = \begin{pmatrix} \Phi_{11}(1) & \Phi_{12}(1) & \Phi_{13}(1) \\ \Phi_{21}(2) & \Phi_{22}(2) & \Phi_{23}(2) \\ \Phi_{31}(3) & \Phi_{32}(3) & \Phi_{33}(3) \end{pmatrix} \qquad \det(\Phi) = S = S(123)$$

This allows us to write our "equations we have to satisfy to get separability" in matrix form (that is to say, the 3D Helmholtz equation with our assumed ODE's for the X_i)

$$\Phi^T V = W \qquad \text{where} \qquad V = \begin{pmatrix} g_1(23)f_1(1) \\ g_2(31)f_2(2) \\ g_3(12)f_3(3) \end{pmatrix} \qquad W = \begin{pmatrix} H(123) \\ 0 \\ 0 \end{pmatrix}$$

As long as we select our functions so that $S \neq 0$, we can invert our matrix equation to get

$$V = [\Phi^T]^{-1}W$$

where W is a multiple of the unit vector as shown. We then are able to show these two facts:

$1/h_p^2 = M_p(\neq p)/S(123)$ // Fact 1

where $M_p(\neq p) = G_{p1}(\neq p) = \text{cof}(\Phi_{p1})$ = a cofactor of element p in the first column of Φ. The second fact is this

$H(123)/S(123) = f_1(1)\, f_2(2)\, f_3(3)$ " the Robertson Condition" // Fact 2

I think our goal is now to find a viable Φ matrix which satisfies these two Facts. Recall that the functions f_n were determined by our earlier condition

$g_n(\neq n)f_n(n) = H(123)/h_n^2$ // Fact 0

If we look at M&S page 6, we see Fact 1 and Fact 2 written as above, where

$H(123) = h_1h_2h_3 = g^{1/2}$ where $g \equiv \det(g_{ij}) = H^2$ and where $g_{ii} = h_i^2$ as usual.

Thus M&S state

Fact 1: $1/g_{ii} = M_i/S = G_{i1}/S \Rightarrow g_{ii} = S/G_{i1}$

Fact 2: $g^{1/2}/S = f_1(1)\, f_2(2)\, f_3(3)$ p 6 (1.25)

Here is how you are supposed to proceed with a given candidate curvilinear system:

- Determine the three $f_n(\xi_n)$ from Fact 0.

- Determine S(123) from Fact 2.

- Fact 1 then tells you the three $M_p = \text{cof}(\Phi_{p1})$ which are the cofactors of the three elements of the leftmost column of the matrix. We have for example

$M_1 = \text{cof}(\Phi_{11}) = \Phi_{22}(2)\Phi_{33}(3) - \Phi_{23}(2)\Phi_{32}(3)$

We seem then to have only four conditions on the nine Φ functions, this being the fourth

$S(123) = \Phi_{11}\, M_1 + \Phi_{21}\, M_2 + \Phi_{31}\, M_3$

Thus, there will be very many possible matrices that work. But, there are serious "functional form" restrictions, so we don't really have 4 conditions on 9 functions in a general sense.

When all is said and done, it turns out that an important 3D orthogonal curvilinear system that works is the ellipsoidal system involving as it does confocal quadric surfaces (ellipsoids, hyperboloids of one and two sheets). If you don't have confocal quadric surfaces, you don't get separation. Any system that is a degenerate limit of the ellipsoidals will be separable as well. This then is why all 11 classical systems are separable. All along here we are talking about Euclidean coordinate systems, by the way.

The *reason* you need to know a workable Stackel matrix is of course that its 9 member functions then tell you your separation equations! Recall,

$$\mathcal{L}_n X_n = (1/f_n)\partial_n[f_n(\partial_n X_n)] + v_n X_n \qquad n = 1,2,3$$

$$v_n = k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)$$

Question: It seems that different choices for a viable Stackel matrix might give you different separated equations! Is that possible? I don't see either M&F or M&S addressing this question. I cannot quickly find comments on this. Let's write out the equations in more detail:

$$M_1(23) = \Phi_{22}(2)\Phi_{33}(3) - \Phi_{23}(2)\Phi_{32}(3)$$
$$M_2(31) = \Phi_{33}(3)\Phi_{11}(1) - \Phi_{31}(3)\Phi_{13}(1)$$
$$M_3(12) = \Phi_{11}(1)\Phi_{22}(2) - \Phi_{12}(1)\Phi_{21}(2)$$

$$S(123) = \Phi_{11}(1)\, M_1(23) + \Phi_{21}(2)\, M_2(31) + \Phi_{31}(3)\, M_3(12$$

$$= \Phi_{11}(1)[\Phi_{22}(2)\Phi_{33}(3) - \Phi_{23}(2)\Phi_{32}(3)]$$
$$+ \Phi_{21}(2)[\Phi_{33}(3)\Phi_{11}(1) - \Phi_{31}(3)\Phi_{13}(1)]$$
$$+ \Phi_{31}(3)[\Phi_{11}(1)\Phi_{22}(2) - \Phi_{12}(1)\Phi_{21}(2)]$$

Ellipsoidal Coordinates

Let's examine the ellipsoidal system as an example. M&F claim that

$$S = \frac{h_1 h_2 h_3}{f_1 f_2 f_3} = \frac{(\xi_1^2 - \xi_2^2)(\xi_2^2 - \xi_3^2)(\xi_3^2 - \xi_1^2)}{\sqrt{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}}$$

$$M_1 = \frac{S}{h_1^2} = -\frac{\xi_2^2 - \xi_3^2}{\sqrt{(\xi_2^2 - a^2)(\xi_3^2 - a^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}}; \quad \text{etc.}$$

Here is the solution M&F give

$$S = \begin{vmatrix} 1 & \frac{1}{(\xi_1^2 - a^2)} & \frac{1}{(\xi_1^2 - b^2)(a^2 - b^2)} \\ 1 & \frac{1}{(\xi_2^2 - a^2)} & \frac{1}{(\xi_2^2 - b^2)(a^2 - b^2)} \\ 1 & \frac{1}{(\xi_3^2 - a^2)} & \frac{1}{(\xi_3^2 - b^2)(a^2 - b^2)} \end{vmatrix}$$

Notice that

$$\Phi_{n1}(1) = 1$$

$\Phi_{n2}(n) = (\xi_n^2 - a^2)^{-1}$
$\Phi_{n3}(n) = (\xi_n^2 - b^2)^{-1}(a^2 - b^2)^{-1}$

so only three "functional forms" appear, not 9.

$$M_1 = (\xi_2^2 - a^2)^{-1}(\xi_3^2 - b^2)^{-1}(a^2 - b^2)^{-1} - (\xi_3^2 - a^2)^{-1}(\xi_2^2 - b^2)^{-1}(a^2 - b^2)^{-1}$$

$$= (a^2 - b^2)^{-1} \{ (\xi_2^2 - a^2)^{-1}(\xi_3^2 - b^2)^{-1} - (\xi_3^2 - a^2)^{-1}(\xi_2^2 - b^2)^{-1} \}$$

$$= (a^2 - b^2)^{-1} \{(\xi_3^2 - a^2)(\xi_2^2 - b^2) - (\xi_2^2 - a^2)(\xi_3^2 - b^2) \} (\xi_2^2 - a^2)^{-1}(\xi_3^2 - b^2)^{-1} (\xi_3^2 - a^2)^{-1}(\xi_2^2 - b^2)^{-1} \}$$

But

$$\{(\xi_3^2 - a^2)(\xi_2^2 - b^2) - (\xi_2^2 - a^2)(\xi_3^2 - b^2) \} = \xi_2^2(b^2 - a^2) - \xi_3^2(b^2 - a^2) = (a^2 - b^2)(\xi_3^2 - \xi_2)^2$$

so

$$M_1 = (\xi_3^2 - \xi_2)^2 (\xi_2^2 - a^2)^{-1}(\xi_3^2 - b^2)^{-1} (\xi_3^2 - a^2)^{-1}(\xi_2^2 - b^2)^{-1} \}$$

which confirms the M_1 quoted above (note the leading minus sign).

Somehow I think you could argue that the three rows have to have the same three functions due to the usual very heavy ellipsoidal cyclic symmetry everywhere.

The Stackel matrix must have some family of symmetry transformations that leave the 4 conditions unaltered.

$M_1(23) = \Phi_{22}(2)\Phi_{33}(3) - \Phi_{23}(2)\Phi_{32}(3)$
$M_2(31) = \Phi_{33}(3)\Phi_{11}(1) - \Phi_{31}(3)\Phi_{13}(1)$
$M_3(12) = \Phi_{11}(1)\Phi_{22}(2) - \Phi_{12}(1)\Phi_{21}(2)$
$S(123) = \Phi_{11}(1)\, M_1(23) + \Phi_{21}(2)\, M_2(31) + \Phi_{31}(3)\, M_3(12)$

But in this case, I think symmetry forces your hand pretty much. You cannot double every entry. In order to get the right $M_{1,2,3}$ you cannot play much with the last two columns! You might swap constants a and b or negate the two last columns, but it looks pretty constrained to me. So maybe in simpler systems it is easier to see things. [I later found rules for S matrix transformations.]

What do we know about extruded conformal 3D systems (cylindricals)?

If 3 is the z coordinate, we know that $h_1 = h_2$ and $h_3 = 1$ (still Cartesian). This is because in all our normal such systems, the 2D pattern is obtained by a conformal map of Cartesians. This applies to regular cylindricals only if you do it with $r = e^{\xi}$. We also know that $h_1 = h_1(12)$. So Fact 0 tells us

$$g_1(\neq 1)f_1(1) = H(123)/h_1^2 = 1 = g_2(\neq 2)f_2(2) \qquad \text{// Fact 0}$$

so we know then that $f_1 = f_2 = 1$ and $g_1 = g_2 = 1$. Then

$$g_3(\neq 3)f_3(3) = H(123)/h_3^2 = h_1^2/h_3 = h_1^2 = h_1^2(12)$$

so we conclude that $g_3(\neq 3) = h_1^2(12)$ and $f_3 = 1$.

