$$\nabla^2 u = (1/[h_1 h_2 h_3]) \{ \partial_1 [(h_1 h_2 h_3 / h_1^2) (\partial_1 u)] + \text{cyclic} \} \quad (1)$$

Let's define $H = h_1 h_2 h_3$ to make things more compact, then we have

$$\nabla^2 \psi = H^{-1} \{ \partial_1 [(H/h_1^2)(\partial_1 \psi)] + \text{cyclic} \} = H^{-1} \Sigma_n \partial_n [(H/h_n^2)(\partial_n \psi)] \quad (2)$$

If we now assume that

$$\psi(\xi_1,\xi_2,\xi_3) = X_1(\xi_1) X_2(\xi_2) X_3(\xi_3) \ /R(123)$$

$$\nabla^2 \psi = H^{-1} \{ \partial_1 [(H/h_1^2)(\partial_1 \{X_1 X_2 X_3 / R(123)\})] + \text{cyclic} \}$$

Now maybe work backwards with hints from M&S 96,such as (4.01) which adds Q like so

$$Q(123)/h_p^2 = G_{p1}(\neq p)/S = \text{cof}(\Phi_{p1})/S \equiv M_p(\neq p)/S \quad (22)$$

and a new Roberson's condition which looks like this, which is M&S 4.02

$$H(123)/S(123) = f_1(1) f_2(2) f_3(3) R^2(123)/Q(123)$$

Now the quote p 96: there is NO R-separation for any Helmholtz! Only Laplace has a chance.

So maybe I will adjust my tune a little. They are looking for this result

$$-(Q/R) \Sigma_n (1/f_n h_n^2) \partial_n (f_n \partial_n R) = \text{constant}$$

So we better go back to the beginning:

$$\nabla^2 \psi = H^{-1} \{ \partial_1 [(H/h_1^2)(\partial_1 \{X_1 X_2 X_3 / R\})] + \text{cyclic} \}$$

$$\nabla^2 \psi = H^{-1} \{ X_2 X_3 \ \partial_1 [(H/h_1^2)(\partial_1 \{X_1/R\})] + \text{cyclic} \}$$

$$\nabla^2 \psi = H^{-1} \{ X_2 X_3 \ \partial_1 [(H/h_1^2)(X_1(-R^{-2}\partial_1 R) + R^{-1}\partial_1 X_1)] + \text{cyclic} \}$$

$$\nabla^2 \psi = - H^{-1} \{ X_2 X_3 \ \partial_1 [(H/h_1^2) R^{-1} X_1 R^{-1} \partial_1 R)] + \text{cyclic} \}$$

$$+ H^{-1} \{ X_2 X_3 \ \partial_1 [(H/h_1^2) R^{-1} \partial_1 X_1)] + \text{cyclic} \}$$

Looking at the second line, we might propose that

$$(H/h_1^2) R^{-1} = g_1(23) f_1(1)$$

like we did last time, but extra factor of R. Then we could write

$$\nabla^2\psi = - H^{-1}\{ X_2X_3\ \partial_1[g_1(23)f_1(1)X_1R^{-1}\partial_1R)] + \text{cyclic}\}$$

$$+ H^{-1}\{ X_2X_3\ \partial_1[g_1(23)f_1(1)\partial_1X_1)] + \text{cyclic}\}$$

$$\nabla^2\psi = - H^{-1}\{ X_2X_3\ g_1(23)\ \partial_1[f_1X_1R^{-1}\partial_1R)] + \text{cyclic}\}$$

$$+ H^{-1}\{ X_2X_3\ g_1(23)\ \partial_1[\ f_1\partial_1X_1)] + \text{cyclic}\}$$

Now this is all $\nabla^2\psi = 0$ Laplace so we could divide both sides by whatever. Maybe

$$\nabla^2\psi = - \{ g_1(23)\ \partial_1[f_1X_1R^{-1}\partial_1R)]/X_1 + \text{cyclic}\}$$

$$+ \{ g_1(23)\ \partial_1[\ f_1\partial_1X_1)]/X_1 + \text{cyclic}\} = 0$$

Maybe do the derivative in the first term a la

$$\partial_1[(X_1R^{-1})(f_1\partial_1R)] = (X_1R^{-1})\ \partial_1(f_1\partial_1R) + \partial_1(X_1/R)\ (f_1\partial_1R)$$

Maybe it is too time wasteful to reverse engineer this. I just need to get a start from somebody here!

