This is the Title PhL 3.26.05

1. How Helmholtz Equation Separation Works

First, we have to review a little of our simple separation theory. In order to get IT to work, we had to make certain assumptions concerning the CL coordinate scale factors $h_i(123)$. The first one was this:

$$(H/h_n^2) = g_n(\neq n)f_n(n) \qquad n = 1,2,3 \qquad \text{where} \qquad H = h_1h_2h_2 = H(123) \qquad (1)$$

Recall that F(123) means F is an arbitrary function of the three orthogonal variables. The assumption shown above is that the scale factor combination on the LHS has the "functional form" shown on the right. This is all a game of "functional form". We are saying that the dependence on ξ_n "factors out" as shown. This is of course three conditions, one for each value of n, so it is quite a tight restriction on the scale factors. For n = 1 the condition happens to be $h_2(123)h_3(123)/h_1(123) = g_1(23)f_1(1)$.

When we then introduce the Φ Stackel matrix with its cofactors $M_n(\neq n)$, we then show (we do this by inverting a little 3x3 matrix equation) that the Φ elements are constrained by these three conditions

$$1/h_n^2 = M_n(\neq n)/S \qquad (2)$$

where remember that the M_n and S are both functions of the Φ elements. This is not a new restriction; it just arises from giving a new name to the linear term $v_n(n)$ in our assumed form for the separated ODE

$$\mathcal{L}_nX_n = (1/f_n)\partial_n[f_n(\partial_nX_n)] + v_nX_n = 0$$
$$\mathcal{L}_nX_n = (1/f_n)\partial_n[f_n(\partial_nX_n)] + [\,k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0$$

In fact $M_p(\neq p) = \text{cof}(\Phi_{p1})/S$ where $S = \det(\Phi)$. We can combine (1) and (2) to get

$$H\, M_n(\neq n)/S = g_n(\neq p)f_n(n) \qquad (3)$$

and this then shows you the connection between M_n and g_n. So no new assumptions here either. We can rewrite (3) in this manner

$$H(123)/S(123) = g_1(\neq 1)f_1(1)/\, M_1(\neq 1) = g_2(\neq 2)f_2(2)/\, M_2(\neq 2) = g_3(\neq 3)f_3(3)/\, M_3(\neq 3)$$

To simplify notation for the moment, let's define

$$N_1(\neq 1) \equiv g_1(\neq 1)/\, M_1(\neq 1) \qquad F(123) \equiv H(123)/S(123)$$

Then our equations above say

$$F(123) = f_1(1)\, N_1(23)$$
$$F(123) = f_2(2)\, N_2(31)$$
$$F(123) = f_3(3)\, N_3(12)$$

$$\Rightarrow\ F(123) = f_1(1)\, f_2(2)\, f_3(3)\ * [N_1(23)\, N_2(31)\, N_3(12)]/3$$

This is a mistake! The correct equation here should be

$$F(123)^3 = f_1(1)\, f_2(2)\, f_3(3)\ * [N_1(23)\, N_2(31)\, N_3(12)]$$

so what follows is wrong.

Notice that these $N_n(\neq n)$ functions are determined in part by our CL system (which determines g_i and f_i from (1) above), and in part by our Φ matrix of unknown functions. We now make our second assumption which is that

$$[N_1(23)\, N_2(31)\, N_3(12)]/3 = 1$$

That is to say, we put this restriction on the Φ matrix components:

$$3\, M_1(\neq 1)\, M_2(\neq 2)\, M_3(\neq 3) = g_1(\neq 1)\, g_2(\neq 3)\, g_3(\neq 3)$$

This restriction is the same as this restriction

$$F(123) = f_1(1)\, f_2(2)\, f_3(3)$$

which is to say the following, which M&F call "the Robertson condition",

$$H(123)/S(123) = f_1(1)\, f_2(2)\, f_3(3) \qquad (4)$$

In this last form, we see that the restriction on the elements of Φ is really just a restriction on the value that S(123) = det(Φ) can have.