Fact 2 then says

$$S(123) = H(123)/(f_1 f_2 f_3) = h_1^2$$

So we arrive at some pretty simple results:

$f_1 = f_2 = f_3 = 1$
$g_1 = g_2 = 1 \qquad g_3 = h_1^2(12)$

$S(123) = h_1^2(12)$ does not depend on z.

Now don't let's forget Fact 1

$$1/h_p^2 = M_p(\neq p)/S(123) \qquad \text{// Fact 1}$$

which says

$1 = M_3/h_1^2(12) \qquad M_3 = h_1^2(12)$
$1/h_1^2 = M_1/h_1^2(12) \qquad M_1 = 1$
$1/h_2^2 = M_2/h_1^2(12) \qquad M_2 = 1$
$S = h_1^2(12)$

This suggests to me a matrix of this form

$$\begin{pmatrix} 0 & -1 & \alpha(1) \\ 0 & 1 & \beta(2) \\ 1 & 0 & 1 \end{pmatrix}$$

Then $M_1 = 1$ and $M_2 = 1$ and $M_3 = -\beta(2) - \alpha(1)$ and $S = M_3$ so we end up with hopefully writing

$$-\beta(2) - \alpha(1) = h_1^2(12).$$

Important fact: this is only going to work in *some* cylinder systems!! If you cannot write $h_1^2(12)$ as the sum of two functions in this manner, you cannot get separation! An example is M&S p 86 for tangent-cylinder coordinates where $h_1^2(12) = 1/(\xi_1^2 - \xi_2^2)^2$. I think only the classical cylinder systems are separable! That is why they are classical!

Now look at M&S p 17 on elliptical cylinder $\eta = \xi_1$. This has $h_1^2 = a^2(ch^2\eta - \cos^2\psi)$ so we get

$$-\beta(\psi) - \alpha(\eta) = a^2 ch^2\eta - a^2\cos^2\psi$$

$\beta(\psi) = a^2\cos^2\psi \qquad \alpha(\eta) = -a^2 ch^2\eta$

and I then get

$$\begin{pmatrix} 0 & -1 & -a^2ch^2\eta \\ 0 & 1 & a^2\cos^2\psi \\ 1 & 0 & 1 \end{pmatrix}$$

and this agrees exactly with M&S page 17.

What do we know about extruded other 3D systems (cylindricals)?

Suppose we had a cylindrical system in which $h_1(12) \neq h_2(12)$ and $h_3 = 1$. Then what happens?

$$g_1(\neq 1)f_1(1) = H(123)/h_1^2 = h_2(12)/h_1(12) \qquad \text{// Fact 0}$$

Well the only case I know of is regular cylindrical in which case r,ψ,z we gave

$$h_1(12) = h_r = 1$$
$$h_2(12) = h_\psi = r = h_2(1)$$

so we really then have

$$g_1(\neq 1)f_1(1) = h_2(1)/1 \quad \Rightarrow \quad f_1(1) = h_2(1) = r$$

and everything gets pretty trivial as on page 12 M&S. This does have the general form of p 7 1.28, and so do my simpler $h_1 = h_2$ cases.

Symmetry operations on matrix Φ revisited. We know that negating a column OR swapping two columns negates the determinant. Thus, we have a few simple symmetry operations we can perform on S. On is to swap the second two columns and negate one of them. The second is to negate both columns. We *cannot* swap two rows because that shuffles the M_i. If two of the M's are the same you might do something, but I don't think in the general case. So I guess symmetry is limited to the first thing I said in the general case.

Comments on the "separation constants". In this analysis, they don't appear in the pleasing obvious way that they appear, say, in the separation of the H atom in QM. Our SE there becomes something like this

$$f(r) + g(\theta,\varphi) = 0$$

and then we are allowed to say [functions here represent things like $L_rR(r) / R(r)$]

$$f(r) = -k$$
$$g(\theta,\varphi) = k$$

and we refer to k as a "separation constant". Quantization from the θ,φ system later causes k = n(n+1). Then within that g(θ,φ) equation we find

$$h_k(\theta) + w(\varphi) = 0$$

so we are allowed to say

$$w(\varphi) = m$$
$$h_k(\theta) = -m$$

and that m is our second separation constant, and we end up with a solution of the form

$$a_k(r)\, b_{k,m}(\theta)\, c_m(\varphi)$$

and we get a characteristic "linkage" of the separation constants. This is the "traditional method" of showing that there ARE two separation constants, and of finding out "how they quantize" in the SL problems of the individual coordinates.

In contrast, in our method above we just ASSUME there are two separation constants when we write our candidate separated ODE in this manner involving 9 functions of a matrix of functions Φ,

$$\mathcal{L}_n X_n = (1/f_n)\partial_n[f_n(\partial_n X_n)] + [\, k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0 \qquad (16)$$

With this assumption, and few other assumptions made along the path, the Helmholtz equation becomes a matrix equation $\Phi^T V = W$ where

$$V = \begin{pmatrix} g_1(23)f_1(1) \\ g_2(31)f_2(2) \\ g_3(12)f_3(3) \end{pmatrix} \qquad W = \begin{pmatrix} H(123) \\ 0 \\ 0 \end{pmatrix}$$

When we simply invert this equation to get $V = [\Phi^T]^{-1}W$ we find that

$$g_p(\neq p)f_p(p) = M_p(\neq p)\, H(123)/S \qquad p = 1,2,3 \qquad (20)$$

This and the earlier assumptions lead us to this set of equations which must be true

$$g_p(\neq p)f_p(p) = H(123)/h_p^2$$
$$M_p(\neq p) = S(123)/h_p^2$$
$$S(123) = H(123)/[\, f_1(1)\, f_2(2)\, f_3(3)]$$

If the system is viable, the first equation tells us the three $f_p(p)$ [and in passing the $g_p(\neq p)$], the second equation gives us the three cofactors M_p, and the third gives us S = det(Φ). We then do trial and error to find a viable matrix Φ that has these cofactors and this determinant.

How to find the matrix Φ

Let's assume now that Φ has this form (h and g are now overloaded)

$$\Phi = \begin{pmatrix} a(1) & b(1) & c(1) \\ d(2) & e(2) & f(2) \\ g(3) & h(3) & i(3) \end{pmatrix}$$

$$M_1 = e(2)i(3)-f(2)h(3)$$
$$-M_2 = b(1)i(3)-c(1)h(3)$$
$$M_3 = b(1)f(2)-c(1)e(2)$$

Let's focus on the first equation:

$$e(2)i(3)-f(2)h(3) = M_1(23) = [S(123)/H(123)]\, g_1(\neq 1)f_1(1) = [\, f_1(1)\, f_2(2)\, f_3(3)]^{-1}\, g_1(\neq 1)f_1(1)$$
$$= g_1(23)/\, [f_2(2)\, f_3(3)]$$

At least we have the RHS in a form that shows it depends only on 2 and 3. *If* at this point we discovered that $g_1(23) = \sqrt{2^2-3^2}$, we would conclude that things were non-separable. So just getting this far does not guarantee success. What we need to assume is that $g_1(23)$ can be written as a sum of *at most two terms* each of which factors [try putting 3 terms and see what happens] . So write

$$g_1(23) = j_1(2)k_1(3) + m_1(2)n_1(3)$$ // we assume this can be done

If this is the case, then we have

$$e(2)i(3)-f(2)h(3) = [j_1(2)k_1(3) + m_1(2)n_1(3)]/\, [f_2(2)\, f_3(3)]$$

$$= (j_1(2)/f_2(2))\, (k_1(3)/f_3(3)) + (m_1(2)/f_2(2))\, (n_1(3)/f_3(3))$$

Later on where I worked on this more, I realized you can think of this last equation as

$$e_2i_3 - f_2h_3 = q_2r_3 - s_2\, t_3$$

If you are given the 4 functions of the RHS, it at first seems there are *very many* choices for the 4 functions of the LHS. In fact, if you think of each side as the det of a 2x2 matrix,

$$\det\begin{pmatrix} e_2 & f_2 \\ h_3 & i_3 \end{pmatrix} = \det\begin{pmatrix} q_2 & s_2 \\ t_3 & r_3 \end{pmatrix}$$

then the left matrix could be the right matrix times any unimodular third matrix you choose (right or left multiplied). But this turns out to be wrong due to the functional form requirements. In fact, I think the possibilities are very limited. For example, we can write

$$\det\begin{pmatrix} e_2 & f_2 \\ h_3 & i_3 \end{pmatrix} = \det\begin{pmatrix} \alpha q_2 & \beta s_2 \\ t_3/\beta & r_3/\alpha \end{pmatrix} \quad \Rightarrow \quad e_2i_3 - f_2h_3 = (\alpha q_2)(r_3/\alpha) - (\beta s_2)(\, t_3/\beta)$$

where we now have to arbitrary scaling constants which obviously do not change the 2x2 determinant. So I think this is the most general solution you can have to our problem

$$e_2 i_3 - f_2 h_3 = q_2 r_3 - s_2 t_3$$

Solution #1 is this:

$$e_2 i_3 = (\alpha q_2)(r_3/\alpha) \quad \text{and} \quad f_2 h_3 = (\beta s_2)(t_3/\beta)$$

Solution #2 is this:

$$-f_2 h_3 = (\alpha q_2)(r_3/\alpha) \quad \text{and} \quad -e_2 i_3 = (\beta s_2)(t_3/\beta)$$

So we can write out these solutions this way

solution #1:	$e_2 = \alpha q_2$	$i_3 = r_3/\alpha$	$f_2 = \beta s_2$	$h_3 = t_3/\beta$
solution #2:	$-f_2 = \alpha q_2$	$i_3 = t_3/\beta$	$-e_2 = \beta s_2$	$h_3 = r_3/\alpha$

In each solution we are free to select α,β as any constants, and these solutions really are different.