M&F Have it. Their buzzword is the R is the "modulation factor", see page 518 vol I.

Start with these "assumptions" within our Stackel matrix formalism: // u = Q of M&S

$$1/h_1^2 \equiv M_1(\neq p)/(Su) \qquad (22)$$
$$H/S = f_1(1)\ f_2(2)\ f_3(3)\ R^2u$$

$$h_1^2 \equiv Su/M_1$$
$$h_1^2/S \equiv u/M_1$$

$$1/S = f_1(1)\ f_2(2)\ f_3(3)\ R^2u/H$$
$$1/S = u/(M_1h_1^2)$$

$$f_1\ f_2\ f_3\ R^2u/H = u/(M_1h_1^2)$$

$$f_1\ f_2\ f_3\ R^2/H = 1/(M_1h_1^2)$$

$$f_1\ f_2\ f_3\ M_1\ R^2 = H/(h_1^2)$$

So using their two suggestions, I get the last item here which could be maybe useful.

Derive the M&F Waypoint stated after 5.1.46 on page 519.

$(\nabla^2 + K_1^2)\psi$

$= H^{-1}\{ \partial_1[(H/h_1^2)(\partial_1\{X_1X_2X_3/R\})] + \text{cyclic}\} + K_1^2X_1X_2X_3/R$

$= H^{-1}\{ \partial_1[f_1\, f_2\, f_3\, M_1\; R^2\, \partial_1\{X_1X_2X_3/R\}] + \text{cyclic}\} + K_1^2X_1X_2X_3/R$

$= H^{-1}\{ f_2\, f_3\, X_2X_3\; M_1(23)\partial_1[f_1R^2(\partial_1\{X_1/R\}] + \text{cyclic}\} + K_1^2X_1X_2X_3/R = 0$

This looks at least a little promising. Multiply by H

$f_2\, f_3\, X_2X_3\; M_1(23)\partial_1[f_1R^2\partial_1(X_1/R)] \;+\; \text{cyclic} + K_1^2H\, X_1X_2X_3/R = 0$

Now ready to do the derivative

$\partial_1(X_1/R) = \partial_1(R^{-1}X_1) = R^{-1}\partial_1X_1 - R^{-2}X_1\partial_1R = R^{-2}\{ (\partial_1X_1)R - X_1(\partial_1R) \}$

Then we get

$f_2\, f_3\, X_2X_3\; M_1(23)\partial_1[f_1\{ (\partial_1X_1)R - X_1(\partial_1R) \}] \;+\; \text{cyclic} \;= - K_1^2H\, X_1X_2X_3/R$

Multiply and divide by f_1 to get

$f_1f_2\, f_3\, X_2X_3\; (1/f_1)M_1(23)\partial_1[f_1\{ (\partial_1X_1)R - X_1(\partial_1R) \}] \;+\; \text{cyclic} \;= - K_1^2H\, X_1X_2X_3/R$

Now divide all terms by the three f_i

$X_2X_3\; (1/f_1)M_1(23)\partial_1[f_1\{ (\partial_1X_1)R - X_1(\partial_1R) \}] \;+\; \text{cyclic} \;=- K_1^2H\, X_1X_2X_3/(f_1f_2f_3R)$

Now pause to use our result above

$H/S \;=\; f_1(1)\, f_2(2)\, f_3(3)\, R^2u \qquad \Rightarrow \qquad H/(f_1f_2f_3R) = SRu$

and we then have

$X_2X_3\; (1/f_1)M_1(23)\partial_1[f_1\{ (\partial_1X_1)R - X_1(\partial_1R) \}] \;+\; \text{cyclic} \;= - K_1^2SRu\, X_1X_2X_3$

Do the same thing with X_1 to get

$(1/[X_1f_1])M_1(23)\partial_1[f_1\{ (\partial_1X_1)R - X_1(\partial_1R) \}] \;+\; \text{cyclic} \;= - K_1^2SRu$