We could have introduced a new constant C in this last step like so

$$[N_1(23)\, N_2(31)\, N_3(12)]/3 = C \qquad N = g/M$$

$$3C\, M_1(\neq 1)\, M_2(\neq 2)\, M_3(\neq 3) = g_1(\neq 1)\, g_2(\neq 3)\, g_3(\neq 3)$$

$$H(123)/S(123) = C\, f_1(1)\, f_2(2)\, f_3(3)$$

Now recall that $f_n g_n$ only appear as a product pair in our "first restriction" $(H/h_n^2) = g_n f_n$. We could for example replace $g_1 f_1$ by $(g_1/C)(f_1 C) = g_1 f_1'$. Then our last two equations above would be

$$3M_1(\neq 1)\, M_2(\neq 2)\, M_3(\neq 3) = g_1'(\neq 1)\, g_2(\neq 3)\, g_3(\neq 3)$$

$$H(123)/S(123) = f_1'(1)\, f_2(2)\, f_3(3)$$

Our point is that were we to add a constant C to our picture, it does not buy us anything. It does not give us any more flexibility on solving for the Φ, so we just set C = 1 and be done with it.

So now let us review things:

• We first determine the g_n and f_n from these three equations

$$(H/h_n^2) = g_n(\neq n)f_n(n)$$

and this is our "first restriction" on things. We assume that we can find g_n and f_n , otherwise we give up on simple separation.

• We assume that the linear term v_n in our assumed separated ODE is a function of Φ matrix elements in a certain way, so we have

$$v_n(n) = k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)$$

and this is how Φ first enters the discussion. Notice that we have quietly put a functional-form restriction on the 9 elements of Φ, in that the Φ_{nm} are functions only of the variable ξ_n.

• By inverting a 3x3 matrix equation, we find this connection between the g_n and Φ

$$M_n/S = g_n f_n/H$$

where the Φ elements appear in both M_n and S. Using our "first restriction" we can write this as

$$M_n/S = 1/h_n^2$$

We can think of *this* equation as a condition on the elements of Φ and it results from our "first restriction".

• We then make the following "second restriction"

$$M_1M_2M_3 = g_1\, g_2\, g_3$$

which can be rewritten as the Robertson condition

$$H/S = f_1\, f_2\, f_3$$

and we can interpret this as a second condition on the elements of Φ. We could have put a constant in here, like this

$$M_1M_2M_3 = K\, g_1\, g_2\, g_3$$
$$H/S = K\, f_1\, f_2\, f_3$$

So here is where we stand. We need to find the 9 elements of Stackel matrix Φ. If we can do so, then we will have obtained our separated equations, which is our ultimate goal in "separation analysis":

$$\mathcal{L}_n X_n = (1/f_n)\partial_n[f_n(\partial_n X_n)] + [\, k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0$$

One condition on the Φ elements is the functional dependence form shown here. The other two conditions on Φ are these

$$M_n(\Phi)/S(\Phi) = 1/h_n^2$$
$$H/S(\Phi) = f_1 f_2 f_3 \qquad \text{where } f_n \text{ come from} \qquad (H/h_n^2) = g_n f_n$$

Now, even assuming we can meet our "first restriction" and come up with g_n and f_n, it may still be that our three conditions on Φ are such that there is no solution Φ ! It turns out that only the 11 classical systems can be solved for Φ in this manner. Since 10 derive from 1 (the ellipsoidal), the significant face is that the ellipsoidal system can be solved for Φ.

2. How R separation Works

We just noted in the previous section that our search for a solution Stackel matrix Φ is limited by these conditions

$$M_n(\Phi)/S(\Phi) = 1/h_n^2$$
$$H/S(\Phi) = f_1 f_2 f_3 \qquad \text{where } f_n \text{ come from} \qquad (H/h_n^2) = g_n f_n$$

Only if the above conditions are satisfied can we find a solution Φ matrix and Helmholtz separates.

In R separation theory, we "soften" these restricting conditions by adding two new functions R and Q so these conditions then become (I am here quoting these, not deriving them. They should emerge from our analysis below.)

$$M_n(\Phi)/S(\Phi) = Q/h_n^2$$
$$H/S(\Phi) = f_1 f_2 f_3 R^2 Q \qquad \text{where } f_n \text{ come from} \qquad (H/h_n^2) = g_n f_n R^2$$

and at the same time, we assume a "softer" form for our separated Helmholtz solution

$$\psi(123) = X_1X_2X_3/R(123) \qquad (2.1)$$

where R=R(123) and Q=Q(123). So this is no longer true "simple separation" because of this extra function R, and M&S call it "R separation", whereas M&F refer to R as a "modulation factor".