<u>Let us now summarize what we have just learned:</u>

If we assume these labels for the functions of Φ

$$\Phi = \begin{pmatrix} a(1) & b(1) & c(1) \\ d(2) & e(2) & f(2) \\ g(3) & h(3) & i(3) \end{pmatrix}$$

we have to select the rightmost 6 functions so they satisfy these three equations

$$M_1 = e(2)i(3)-f(2)h(3)$$
$$-M_2 = b(1)i(3)-c(1)h(3)$$
$$M_3 = b(1)f(2)-c(1)e(2)$$

where, from (2.5) above, we know that

$$M_1(\neq 1) = g_1(\neq 1)/[\, f_2(2)\, f_3(3)]$$
$$M_2(\neq 2) = g_2(\neq 2)/[\, f_3(3)\, f_1(1)]$$
$$M_3(\neq 3) = g_3(\neq 3)/[\, f_1(1)\, f_2(2)] \qquad (2.5)$$

So our problem then is to solve

$$e(2)i(3)-f(2)h(3) = + g_1(\neq 1)/[\, f_2(2)\, f_3(3)]$$
$$b(1)i(3)-c(1)h(3) = - g_2(\neq 2)/[\, f_3(3)\, f_1(1)]$$
$$b(1)f(2)-c(1)e(2) = + g_3(\neq 3)/[\, f_1(1)\, f_2(2)]$$

In order to even stand a chance, we must be able to write the g_n in this manner

$g_1(23)/(f_2f_3) = u_{12}(2)u_{13}(3) - v_{12}(2)v_{13}(3)$
$g_2(31)/(f_3f_1) = u_{23}(3)u_{21}(1) - v_{23}(3)v_{21}(1)$
$g_3(12)/(f_1f_2) = u_{31}(1)u_{32}(2) - v_{31}(1)v_{32}(2)$

which basically says we can write $g_n(\neq n)$ as a sum of at most two factored terms. Assuming we can do this (if we cannot, they we cannot separate!), we then have

$e(2)i(3)-f(2)h(3) = + u_{12}(2)u_{13}(3) - v_{12}(2)v_{13}(3)$
$b(1)i(3)-c(1)h(3) = - u_{23}(3)u_{21}(1) + v_{23}(3)v_{21}(1)$
$b(1)f(2)-c(1)e(2) = + u_{31}(1)u_{32}(2) - v_{31}(1)v_{32}(2)$

At this point it is probably good to negate the LHS of the middle equation, and also to reverse the order in each of its products. Now we have the three RHS's all looking the same

$e(2)\,i(3)-f(2)h(3) = + u_{12}(2)u_{13}(3) - v_{12}(2)v_{13}(3)$ // = $q_2r_3 - s_2\,t_3$ of previous analysis
$h(3)c(1)- i(3)b(1) = + u_{23}(3)u_{21}(1) - v_{23}(3)v_{21}(1)$
$b(1)f(2)-c(1)e(2) = + u_{31}(1)u_{32}(2) - v_{31}(1)v_{32}(2)$

We can now state the most general solutions of the first equation:

solution #1: $e_2 = \alpha_1\, u_{12}(2)$ $i_3 = u_{13}(3)/\alpha_1$ $f_2 = \beta_1\, v_{12}(2)$ $h_3 = v_{13}(3)/\beta_1$
solution #2: $-f_2 = \alpha_1\, u_{12}(2)$ $i_3 = v_{13}(3)/\beta_1$ $-e_2 = \beta_1\, v_{12}(2)$ $h_3 = u_{13}(3)/\alpha_1$

where we have now subscripted the general constants α and β, and installed our more generalized function names. We can similarly write the possible solutions for second equation. To do this, we make these changes: $e_2 \rightarrow h_3$, $i_3 \rightarrow c_1$, $f_2 \rightarrow i_3$, $h_3 \rightarrow b_1$ then cyclic all the RHS's

solution #1: $h_3 = \alpha_2\, u_{23}(3)$ $c_1 = u_{21}(1)/\alpha_2$ $i_3 = \beta_1\, v_{23}(3)$ $b_1 = v_{31}(1)/\beta_2$
solution #2: $-i_3 = \alpha_2\, u_{23}(3)$ $c_1 = v_{21}(1)/\beta_2$ $-h_3 = \beta_2\, v_{23}(3)$ $b_1 = u_{21}(1)/\alpha_2$

When you make the substitutions with the arrows, it is good to turn each one red after you have done it, then you won't get confused, then make them black again when done.

Now, rather than write the general solutions for the third equation, we will just insert the solutions we have already found and "see what happens". Let's use our "solution #1" for both the first rows. Then the third row becomes

$b(1)f(2)-c(1)e(2) = + u_{33}(1)u_{32}(2) - v_{31}(1)v_{32}(2)$

$[v_{31}(1)/\beta_2][\,\beta_1\, v_{12}(2)] - [u_{21}(1)/\alpha_2][\,\alpha_1\, u_{12}(2)] = + u_{31}(1)u_{32}(2) - v_{31}(1)v_{32}(2)$

This is one of four choices we could have made for the LHS. This choice requires that

$[v_{31}(1)/\beta_2][\ \beta_1\ v_{12}(2)] = -v_{31}(1)v_{32}(2)$ => $(\beta_1/\beta_2)\ v_{12}(2) = -v_{32}(2)$
$[u_{21}(1)/\alpha_2][\ \alpha_1\ u_{12}(2)] = +u_{31}(1)u_{32}(2)$

I think we have now reached the end of the road. In *general*, $u_{31}(1)$ could be completely unrelated to all other functions with a 1 argument in its functional form. So no matter which of the four choices of solutions to the first equations we take, we end up with an equation involving $u_{31}(1)$. It might be that this cannot be expressed as constant multiple of the other argument-1 function appearing in the second equation, and then there will be no solution.

So I was hoping to show that a general solution was always possible, but I see that is not the case if we think of *completely general* $g_n(\neq n)$ functional forms as sums of two factored terms.

The upshot is this: we will solve the problem in ellipsoidals using trial and error, and then we will know that a solution must exist in all systems obtained from ellipsoidals, which includes all the classical 11 systems. These are the only systems in which Helmholtz is separable! In the ellipsoidal case, we will find that we don't need the most general form of two terms for the g_i as used above, one term will do, which means we can set all the v's to 0 in the above.

Finding the Stackel matrix for the ellipsoidal coordinate system

The need to use F_1,F_2, F_3 for our Stackel analysis

Before continuing, let's quote values from the ellipsoidal system, just to "keep us honest". Here are some facts from that system as presented in MF:

$f_1^2 = (\xi_1^2-a^2)(\ \xi_1^2-b^2) = ++ = +$ $\quad G_1 = (\xi_2^2-\xi_3^2) > 0$
$f_2^2 = (\xi_2^2-a^2)(\ \xi_2^2-b^2) = -+ = -$ $\quad G_2 = (\xi_3^2-\xi_1^2) < 0$
$f_3^2 = (\xi_3^2-a^2)(\ \xi_3^2-b^2) = -- = +$ $\quad G_3 = (\xi_1^2-\xi_2^2) > 0$

$Q_1^2 = h_1^2 = G_3(-G_2)/f_1^2$
$Q_2^2 = h_2^2 = G_3G_1/(-f_2^2)$
$Q_3^3 = h_3^2 = (-G_2)G_1/f_3^2$

$h_1 = \sqrt{G_3(-G_2)/f_1^2}$
$h_2 = \sqrt{G_3G_1/(-f_2^2)}$
$h_3 = \sqrt{(-G_2)G_1/f_3^2}$

where all radical arguments are positive. M&F like these forms because they are regular and cyclic. The problem however is that f_2 is imaginary and this causes problems with the Stackel matrix analysis which basically thinks of things as real. So let us make these replacements

$F_1^2 = (\xi_1^2-a^2)(\ \xi_1^2-b^2)$
$F_2^2 = -(\xi_2^2-a^2)(\ \xi_2^2-b^2) = -f_2^2$

$$F_3^2 = (\xi_3^2 - a^2)(\xi_3^2 - b^2)$$

so that our three F_n are now all real. We continue to let G_2 be negative so the G's maintain their cyclic form. Looking at the MF data, we can write

$$h_1 = \sqrt{G_3(-G_2)/f_1^2} = \sqrt{G_3(-G_2)}/F_1 \qquad h_1^2 = -G_2G_3/F_1^2$$
$$h_2 = \sqrt{G_3G_1/(-f_2^2)} = \sqrt{G_1G_3}/F_2 \qquad h_2^2 = +G_3G_1/F_2^2$$
$$h_3 = \sqrt{(-G_2)G_1/f_3^2} = \sqrt{(-G_2)G_1}/F_3 \qquad h_3^2 = -G_1G_2/F_3^2$$

Things are NOT cyclic now with the h_i so be careful. For the h_i^2 things are cyclic except we have to change the sign of on the 2 equation. For H we get

$$H = \sqrt{G_3(-G_2)}/F_1 * \sqrt{G_1G_3}/F_2 * \sqrt{(-G_2)G_1}/F_3$$

$$= G_1(-G_2)G_3/[F_1F_2F_3]$$

which is real and positive, the way we like it.