Now I will try to get rid of M_1 using their hint

$h_1^2 \equiv Su/M_1 \qquad \Rightarrow \qquad M_1 = Su/h_1^2$

so the above then becomes

$$\text{Su } (1/[h_1^2 X_1 f_1]))\partial_1[f_1\{ (\partial_1 X_1)R - X_1(\partial_1 R) \}] \;+\; \text{cyclic} \;= -K_1^2 SRu$$

Question: when we say S, we mean S(123) always. So in the cyclic sum, it does not "cycle". I guess the same for u(123), so maybe we can divide them out to get

$$\Sigma_n (1/[h_n^2 X_n f_n])\, \partial_n[f_n\{ (\partial_n X_n)R - X_n(\partial_n R) \}] = -K_1^2 R$$

$$\Sigma_n (1/[h_n^2 X_n f_n])\, \partial_n[f_n(\partial_n X_n)R] = \Sigma_n (1/[h_n^2 X_n f_n])\, \partial_n[f_n X_n(\partial_n R)] - K_1^2 R$$

Now we are trying to get to our next waypoint on page 519 and we seem not *too* bad.

Now let's "do out" the two derivatives:

$$\partial_n[f_n(\partial_n X_n)R] = \partial_n[R\{f_n(\partial_n X_n)\}]$$

$$= R\partial_n\{f_n(\partial_n X_n)\} + (\partial_n R)\ \{f_n(\partial_n X_n)\}$$

$$\partial_n[f_n X_n(\partial_n R)] = \partial_n[X_n\{f_n(\partial_n R)\}]$$

$$= X_n\partial_n\{f_n(\partial_n R)\} + (\partial_n X_n)\ \{f_n(\partial_n R)\}$$

Now insert this into our equation above to get

$$\text{LHS} = \Sigma_n (1/[h_n^2 X_n f_n])[\;\; R\partial_n\{f_n(\partial_n X_n)\} + (\partial_n R)\ \{f_n(\partial_n X_n)\} \;\;]$$

$$= \Sigma_n (1/[h_n^2 X_n f_n])R\ \partial_n\{f_n(\partial_n X_n)\} \;+\; \Sigma_n (1/[h_n^2 X_n f_n])\ (\partial_n R)\ \{f_n(\partial_n X_n)\}$$

$$= \Sigma_n (1/[h_n^2 X_n f_n])\ R\partial_n\{f_n(\partial_n X_n)\} \;+\; \Sigma_n (1/[h_n^2 X_n])\ (\partial_n R)\ (\partial_n X_n)$$

$$\text{RHS} = \Sigma_n (1/[h_n^2 X_n f_n])\ [\;\; X_n\partial_n\{f_n(\partial_n R)\} + (\partial_n X_n)\ \{f_n(\partial_n R)\} \;\;] - K_1^2 R$$

$$= \Sigma_n (1/[h_n^2 X_n f_n])\ X_n\partial_n\{f_n(\partial_n R)\} + \Sigma_n (1/[h_n^2 X_n f_n])\ (\partial_n X_n)\ \{f_n(\partial_n R)\} - K_1^2 R$$

$$= \Sigma_n (1/[h_n^2 f_n])\ \partial_n\{f_n(\partial_n R)\} + \Sigma_n (1/[h_n^2 X_n])\ (\partial_n X_n)(\partial_n R) - K_1^2 R$$

The two red sums cancel and we then get

$$\Sigma_n (1/[h_n^2 X_n f_n])\ R\partial_n\{f_n(\partial_n X_n)\} = \Sigma_n (1/[h_n^2 f_n])\ \partial_n\{f_n(\partial_n R)\} - K_1^2 R$$

Divide both sides by R to get

$$\Sigma_n (1/[h_n^2 f_n X_n])\ \partial_n\{f_n(\partial_n X_n)\} = \Sigma_n (1/[h_n^2 f_n R])\ \partial_n\{f_n(\partial_n R)\} - K_1^2$$

and we have finally arrived at the page 519 waypoint (if we set $K_1^2 = 0$ which I think they did all along)

$$\Sigma_n \, (1/[h_n^2 f_n X_n]) \, \partial_n\{f_n(\partial_n X_n)\} = \Sigma_n \, (1/[h_n^2 f_n R]) \, \partial_n\{f_n(\partial_n R)\} \qquad \text{// the waypoint}$$