Now let's review what happens in our R separation theory. We first assume $\psi(123) = X_1X_2X_3/R(123)$ and we put this into our Helmholtz equation and start our "hunt" for separation.:

$$(\nabla^2 + K_1^2)\psi$$

$$= H^{-1}\{ \partial_1[(H/h_1^2)(\partial_1\{X_1X_2X_3/R\})] + \text{cyclic}\} + K_1^2X_1X_2X_3/R = 0$$

$$= H^{-1}\{ X_2X_3\ \partial_1[(H/h_1^2)(\partial_1\{X_1/R\})] + \text{cyclic}\} + K_1^2X_1X_2X_3/R = 0$$

$$= H^{-1}\{ X_2X_3\ \partial_1[\{(H/h_1^2)(1/R^2)\}\{ (\partial_1X_1)R - X_1(\partial_1R) \})] + \text{cyclic}\} + K_1^2X_1X_2X_3/R = 0$$

Here is where we **must** make our "first restriction" which is

$$\{(H/h_1^2)(1/R^2)\} = g_1f_1 \qquad \text{or} \qquad (H/h_1^2) = g_1f_1R^2 \qquad (2.2)$$

Now pause here for a moment. Write this out in more functional-form detail

$$(H(123)/[h_n(123)]^2 = g_1(23)f_1(1)\ [\ R(123)]^2$$

The LHS is completely determined by the CL system. The only reason we are adding $R \neq 1$ is for the case that the LHS does not match the functional form $g_1(23)f_1(1)$. In the worst case, we might have to set both $f_1=1$ and $g_1 = 1$ and let R "carry the whole load". However the 1-equation resolves itself, we still of course have the 2- and 3-equations, and this same R(123) might now work in those equations! So (2.2) is not going to work for ANY CL coordinate system, it is indeed a restriction. I think we should think of (2.2) as determining R(123), so this is not a function you can fiddle with in a significant way. So think then of equation (2.2) as determining all these 6 objects: g_n, f_n and R. We have to assume that our CLS allows this to happen, otherwise we give up.

Now we can *resume* processing our Helmholtz equation from above:

$$= H^{-1}\{ X_2X_3\ \partial_1[g_1f_1\{ (\partial_1X_1)R - X_1(\partial_1R) \})] + \text{cyclic}\} + K_1^2X_1X_2X_3/R = 0$$

$$= H^{-1}\{ X_2X_3\ g_1\partial_1[f_1\{ (\partial_1X_1)R - X_1(\partial_1R) \})] + \text{cyclic}\} + K_1^2X_1X_2X_3/R = 0$$

$$= \{ X_2X_3\ (g_1/H)\ \partial_1[f_1\{ (\partial_1X_1)R - X_1(\partial_1R) \})] + \text{cyclic}\} + K_1^2X_1X_2X_3/R = 0$$

$$= \{ X_2X_3\ (1/[h_1^2f_1R^2]\ \partial_1[f_1\{ (\partial_1X_1)R - X_1(\partial_1R) \})] + \text{cyclic}\} + K_1^2X_1X_2X_3/R = 0$$

$$\Rightarrow \{ (1/[h_1^2f_1X_1R^2]\ \partial_1[f_1\{ (\partial_1X_1)R - X_1(\partial_1R) \})] + \text{cyclic}\} + K_1^2/R = 0$$

$$\Sigma_n(1/[h_n^2f_nX_nR^2]\ \partial_n[f_n\{ (\partial_nX_n)R - X_n(\partial_nR) \})] + K_1^2/R = 0 \qquad \text{// mult by } R^2$$

$$\Sigma_n(1/[h_n^2f_nX_n]\ \partial_n[f_n\{ (\partial_nX_n)R - X_n(\partial_nR) \})] + K_1^2R = 0 \qquad (2.3)$$

where each line is just trivial algebra the reader can check by inspection. Notice in the 4th line that we again made use of (2.2) but now we are outside the ∂_1 operator. I don't see any alternative whatsoever to the above processing. In particular, we really must have $(H/h_n^2) = g_nf_nR^2$ with no other factors (such as a Q or S), because this is inside the outer ∂_1 operator and we don't want to pick up any other messy terms.