The equations for the M_n cofactors

We now turn to (2.1) which says

$$g_1(\neq 1)F_1(1) = H/h_1^2 = G_1(-G_2)G_3/[F_1F_2F_3] / [-G_2G_3/F_1^2] = G_1F_1/[F_2F_3]$$

Keeping the 2-equation sign in mind, we get (the g_i are all real)

$$g_1(\neq 1) = +G_1/[F_2F_3]$$
$$g_2(\neq 2) = -G_2/[F_3F_1]$$
$$g_3(\neq 3) = +G_3/[F_1F_2]$$

Now (2.5) tells us that

$$M_1(\neq 1) = g_1(\neq 1)/[F_2(2)\,F_3(3)]$$
$$M_2(\neq 2) = g_2(\neq 2)/[F_3(3)\,F_1(1)]$$
$$M_3(\neq 3) = g_3(\neq 3)/[F_1(1)\,F_2(2)]$$

from which we conclude that (these are all real and positive)

$$M_1 = +G_1/(F_2^2F_3^2) = -(\xi_2^2 - \xi_3^2)/[f_2^2f_3^2]$$
$$M_2 = -G_2/(F_3^2F_1^2) = -(\xi_3^2 - \xi_1^2)/[f_3^2f_1^2]$$
$$M_3 = +G_3/(F_1^2F_2^2) = -(\xi_1^2 - \xi_2^2)/[f_1^2f_2^2] \qquad \text{// cyclic when using f's}$$

Trial and Error Solution for the last 2 columns of the Stackel matrix.

Now recall from above that (checked)

$M_1 = e(2)i(3) - f(2)h(3)$
$-M_2 = b(1)i(3) - c(1)h(3)$
$M_3 = b(1)f(2) - c(1)e(2)$

So we end up with these 3 equations on our 6 members of the Φ matrix

$e(2)i(3) - f(2)h(3) = G_1/(F_2^2F_3^2) = (\xi_2^2 - \xi_3^2)/(F_2^2F_3^2)$
$b(1)i(3) - c(1)h(3) = G_2/(F_3^2F_1^2) = (\xi_3^2 - \xi_1^2)/(F_3^2F_1^2)$
$b(1)f(2) - c(1)e(2) = G_3/(F_1^2F_2^2) = (\xi_1^2 - \xi_2^2)/(F_1^2F_2^2)$

where we no longer have a minus sign in the middle equation.

Now come the Big Questions:
(1) To what extent do these three equations nail down our 6 functions?
(2) Can we even *find* 6 functions that solve these 3 equations?
As we shall learn later, the equations do not completely nail down the 6 functions, despite the rather tight functional form requirements. So later we shall deal indirectly with question (1). Now try (2).

Let's try to find *a solution* by trial and error. First, write the three equations like this:

$e(2)i(3) - f(2)h(3) = (\xi_2^2/F_2^2)(1/F_3^2) - (1/F_2^2)(\xi_3^2/F_3^2)$
$b(1)i(3) - c(1)h(3) = (\xi_3^2/F_3^2)(1/F_1^2) - (1/F_3^2)(\xi_1^2/F_1^2)$
$b(1)f(2) - c(1)e(2) = (\xi_1^2/F_1^2)(1/F_2^2) - (1/F_1^2)(\xi_2^2/F_2^2)$

where the RHS is all cyclic. Now define the obvious factors and compress this to

$e_2i_3 - f_2h_3 = q_2r_3 - r_2q_3$ $\qquad q_i \equiv (\xi_i^2/F_i^2)$ $\qquad r_i = (1/F_i^2)$
$b_1i_3 - c_1h_3 = q_3r_1 - r_3q_1$
$b_1f_2 - c_1e_2 = q_1r_2 - r_1q_2$

Now *try* this solution of the first equation

$e_2 = sq_2 \quad \Rightarrow \quad i_3 = sr_3$ $\qquad$ where s is a sign, TBD
$f_2 = t\,r_2 \quad \Rightarrow \quad h_3 = t\,q_3$ $\qquad$ t is another sign TBD

Now insert this solution into the second equation to get

$b_1\,sr_3 - c_1\,t\,q_3 = q_3r_1 - r_3q_1 \quad \Rightarrow \quad sb_1 = -q_1$ and $-c_1\,t = r_1$

$\Rightarrow \quad b_1 = -sq_1 \quad c_1 = -tr_1$

Now insert all these solutions into the third equation to get

$$LHS = -sq_1\, t\, r_2 + tr_1\, sq_2 = -st(q_1 r_2 - r_1\, q_2)$$

This matches the RHS's if we select phases so that st=-1. So arbitrarily I will take s=1 and t=-1 so we then have

$$e_2 = q_2 \qquad i_3 = r_3 \qquad f_2 = -\, r_2 \qquad h_3 = -\, q_3 \qquad b_1 = -q_1 \qquad c_1 = r_1$$

$$\text{with } q_i \equiv (\xi_i^2/F_i^2) \qquad r_i = (1/F_i^2)$$

These then are our proposed 6 elements of the matrix Φ.

Finding the 1st column of the Stackel matrix.

But now we have to find three first-column entries such that

$$a(1)M_1 + d(2)M_2 + g(3)M_3 = S$$

From 2.4 we know that

$$S = H/[F_1F_2F_3] = \{G_1(-G_2)G_3/[F_1F_2F_3]/\,[F_1F_2F_3]\} = G_1(-G_2)G_3/[F_1F_2F_3]^2$$

Recall again that

$$M_1 = +\, G_1/(F_2^2F_3^2)$$
$$M_2 = -\, G_2/(F_3^2F_1^2)$$
$$M_3 = +\, G_3/(F_1^2F_2^2)$$

So this says

$$a(1)M_1 + d(2)M_2 + g(3)M_3 = S$$

$$a(1)\, G_1/(F_2^2F_3^2) - d(2)\, G_2/(F_3^2F_1^2) + g(3)\, G_3/(F_1^2F_2^2) = G_1(-G_2)G_3/[F_1F_2F_3]^2$$

$$a(1)\, G_1F_1^2 - d(2)\, G_2F_2^2 + g(3)\, G_3F_3^2 = G_1(-G_2)G_3$$

It is not obvious that there even is a solution. Let's expand everything, and I install the f_i functions at this point for calculations:

$$a(1)\, G_1f_1^2 + d(2)\, G_2f_2^2 + g(3)\, G_3f_3^2 = -\, G_1G_2G_3$$

$$\begin{aligned} &a(1)\,(\xi_2^2-\xi_3^2)\,(\xi_1^2-a^2)(\,\xi_1^2-b^2) \\ &+ d(2)\,(\xi_3^2-\xi_1^2)\,(\xi_2^2-a^2)(\,\xi_2^2-b^2) \\ &+ g(3)\,(\xi_1^2-\xi_2^2)\,(\xi_3^2-a^2)(\,\xi_3^2-b^2) \quad = -\,(\xi_2^2-\xi_3^2)\,(\xi_3^2-\xi_1^2)\,(\xi_1^2-\xi_2^2) \end{aligned}$$

Let's now define

$A(1) = a(1)\,(\xi_1^2-a^2)(\,\xi_1^2-b^2)$
$D(2) = d(2)\,(\xi_2^2-a^2)(\,\xi_2^2-b^2)$
$G(3) = g(3)\,(\xi_3^2-a^2)(\,\xi_3^2-b^2)$

Then our equation reads

$$A(\xi_1)\,(\xi_2^2-\xi_3^2) + D(\xi_2)\,(\xi_3^2-\xi_1^2) + G(\xi_3)\,(\xi_1^2-\xi_2^2) = -(\xi_2^2-\xi_3^2)\,(\xi_3^2-\xi_1^2)\,(\xi_1^2-\xi_2^2)$$

The RHS is this

$$-(\xi_2^2-\xi_3^2)\,(\xi_3^2-\xi_1^2)\,(\xi_1^2-\xi_2^2) = -(\xi_2^2-\xi_3^2)[\,\xi_3^2\,\xi_1^2 - \xi_1^4 - \xi_2^2\,\xi_3^2 + \xi_1^2\,\xi_2^2]$$

$$= -[\,\xi_3^2\,\xi_1^2\,\xi_2^2 - \xi_1^4\,\xi_2^2 - \xi_2^4\,\xi_3^2 + \xi_1^2\,\xi_2^4]$$
$$+ \;[\,\xi_3^4\,\xi_1^2 - \xi_1^4\,\xi_3^2 - \xi_2^2\,\xi_3^4 + \xi_1^2\,\xi_2^2\,\xi_3^2]$$

$$= -\xi_3^2\,\xi_1^2\,\xi_2^2 + \xi_1^4\,\xi_2^2 + \xi_2^4\,\xi_3^2 - \xi_1^2\,\xi_2^4$$
$$+ \;\xi_3^4\,\xi_1^2 - \xi_1^4\,\xi_3^2 - \xi_2^2\,\xi_3^4 + \xi_1^2\,\xi_2^2\,\xi_3^2$$

$$= +\xi_1^4\,\xi_2^2 + \xi_2^4\,\xi_3^2 - \xi_1^2\,\xi_2^4 + \;\xi_3^4\,\xi_1^2 - \xi_1^4\,\xi_3^2 - \xi_2^2\,\xi_3^4$$

$= \xi_1^4\,(\xi_2^2 - \xi_3^2) + \xi_2^4(\xi_3^2-\xi_1^2) + \xi_3^4(\xi_1^2 - \xi_2^2)$ // note is cyclic