Now suppose we find R that solves this

$$\Sigma_n \, (1/[h_n^2 f_n R]) \, \partial_n\{f_n(\partial_n R)\} = - k_1^2/u$$

or

$$\Sigma_n \, (1/[h_n^2 f_n]) \, \partial_n\{f_n(\partial_n R)\} + k_1^2 R/u \; = 0 \qquad \text{// agrees with 5.1.47}$$

WARNING: this k_1^2 is not the Helmholtz parameter K_1^2, it is just "some constant" (I think). Then we have
from the waypoint that

$$\Sigma_n \, (1/[h_n^2 f_n X_n]) \, \partial_n\{f_n(\partial_n X_n)\} = - k_1^2/u$$

or

$$\Sigma_n \, (u/[h_n^2 f_n X_n]) \, \partial_n\{f_n(\partial_n X_n)\} + k_1^2 \; = 0$$

But now we use

$$h_n^2/S \equiv u/M_n \quad \Rightarrow \quad u/h_n^2 = M_n/S$$

and this becomes

$$\Sigma_n \, (M_n/[S f_n X_n]) \, \partial_n\{f_n(\partial_n X_n)\} + k_1^2 \; = 0 \qquad \text{// agrees with 5.1.48}$$

Now, **beam back** to the PL Stackel notes where equation (9) said

$$(\nabla^2 + k_1^2)\psi = H^{-1} \{ g_1(23)\partial_1[f_1(\partial_1 X_1)]/X_1 + \text{cyclic}\}\psi + k_1^2\psi \; = 0 \qquad (9)$$

or

$$H^{-1} \{ g_1(23)\partial_1[f_1(\partial_1 X_1)]/X_1 + \text{cyclic}\} + k_1^2 \; = 0$$

or

$$H^{-1} \{ g_1(23)/[X_1] \; \partial_1[f_1(\partial_1 X_1)] \; + \text{cyclic}\} + k_1^2 \; = 0$$

and where we eventually in those notes had in (2.2)

$$H/S = g_1(23) f_1/M_1 \qquad \Rightarrow \qquad g_1(23) = M_1 H/[S f_1]$$

so we can write our last result as

$$H^{-1} \{ M_1 H/[S f_1]/[X_1] \; \partial_1[f_1(\partial_1 X_1)] \; + \text{cyclic}\} + k_1^2 \; = 0$$

or

$$\{ M_1/[S f_1]/[X_1] \; \partial_1[f_1(\partial_1 X_1)] \; + \text{cyclic}\} + k_1^2 \; = 0$$

or

$$\{ M_1/[Sf_1X_1] \ \partial_1[f_1(\partial_1X_1)] \ + \text{cyclic}\} + k_1^2 = 0$$

and this last result matches 5.1.48.

So here is the point: if we find R that solves 5.1.47 with some arbitrary constant we call k_1^2, then we find that we obtain

$$\Sigma_n \ (M_n /[Sf_nX_n]) \ \partial_n\{f_n(\partial_nX_n)\} + k_1^2 = 0$$

and we see that this is the Helmholtz equation where the parameter is that constant k_1^2 and where the separated equations for the X_n are given in the usual Stackel theory form

$$(1/f_n)\partial_n[f_n(\partial_nX_n)] + [\ k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0$$

where now we seem to have three arbitrary constants.

Now one more time write the assumed R equation

$$\Sigma_n \ (1/[h_n^2f_nR]) \ \partial_n\{f_n(\partial_nR)\} = - \ k_1^2/u$$

$$\Sigma_n \ (u/[h_n^2f_nR]) \ \partial_n\{f_n(\partial_nR)\} = - \ k_1^2$$

$$k_1^2 = \ - \ (u/R) \ \Sigma_n \ (1/[h_n^2f_n]) \ \partial_n\{f_n(\partial_nR)\}$$

And I am happy to see this agree with M&S 4.03 where they call $k_1^2 = \alpha_2$ and u = Q.

Now let's go back go our M&F assumptions at the start:

$$H/S = f_1(1) \ f_2(2) \ f_3(3) \ R^2u$$
$$h_1^2 \equiv Su/M_1$$

and these are seen to agree with M&S page 96 equations (4.01) and (4.02), so we have made contact! Now M&S reference their own "Field theory for Engineers" 1961 for this stuff, but I use M&F which is just fine.