We then have

$$A(\xi_1)\,(\xi_2^2-\xi_3^2) + D(\xi_2)\,(\xi_3^2-\xi_1^2) + G(\xi_3)\,(\xi_1^2-\xi_2^2)$$

$$= \xi_1^4\,(\xi_2^2 - \xi_3^2) + \xi_2^4(\xi_3^2-\xi_1^2) + \xi_3^4(\xi_1^2 - \xi_2^2)$$

and we can luckily read off our solution

$A(\xi_1) = \xi_1^4$ $D(\xi_2) = \xi_2^4$ $G(\xi_3) = \xi_3^4$

and this tells us our last three Φ matrix elements

$a(1) = \xi_1^4/\,[(\xi_1^2-a^2)(\,\xi_1^2-b^2)] = \xi_1^4/f_1^2$
$d(2) = \xi_2^4/\,[(\xi_2^2-a^2)(\,\xi_2^2-b^2)] = \xi_2^4/f_2^2$
$g(3) = \xi_3^4/\,[(\xi_3^2-a^2)(\,\xi_3^2-b^2)] = \xi_3^4/f_3^2$

Here then is our Stackel matrix

$$\Phi = \begin{pmatrix} a(1) & b(1) & c(1) \\ d(2) & e(2) & f(2) \\ g(3) & h(3) & i(3) \end{pmatrix}$$

$$\begin{matrix} \xi_1^4/f_1^2 & -(\xi_1^2/f_1^2) & (1/f_1^2) \\ \xi_2^4/f_2^2 & -(\xi_2^2/f_2^2) & (1/f_2^2) \\ \xi_2^4/f_3^2 & -(\xi_3^2/f_3^2) & (1/f_3^2) \end{matrix}$$

$$\begin{matrix} \xi_1^4/(\xi_1^2-a^2)(\xi_1^2-b^2) & -(\xi_1^2/(\xi_1^2-a^2)(\xi_1^2-b^2)) & (1/(\xi_1^2-a^2)(\xi_1^2-b^2)) \\ \xi_2^4/(\xi_2^2-a^2)(\xi_2^2-b^2) & -(\xi_2^2/(\xi_2^2-a^2)(\xi_2^2-b^2)) & (1/(\xi_2^2-a^2)(\xi_2^2-b^2)) \\ \xi_2^4/(\xi_3^2-a^2)(\xi_3^2-b^2) & -(\xi_3^2/(\xi_3^2-a^2)(\xi_3^2-b^2)) & (1/(\xi_3^2-a^2)(\xi_3^2-b^2)) \end{matrix}$$

which we can compare to the official M&F matrix

$$S = \begin{vmatrix} 1 & \frac{1}{(\xi_1^2 - a^2)} & \frac{1}{(\xi_1^2 - b^2)(a^2 - b^2)} \\ 1 & \frac{1}{(\xi_2^2 - a^2)} & \frac{1}{(\xi_2^2 - b^2)(a^2 - b^2)} \\ 1 & \frac{1}{(\xi_3^2 - a^2)} & \frac{1}{(\xi_3^2 - b^2)(a^2 - b^2)} \end{vmatrix}$$

How can these be so different? Let's compute the cofactor of MY top left entry

$M_1 = -(\xi_2^2/f_2^2)(1/f_3^2) - (1/f_2^2)(-1)(\xi_3^2/f_3^2)$

$= (\xi_3^2-\xi_2^2)/(f_2^2 f_3^2) = (\xi_3^2-\xi_2^2)/[(\xi_2^2-a^2)(\xi_2^2-b^2)(\xi_3^2-a^2)(\xi_3^2-b^2)]$

which agrees with my earlier computation of this same quantity.

Now do the same for THEIR top left entry

$(\xi_2^2-a^2)^{-1}(\xi_3^2-b^2)^{-1}(a^2-b^2)^{-1} - (\xi_3^2-a^2)^{-1}(\xi_2^2-b^2)^{-1}(a^2-b^2)^{-1}$

$= -(a^2-b^2)^{-1}\{(\xi_2^2-b^2)^{-1}(\xi_3^2-a^2)^{-1} - (\xi_2^2-a^2)^{-1}(\xi_3^2-b^2)^{-1}\}$

$= -(a^2-b^2)^{-1}\{(\xi_2^2-a^2)(\xi_3^2-b^2) - (\xi_2^2-b^2)(\xi_3^2-a^2)\} / [(\xi_2^2-a^2)(\xi_2^2-b^2)(\xi_3^2-a^2)(\xi_3^2-b^2)]$

$= -(a^2-b^2)^{-1}\{(\xi_2^2-a^2)(\xi_3^2-b^2) - (\xi_2^2-b^2)(\xi_3^2-a^2)\} / [(\xi_2^2-a^2)(\xi_2^2-b^2)(\xi_3^2-a^2)(\xi_3^2-b^2)]$

```
f := (x2^2-a^2)*(x3^2-b^2) - (x2^2-b^2)*(x3^2-a^2);
```

$$f := (x2^2 - a^2)(x3^2 - b^2) - (x2^2 - b^2)(x3^2 - a^2)$$

```
expand(%):factor(%);
```

$$(x2 - x3)(x2 + x3)(a - b)(a + b)$$

$= -(a^2-b^2)^{-1}\{(\xi_2^2-\xi_3^2)(a^2-b^2)\} / [(\xi_2^2-a^2)(\xi_2^2-b^2)(\xi_3^2-a^2)(\xi_3^2-b^2)]$

$= (\xi_3^2-\xi_2^2) / [(\xi_2^2-a^2)(\xi_2^2-b^2)(\xi_3^2-a^2)(\xi_3^2-b^2)]$

We agree! So I think both of our matrices have the same cofactors M_i . We are still alive!

The alternate and simpler solution for the 1st column

Before continuing, we point out that there is in fact an alternate and in fact much simpler solution for the first column! Go back to where we were

$$\begin{aligned} &a(1)\,(\xi_2^2-\xi_3^2)\,(\xi_1^2-a^2)(\,\xi_1^2-b^2) \\ &+\,d(2)\,(\xi_3^2-\xi_1^2)\,(\xi_2^2-a^2)(\,\xi_2^2-b^2) \\ &+\,g(3)\,(\xi_1^2-\xi_2^2)\,(\xi_3^2-a^2)(\,\xi_3^2-b^2) \quad = -(\xi_2^2-\xi_3^2)\,(\xi_3^2-\xi_1^2)\,(\xi_1^2-\xi_2^2) \end{aligned}$$

My claim is that the simpler solution is a(1) = d(2) = g(3) = 1. If this is indeed a solution, then this must be true

$$\begin{aligned} &(\xi_2^2-\xi_3^2)\,(\xi_1^2-a^2)(\,\xi_1^2-b^2) \\ &+\,(\xi_3^2-\xi_1^2)\,(\xi_2^2-a^2)(\,\xi_2^2-b^2) \\ &+\,(\xi_1^2-\xi_2^2)\,(\xi_3^2-a^2)(\,\xi_3^2-b^2) \quad = -(\xi_2^2-\xi_3^2)\,(\xi_3^2-\xi_1^2)\,(\xi_1^2-\xi_2^2) \end{aligned}$$

or

$$(\xi_2^2-\xi_3^2)\,(\xi_1^2-a^2)(\,\xi_1^2-b^2)+\text{cyclic} = -(\xi_2^2-\xi_3^2)\,(\xi_3^2-\xi_1^2)\,(\xi_1^2-\xi_2^2)$$

This seems rather preposterous because the LHS appears to be a function of a and b. But let's examine that appearance by looking at the coefficient of a^2 of the LHS

$$\text{coeff}(a^2) = -(\xi_2^2-\xi_3^2)\,(\,\xi_1^2-b^2) + \text{cyclic}$$

$$= -[\,\xi_1^2(\xi_2^2-\xi_3^2) + \text{cyclic}] + b^2[(\xi_2^2-\xi_3^2) + \text{cyclic}\,]$$

But both square brackets vanish due to the following two trivial theorems:

$$f(1)(f(2) - f(3)) + \text{cyclic} = 0 \qquad \text{// makes first [] = 0}$$
$$(f(2) - f(3)) + \text{cyclic} = 0 \qquad \text{// makes second [] = 0}$$

Therefore, since we know the coefficient of a^2 (and by similar argument that of b^2) on the LHS above is 0, we can select *any values we want* for a and b to do the evaluation. So take $a = \xi_1^2$ and $b = \xi_2^2$, then the LHS has only the third term which is

$$\text{third term of LHS} = (\xi_1^2-\xi_2^2)\,(\xi_3^2-a^2)(\,\xi_3^2-b^2) = (\xi_1^2-\xi_2^2)\,(\xi_3^2-\xi_1^2)(\,\xi_3^2-\xi_2^2)$$

$$= (\,\xi_3^2-\xi_2^2)\,(\xi_3^2-\xi_1^2)\,(\xi_1^2-\xi_2^2) = -(\xi_2^2-\xi_3^2)\,(\xi_3^2-\xi_1^2)\,(\xi_1^2-\xi_2^2) \qquad = \text{RHS}$$

Let this be a reminder that "functional form" equations can have multiple solution, some of which might be simpler than others!

Therefore, we can take 1,1,1 for our first Stackel column in place of what we got above. Our derived Stackel matrix is then

$$\Phi = \begin{pmatrix} a(1) & b(1) & c(1) \\ d(2) & e(2) & f(2) \\ g(3) & h(3) & i(3) \end{pmatrix}$$

$$\begin{matrix} 1 & -(\xi_1^2/f_1^2) & (1/f_1^2) \\ 1 & -(\xi_2^2/f_2^2) & (1/f_2^2) \\ 1 & -(\xi_3^2/f_3^2) & (1/f_3^2) \end{matrix}$$

$$\begin{matrix} 1 & -(\xi_1^2/(\xi_1^2-a^2)(\xi_1^2-b^2)) & (1/(\xi_1^2-a^2)(\xi_1^2-b^2)) \\ 1 & -(\xi_2^2/(\xi_2^2-a^2)(\xi_2^2-b^2)) & (1/(\xi_2^2-a^2)(\xi_2^2-b^2)) \\ 1 & -(\xi_3^2/(\xi_3^2-a^2)(\xi_3^2-b^2)) & (1/(\xi_3^2-a^2)(\xi_3^2-b^2)) \end{matrix}$$

which we again compare to the "official" matrix given in MF:

$$S = \begin{vmatrix} 1 & \frac{1}{(\xi_1^2-a^2)} & \frac{1}{(\xi_1^2-b^2)(a^2-b^2)} \\ 1 & \frac{1}{(\xi_2^2-a^2)} & \frac{1}{(\xi_2^2-b^2)(a^2-b^2)} \\ 1 & \frac{1}{(\xi_3^2-a^2)} & \frac{1}{(\xi_3^2-b^2)(a^2-b^2)} \end{vmatrix}$$

at least now the first column agrees, though the other two do not.

What are the "symmetry" operations for a Stackel matrix?

What transformation on a matrix preserves the cofactors of the first column of elements, and also preserves the determinant! I flailed away before getting to **Plan G** !! I will keep the earlier "plans" here, but reader should skip to Plan G.

Plan A: Here is an idea. Suppose you take a matrix of this shape $Q = \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & 0 & 0 \end{pmatrix}$ where the upper right matrix is unitary. Then we would expect that $Q^{-1}SQ$ would not alter M_1 which is the det of the upper right of S. But it would surely alter the two other minors.

Plan B: Can you make a 3x3 unitary matrix such that the three right-side 2x2 matrices are unitary?
I tried, I think not.

Plan C: Make a 3x3 matrix with the three right-side 2x2 matrices all unimodular.

$$\begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & e & f \end{pmatrix}$$

Suppose we select the 6 numbers such that the three minors here are 1. that is to say

ad-bc = 1
af-be =1
cf-de = 1

This is 3 conditions on 6 numbers, should not be too hard to solve. Then consider

Plan D: Start with a matrix with unimodular upper right piece (note $\det A_3 = 1$)

$$A_3 \equiv \begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 1 & 0 & 0 \end{pmatrix} \qquad ad-bc = 1$$

Apply this to Stackel matrix S like so

$$R_1 = A_3 S A_3^{-1}$$

The resulting matrix R_1 will have the same M_1 as S, but M_2 and M_3 will be damaged.

Plan E: Consider

$$e_2 i_3 - f_2 h_3 = q_2 r_3 - r_2 q_3$$

You are given the RHS. What is the most general solution set of functions on the left? Think of it this way

$$\det\begin{pmatrix} e_2 & f_2 \\ h_3 & i_3 \end{pmatrix} = \det\begin{pmatrix} q_2 & r_2 \\ q_3 & r_3 \end{pmatrix}$$

It seems clear that you can have

$$\begin{pmatrix} e_2 & f_2 \\ h_3 & i_3 \end{pmatrix} = U \begin{pmatrix} q_2 & r_2 \\ q_3 & r_3 \end{pmatrix} V$$

where U and V are any unimodular matrices, *except* you have to mind "the functional form"

$$U = \begin{pmatrix} \alpha_2 & \beta_2 \\ \delta_3 & \gamma_3 \end{pmatrix} \text{ is too general}$$

$$\begin{pmatrix} \alpha_2 & 0 \\ 0 & \gamma_3 \end{pmatrix}\begin{pmatrix} q_2 & r_2 \\ q_3 & r_3 \end{pmatrix} = \begin{pmatrix} \alpha_2 q_2 & \alpha_2 r_2 \\ q_3/\alpha_2 & \gamma_3 r_3/\alpha_2 \end{pmatrix} \qquad \det U=1 \text{ so } \gamma_3 = 1/\alpha_2$$

So all you really can do is scale up one row and scale down its partner row. But even this doesn't work if α_2 is a function of ξ_2, so α_2 can only be a constant.

Plan F. Consider again

$$\begin{array}{lll} \xi_1^4/f_1^2 & -(\xi_1^2/f_1^2) & (1/f_1^2) \\ \xi_2^4/f_2^2 & -(\xi_2^2/f_2^2) & (1/f_2^2) \\ \xi_2^4/f_3^2 & -(\xi_3^2/f_3^2) & (1/f_3^2) \end{array}$$

$$\begin{array}{lll} \xi_1^4/(\xi_1^2-a^2)(\xi_1^2-b^2) & -(\xi_1^2/(\xi_1^2-a^2)(\xi_1^2-b^2)) & (1/(\xi_1^2-a^2)(\xi_1^2-b^2)) \\ \xi_2^4/(\xi_2^2-a^2)(\xi_2^2-b^2) & -(\xi_2^2/(\xi_2^2-a^2)(\xi_2^2-b^2)) & (1/(\xi_2^2-a^2)(\xi_2^2-b^2)) \\ \xi_2^4/(\xi_3^2-a^2)(\xi_3^2-b^2) & -(\xi_3^2/(\xi_3^2-a^2)(\xi_3^2-b^2)) & (1/(\xi_3^2-a^2)(\xi_3^2-b^2)) \end{array}$$

and we can compare this with the official matrix which is

$$S = \begin{vmatrix} 1 & \dfrac{1}{(\xi_1^2-a^2)} & \dfrac{1}{(\xi_1^2-b^2)(a^2-b^2)} \\ 1 & \dfrac{1}{(\xi_2^2-a^2)} & \dfrac{1}{(\xi_2^2-b^2)(a^2-b^2)} \\ 1 & \dfrac{1}{(\xi_3^2-a^2)} & \dfrac{1}{(\xi_3^2-b^2)(a^2-b^2)} \end{vmatrix}$$

I can make his look a little like mine as follows

$$\begin{array}{lll} S = (\xi_1^2-a^2)(\xi_1^2-b^2)/f_1^2 & (\xi_1^2-b^2)/f_1^2 & (a^2-b^2)^{-1}(\xi_1^2-a^2)/f_1^2 \\ me = \xi_1^4/f_1^2 & -(\xi_1^2/f_1^2) & (1/f_1^2) \end{array}$$

Plan G: The symmetry transformations

Theorem: If you add a multiple of one of the last two columns to the other, then
(1) none of the minors of the elements of the first column change;
(2) the overall determinant does not change.

Item (2) is a standard matrix theorem. Item (1) is easy to demonstrate

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightarrow \begin{pmatrix} a & b+kc & c \\ d & e+kf & f \\ g & h+ki & i \end{pmatrix} \qquad M_1 = (b+kc)f - (e+kf)c = bf - ec \quad \text{QED}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightarrow \begin{pmatrix} a & b & c+kb \\ d & e & f+ke \\ g & h & i+kh \end{pmatrix} \qquad M_1 = b(f+ke)-e(c+kb) = bf - ec \quad \text{QED}$$

Seeing is believing! So this is something you can do to a Stackel matrix to change its form, and you can do it multiple times.

You can of course also add a multiple of either of the last two columns to the *first* without changing either of these things! (1) is obvious since you haven't changed the last two columns, and (2) is the same matrix theorem. So we generalize the theorem

Theorem 1: If you add a *constant* multiple of one of the last two columns to any other column, then
(1) none of the minors of the elements of the first column change;
(2) the overall determinant does not change.

What happens if k instead of being a constant is a function of the variable of its row? Well, then in each row k would be different, since $k(1) \neq k(2)$, and then item (2) fails. Item (1) also fails.

Theorem 2: You can change the signs of the last two columns without affecting the determinant or the minors of the first column.

(1)

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightarrow \begin{pmatrix} a & -b & -c \\ d & -e & -f \\ g & -h & -i \end{pmatrix} \qquad M_1 = (-b)(-f) - (-c)(-e) = bf-ce \quad \text{QED}$$

(2) We know the negating one column negates the determinant, so doing two column negations leaves it the same.

Theorem 3 ?

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightarrow \begin{pmatrix} a & b+ka & c+ka \\ d & e+kd & f+kd \\ g & h+kg & i+kg \end{pmatrix}$$

$$M_1 = (b+ka)(f+kd) - (c+ka)(e+kd)$$

$$= bf-ce + \{ bkd+kaf+k^2ad - kae-ckd- k^2ad \}$$

$$= bf-ce + \{ kbd+kaf - kae-kcd \} = \text{NOGO}$$

So let's try to transform my form into their form. We only need to deal with the first row.
Here are our starting positions:

they = 1 $\quad (\xi_1^2-b^2)/f_1^2 \quad (a^2-b^2)^{-1}(\xi_1^2-a^2)/f_1^2$

me = 1 $\quad -(\xi_1^2/f_1^2) \quad (1/f_1^2)$

Let's start by negating my last two columns:

me_1 = 1 $\quad (\xi_1^2/f_1^2) \quad -(1/f_1^2)$

Next, let's add a multiple b^2 of the third column to the second column

me_2 = 1 $\quad (\xi_1^2-b^2)/f_1^2 \quad -(1/f_1^2)$

This gets the middle columns to agree. Let's try another column operation on me_2 :

$-(1/f_1^2) \rightarrow -(1/f_1^2) + k\,(\xi_1^2-b^2)/f_1^2 = (-1 + k\xi_1^2-kb^2)/f_1^2 =? (a^2-b^2)^{-1}(\xi_1^2-a^2)/f_1^2$

I would then need: $(-1 + k\xi_1^2-kb^2) = (a^2-b^2)^{-1}(\xi_1^2-a^2)$

This requires that $k = (a^2-b^2)^{-1}$ to make the ξ_1^2 match, then LHS is

$(-1 + (a^2-b^2)^{-1}\xi_1^2-(a^2-b^2)^{-1}b^2) = (-(a^2-b^2) + \xi_1^2-b^2)\,(a^2-b^2)^{-1}$

$= (-a^2+b^2 + \xi_1^2-b^2)\,(a^2-b^2)^{-1} = (\xi_1^2-a^2)\,(a^2-b^2)^{-1}$ and it works!

So this is then where we stand: (I now show both possible first columns)

they =	1	$(\xi_1^2-b^2)/f_1^2$	$(a^2-b^2)^{-1}(\xi_1^2-a^2)/f_1^2$
me_3 =	1 or ξ_1^4/f_1^2	$(\xi_1^2-b^2)/f_1^2$	$(a^2-b^2)^{-1}(\xi_1^2-a^2)/f_1^2$

So we are done, we have derived the official Stackel matrix for ellipsoidal coordinates.

Verify that my two alternate 1st column forms have the same determinant

It certainly seems unlikely my two matrices have the same determinant if I use the radically different first column values. . But let's just check : (I am still using M to mean cofactor

$S_{me31} = M_1 + M_2 + M_3$

$S_{me32} = \xi_1^4/f_1^2\,M_1 + \xi_2^4/f_2^2\,M_2 + \xi_3^4/f_3^2\,M_3$

If these are the same, then this must be true

$(\xi_1^4/f_1^2 - 1)M_1 + (\xi_2^4/f_2^2 - 1)M_2 + (\xi_3^4/f_3^2 - 1)M_3 = 0$
$(\xi_1^4/f_1^2 - 1)(-M_1) + (\xi_2^4/f_2^2 - 1)(-M_2) + (\xi_3^4/f_3^2 - 1)(-M_3) = ? \quad 0$

But I know from earlier calculation that

$M_1 = -(\xi_2^2-\xi_3^2)/[f_2^2f_3^2]$
$M_2 = -(\xi_3^2-\xi_1^2)/[f_3^2f_1^2]$
$M_3 = -(\xi_1^2-\xi_2^2)/[f_1^2f_2^2]$ // cyclic when using f's

So we need to evaluate

$(\xi_1^4/f_1^2 - 1)(-M_1) + (\xi_2^4/f_2^2 - 1)(-M_2) + (\xi_3^4/f_3^2 - 1)(-M_3)$

$$(\xi_1^4/f_1^2 - 1)(\xi_2^2 - \xi_3^2)/[f_2^2 f_3^2] + (\xi_2^4/f_2^2 - 1)(\xi_3^2 - \xi_1^2)/[f_3^2 f_1^2] + (\xi_3^4/f_3^2 - 1)(\xi_1^2 - \xi_2^2)/[f_1^2 f_2^2]$$

Multiply through by $[f_1^2 f_2^2 f_3^2]$:

$$(\xi_1^4 - f_1^2)(\xi_2^2 - \xi_3^2) + (\xi_2^4 - f_2^2)(\xi_3^2 - \xi_1^2) + (\xi_3^4 - f_3^2)(\xi_1^2 - \xi_2^2)$$

This is a job for Maple

```
f1s := (x1^2-a^2)*(x1^2-b^2);
```

$$f1s := (x1^2 - a^2)(x1^2 - b^2)$$

```
f2s := (x2^2-a^2)*(x2^2-b^2);
```

$$f2s := (x2^2 - a^2)(x2^2 - b^2)$$

```
f3s := (x3^2-a^2)*(x3^2-b^2);
```

$$f3s := (x3^2 - a^2)(x3^2 - b^2)$$

```
t1 := (x1^4 - f1s)*(x2^2-x3^2);
```

$$t1 := (x1^4 - (x1^2 - a^2)(x1^2 - b^2))(x2^2 - x3^2)$$

```
t2 := (x2^4 - f2s)*(x3^2-x1^2);
```

$$t2 := (x2^4 - (x2^2 - a^2)(x2^2 - b^2))(x3^2 - x1^2)$$

```
t3 := (x3^4 - f3s)*(x1^2-x2^2);
```

$$t3 := (x3^4 - (x3^2 - a^2)(x3^2 - b^2))(x1^2 - x2^2)$$

```
simplify(t1+t2+t3);
```

$$0$$

and there it is! I have just shown that

$$(\xi_1^4 - f_1^2)(\xi_2^2 - \xi_3^2) + (\xi_2^4 - f_2^2)(\xi_3^2 - \xi_1^2) + (\xi_3^4 - f_3^2)(\xi_1^2 - \xi_2^2) = 0$$

which means

$$(\xi_1^4/f_1^2 - 1)M_1 + (\xi_2^4/f_2^2 - 1)M_2 + (\xi_3^4/f_3^2 - 1)M_3 = 0$$

which means

$$me_{31} = \begin{matrix} 1 & (\xi_1^2 - b^2)/f_1^2 & (a^2 - b^2)^{-1}(\xi_1^2 - a^2)/f_1^2 \end{matrix}$$

$$me_{32} = \begin{matrix} \xi_1^4/f_1^2 & (\xi_1^2 - b^2)/f_1^2 & (a^2 - b^2)^{-1}(\xi_1^2 - a^2)/f_1^2 \end{matrix}$$

are both valid Stackel matrices!

Just a little more. Go back to

$$Q(a,b) = (\xi_1^4 - f_1^2)(\xi_2^2 - \xi_3^2) + (\xi_2^4 - f_2^2)(\xi_3^2 - \xi_1^2) + (\xi_3^4 - f_3^2)(\xi_1^2 - \xi_2^2)$$

where $f_1^2 = (\xi_1^2 - a^2)(\xi_1^2 - b^2)$

We can show that Q(a,b) is independent of both a and b, and so can be evaluated at a=b=0 where we clearly get 0. We can examine for example the coefficient of a^2

coef (a^2, Q(a,b)) = (ξ_1^2-b^2) (ξ_2^2- ξ_3^2) + cyclic

and here we get to make use of **two famous results** (from ancient memory)

(ξ_2^2- ξ_3^2) + cyclic = 0 2-3 + 3-1 + 1-2 = 0
ξ_1^2 (ξ_2^2- ξ_3^2) + cyclic = 0 12-13 + 23-21 + 31-32 = 0

(f(2) - f(3)) +cyclic = 0 // a little more general
f(1)(f(2) - f(3)) +cyclic = 0

Thus, coef (a^2, Q(a,b)) = 0. Since Q(a,b) = Q(b,a), we know coef (b^2, Q(a,b)) = 0 as well. QED.

---

just storing this here: $\begin{pmatrix} \Phi_{11}(1) & \Phi_{12}(1) & \Phi_{13}(1) \\ \Phi_{21}(2) & \Phi_{22}(2) & \Phi_{23}(2) \\ \Phi_{31}(3) & \Phi_{32}(3) & \Phi_{33}(3) \end{pmatrix}$

What about Laplace separation?

First of all, any Helmholtz separable CL system will also be Laplace separable, and this includes the 11 classical systems. So what we really wonder is: are there some CL systems that are NOT Helmholtz separable but which are Laplace separable? The answer really is that there are none. There are some that are R separable for Laplace but not R separable for Helmholtz, but that is not our subject in this document.

Method 1. (Limit $k_1^2 \to 0$) We can take the existing analysis and compute the 6 rightmost functions in the Stackel matrix using the M_i information and be done with it. This is because with $k_1^2 = 0$, the first column of the matrix does not play a role in anything other than affecting the Stackel determinant number S. We would proceed as follows. First, use (2.1) to determine the g_n and the f_n

$g_n(\neq n)f_n(n) = H/h_n^2$ n = 1,2,3 *(2.1)*

Then use these and "trial and error" to find the rightmost 6 Stackel matrix functions,

$$\Phi = \begin{pmatrix} a(1) & b(1) & c(1) \\ d(2) & e(2) & f(2) \\ g(3) & h(3) & i(3) \end{pmatrix} = \begin{pmatrix} \Phi_{11}(1) & \Phi_{12}(1) & \Phi_{13}(1) \\ \Phi_{21}(2) & \Phi_{22}(2) & \Phi_{23}(2) \\ \Phi_{31}(3) & \Phi_{32}(3) & \Phi_{33}(3) \end{pmatrix}$$

e(2)i(3)-f(2)h(3) = + $g_1(\neq 1)$/[$f_2(2)$ $f_3(3)$]
b(1)i(3)-c(1)h(3) = – $g_2(\neq 2)$/[$f_3(3)$ $f_1(1)$]
b(1)f(2)-c(1)e(2) = + $g_3(\neq 3)$/[$f_1(1)$ $f_2(2)$]

Our separated equations are then these:

$$\mathcal{L}_n X_n = (1/f_n)\partial_n[f_n(\partial_n X_n)] + [\, k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0 \qquad (16)$$

So here we don't ever mention the number S, although it no doubt still exists.

Method 2, start with $k_1^2 = 0$. Now let's take a look at bit back at our Helmholtz development. We worked our way down to this

$$\Sigma_n\, g_n(\neq n)\, [\, k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]f_n(n) = H(123)k_1^2 \qquad (17)$$

Now if $k_1 = 0$ all along , this would really say

$$\Sigma_n\, g_n(\neq n)\, [k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]f_n(n) = 0$$

If we want this to be true for arbitrary separation constants, then our set of three equations in (18) would be reduced to just two equations

$$\Sigma_n\, g_n(\neq n)\, f_n(n)\Phi_{n2}(n) = 0$$
$$\Sigma_n\, g_n(\neq n)\, f_n(n)\Phi_{n3}(n) = 0 \qquad (18)'$$

We then no longer have a 3x3 matrix equation $\Phi^T V = W$. We can still of course talk about 3x3 matrix Φ. Suppose we define these 6 functions

$$A_n(n) = f_n(n)\Phi_{n2}(n)$$
$$B_n(n) = f_n(n)\Phi_{n3}(n)$$

Then our equations above are these

$$g_1(23)\, A_{12}(1) + g_2(31)\, A(2) + g_3(12)\, A_3(3) = 0$$
$$g_1(23)\, B_1(1) + g_2(31)\, B_2(2) + g_3(12)\, B_3(3) = 0$$

How do we even know solutions exist to these equations? Certainly if the three g_i have completely random functional form, there will be no non-trivial solution and we will say that the Laplace equation is not separable in such a system.

Let's pause here to look at an example. Look at Bicylindrical (=bipolar = E.4) coordinates M&S page 81 where we have

$$h_1^2 = h_2^2 = a^2(\mathrm{ch}\eta - \cos\psi)^{-2} \qquad h_3 = 1$$

Consider then

$$g_n(\neq n)f_n(n) = H/h_n^2 \qquad n = 1,2,3 \qquad (2.1)$$

So that

$$g_1(\neq 1)f_1(1) = H/h_1^2 = h_2h_3/h_1 = h_2/h_1 = 1$$

Then we find that $f_1(1) = 1$ and $g_1(\neq 1) = 1$. Since 1 and 2 are symmetric, same for 2. Then for 3,

$$g_3(\neq 3)f_3(3) = H/h_3^2 = h_1^2 = a^2(ch\eta - cos\psi)^{-2}$$

I suppose we can say (η,ψ,z) - (1,2,3) so we then have

$$g_3(\neq 3)f_3(3) = a^2(cosh\xi_1 - cos\xi_2)^{-2}$$

So here are the things we learn from this very first step

$$f_1(1) = 1 \qquad f_2(2) = 1 \qquad f_3(3) = 1$$
$$g_1(23) = 1 \qquad g_2(31) = 1 \qquad g_3(12) = a^2(cosh\xi_1 - cos\xi_2)^{-2}$$

Now we can look at our two equations

$$g_1(23)\, A_1(1) + g_2(31)\, A_2(2) + g_3(12)\, A_3(3) = 0$$
$$g_1(23)\, B_1(1) + g_2(31)\, B_2(2) + g_3(12)\, B_3(3) = 0$$

$$A_1(1) + A_2(2) + a^2(cosh\xi_1 - cos\xi_2)^{-2}\, A_3(3) = 0$$
$$B_1(1) + B_2(2) + a^2(cosh\xi_1 - cos\xi_2)^{-2}\, B_3(3) = 0$$

Consider the first of this last pair. If we set $A_3(3) = 0$, then we need $A_1(1) + A_2(2) = 0$, but due to coordinate dependence, this means $A_1(1) = c_1$ and $A_2(2) = -c_1$ which gives the kind of solution discussed below. If you assume some non-vanishing $A_3(3)$, you then have $A_1(1) + A_2(2) = stuff(12)\, B_3(3)$ and there is no solution. So the only possibility here is that $A_3(3) = B_3(3) = 0$ and that means

$$\mathcal{L}_3X_3 = (1/f_3)\partial_3[f_3(\partial_3X_3)] + [\, k_2^2\Phi_{32}(3) + k_3^2\Phi_{33}(3)]X_3 = 0 \qquad (16)$$
$$\mathcal{L}_3X_3 = (1/f_3)\partial_3[f_3(\partial_3X_3)] + [\quad 0 \quad + \quad 0 \quad]X_3 = 0 \qquad (16)$$
$$\mathcal{L}_3X_3 = \partial_3[(\partial_3X_3)] = 0 \qquad (16)$$

$$X_3 = a\xi_3 + b$$

So in this case, we have $\nabla^2X_3 = 0$ and X_3 is a linear function at the fanciest.

Now, suppose we know that ψ will have the form just X_1X_2 because we know there will be no dependence at all on $\xi_3 = z$ in this case. Then we have a 2D problem and our equations above would become

$$A_1(1) + A_2(2) = 0$$

$B_1(1) + B_2(2) = 0$

The solutions here would be

$A_1(1) = c_1$ $\quad A_2(2) = -c_1$
$B_1(1) = d_1$ $\quad B_2(2) = -d_1$

But for our system

$A_n(n) = \Phi_{n2}(n)$
$B_n(n) = \Phi_{n3}(n)$

so we have our 4 elements of the 2D case

$\Phi_{12}(n) = c_1$ $\quad \Phi_{22}(n) = -c_1$
$\Phi_{13}(n) = d_1$ $\quad \Phi_{23}(n) = -d_1$

Now look again at

$$\mathcal{L}_n X_n = (1/f_n)\partial_n[f_n(\partial_n X_n)] + [k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0 \qquad (16)$$

$$\mathcal{L}_1 X_1 = (1/f_1)\partial_1[f_1(\partial_1 X_1)] + [k_2^2\Phi_{12}(2) + k_3^2\Phi_{13}(3)]X_1 = 0$$
$$\mathcal{L}_1 X_1 = (1/f_1)\partial_1[f_1(\partial_1 X_1)] + [k_2^2 c_1 + k_3^2 d_1)]X_1 = 0$$

$$\mathcal{L}_2 X_2 = (1/f_2)\partial_2[f_2(\partial_2 X_2)] + [k_2^2\Phi_{22}(2) + k_3^2\Phi_{23}(3)]X_2 = 0$$
$$\mathcal{L}_2 X_2 = (1/f_2)\partial_2[f_2(\partial_2 X_2)] + [k_2^2(-c_1) + k_3^2(-d_1))]\, X_2 = 0$$

So define $\alpha^2 = k_2^2 c_1 + k_3^2 d_1$ and set the f's to one and we have

$$\mathcal{L}_1 X_1 = \partial_1[(\partial_1 X_1)] + \alpha^2 X_1 = 0$$
$$\mathcal{L}_2 X_2 = \partial_2[(\partial_2 X_{12})] - \alpha^2 X_1 = 0$$

and these are our separated equations in this case. But we know these solutions pretty well.

$\alpha^2 > 0$: $\quad X_1 = [\sin(\alpha\xi_1), \cos(\alpha\xi_1)]$ $\quad X_2 = \exp(\pm\alpha\xi_2)$
$\alpha^2 < 0$: $\quad X_2 = [\sin(\alpha\xi_2), \cos(\alpha\xi_2)]$ $\quad X_1 = \exp(\pm\alpha\xi_1)$

This is what M&S are talking about on page 78 equation (3.07).

So the most general solution you can have here is sine and expo and linear in z. In Laplace then the linear in z portion ∂_z^2 of the Laplacian gives 0 on X_3 and contributes nothing to $\nabla^2\psi$. That is

$\psi = X_1 X_2 X_3$

with $X_{1,2}$ as the above atoms, and X_3 = linear.

If you look at M&S, you see in their CYL chapter that none other than the classical ones are separable for either Helmholtz or Laplace. They are all 2D Laplace Separable in the sense outlined above since they are all conformal maps of the Cartesians and therefore only differ from the above example in the $g_3(12)$ function.

Morse & Feshbach sign error search

On page 509 M&F start talking Stackel. Equation 5.1.28 is of the form have always used.

$$(1/f_n)\partial_n(f_n\partial_n X_n) + [\, k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0 \qquad (5.1.28)$$

They then tell us to multiply this equation by factor $(M_n/S)\psi/X_n$. That gives me

$$(\, M_n/S\, f_n X_n)\partial_n(f_n\partial_n X_n)\psi + [\, k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]\,(M_n/S)\psi = 0$$

We can then divide out the ψ and move the M_n to get

$$(\, M_n/S\, f_n X_n)\partial_n(f_n\partial_n X_n) + [\, k_1^2\Phi_{n1}(n)\, M_n + k_2^2\Phi_{n2}(n)\, M_n + k_3^2\Phi_{n3}(n)\, M_n]\,(1/S) = 0$$

Now we are instructed to sum over n, which gives me

$$\Sigma_n(\, M_n/S\, f_n X_n)\partial_n(f_n\partial_n X_n) + [\, k_1^2\, \Sigma_n\, \Phi_{n1}(n)\, M_n + k_2^2\, \Sigma_n\, \Phi_{n2}(n)\, M_n + k_3^2\, \Sigma_n\, \Phi_{n3}(n)\, M_n]\,(1/S) = 0$$

Now the red sum goes down the first column, and therefore gives S, and we know the other sums give 0 due to a weird theorem, so this gives

$$\Sigma_n(\, M_n/S\, f_n X_n)\partial_n(f_n\partial_n X_n) + k_1^2 = 0$$

or

$$\Sigma_n(\, M_n/S\, f_n)\partial_n(f_n\partial_n X_n) + k_1^2 X_n = 0 \qquad (P1)$$

If we then multiply the n=1 equation by X_2X_3 and so on, we then get

$$\Sigma_n(\, M_n/S\, f_n)\partial_n(f_n\partial_n \psi) + k_1^2\psi = 0 \qquad (5.1.29)$$

So there is no sign error at least in this little M&F section. Notice that in (5.1.28) he has a + sign on both LHS terms, as I have always had.

So where do I think I have a problem? In my #5 document, I process the Helmholtz equation which starts having the form $(\nabla^2+k_1^2)\psi = 0$ and I get it to this form:

$$\Sigma_n(1/[h_n^2 X_n])\,(1/f_n)\partial_n[f_n(\partial_n X_n)] + k_1^2/Q + K_1^2 = 0 \qquad (3.10)$$

Signs here agree with my little (P1) intermediate result above. So it seems that (3.10) is compatible with M&F. Now suppose I take M&F 5.1.28 above and write it as

$$(1/f_n)\partial_n(f_n\partial_n X_n) = -\ [\ k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n$$

If I then insert this into my (3.10) I get

$$-\Sigma_n(1/[h_n^2X_n])\ [\ k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n + k_1^2/Q\ + K_1^2\ = 0$$

and now the first term has a minus sign. But this then disagrees with my #5 paper (4.9). THAT is my sign error problem! OK, I have found and fixed the error in my #5 doc!