By computing out the derivatives in the correct threaded manner,

$$\partial_n[f_n(\partial_n X_n)R] = \partial_n[R\{f_n(\partial_n X_n)\}] = R\partial_n\{f_n(\partial_n X_n)\} + (\partial_n R)\ \{f_n(\partial_n X_n)\}$$

$$\partial_n[f_n X_n(\partial_n R)] = \partial_n[X_n\{f_n(\partial_n R)\}] = X_n\partial_n\{f_n(\partial_n R)\} + (\partial_n X_n)\ \{f_n(\partial_n R)\}$$

we find that in (2.3) we get cancellation of two terms containing $(\partial_n X_n)(\partial_n R)$ and we are left with

$$\Sigma_n\ (1/[h_n^2 X_n f_n])\ R\partial_n\{f_n(\partial_n X_n)\} = \Sigma_n\ (1/[h_n^2 f_n])\ \partial_n\{f_n(\partial_n R)\} - K_1^2 R$$

and we divide by R to get

$$\Sigma_n\ (1/[h_n^2 f_n X_n])\ \partial_n\{f_n(\partial_n X_n)\} = \Sigma_n\ (1/[h_n^2 f_n R])\ \partial_n\{f_n(\partial_n R)\} - K_1^2 \qquad (2.4)$$

which agrees with the "waypoint" of M&F on page 519 and with my previous work, except M&F assume from the start that $K_1^2 = 0$ from the start, but I have chosen to keep this term for now.

This (2.4) *is* our processed Helmholtz equation! Remember that we are looking for a separated ODE equation for X_n, and we have complete freedom in trying to find R and Q functions to make this work. As in the simple separation case, suppose we seek a separated equation of the same form,

$$\mathcal{L}_n X_n = (1/f_n)\partial_n[f_n(\partial_n X_n)] + [\ K_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0$$

$$(1/f_n)\partial_n[f_n(\partial_n X_n)] = -\ [\ K_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n$$

$$(1/[f_n\ X_n])\partial_n[f_n(\partial_n X_n)] = -\ [\ K_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]$$

$$(1/[h_n^2 f_n\ X_n])\partial_n[f_n(\partial_n X_n)] = -\ [\ K_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]/h_n^2$$

$$\Sigma_n\ (1/[h_n^2 f_n\ X_n])\partial_n[f_n(\partial_n X_n)] = -\ \Sigma_n\ [\ K_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]/h_n^2 \qquad (2.5)$$

Notice now that LHS(2.5) = LHS(2.6), so we then have

$$\Sigma_n\ [\ K_1^2\Phi_{n1}(n)+k_2^2\Phi_{n2}(n)+k_3^2\Phi_{n3}(n)]/h_n^2+\Sigma_n\ (1/[h_n^2 f_n R])\ \partial_n\{f_n(\partial_n R)\}\ -K_1^2 = 0 \qquad (2.6)$$

and this has the form

$$K_1^2\ (\text{stuff}) + k_2^2(\text{stuff}') + k_3^2(\text{stuff}'') = -\ \Sigma_n\ (1/[h_n^2 f_n R])\ \partial_n\{f_n(\partial_n R)\}$$

which is supposed to be valid for arbitrary values of K_1^2, k_2^2 and k_3^2. But this can only be satisfied if RHS = 0, so we would need to find an R such that

$$\Sigma_n\ (1/[h_n^2 f_n R])\ \partial_n\{f_n(\partial_n R)\} = 0 \qquad (2.7)$$

Let's assume we can find an R satisfying this equation. Then we have

$$K_1^2 [\Sigma_n (1/h_n^2) \Phi_{n1}(n) - 1] + k_2^2 [\Sigma_n (1/h_n^2) \Phi_{n2}(n)] + k_3^2 [\Sigma_n (1/h_n^2) \Phi_{n3}(n)] = 0$$

and setting the coefficients to zero we get

$$\begin{aligned} &\Sigma_n (1/h_n^2) \Phi_{n1}(n) = 1 \\ &\Sigma_n (1/h_n^2) \Phi_{n2}(n) = 0 \\ &\Sigma_n (1/h_n^2) \Phi_{n3}(n) = 0 \end{aligned} \qquad (2.8)$$

Now recall our first restriction(2.2) that

$$(H/h_n^2) = g_n f_n R^2 \qquad \Rightarrow (1/h_n^2) = g_n f_n \, R^2/H$$

so we can write our three equations as

$$\begin{aligned} &(R^2/H) \Sigma_n g_n f_n \Phi_{n1}(n) = 1 \\ &(R^2/H) \Sigma_n g_n f_n \Phi_{n2}(n) = 0 \\ &(R^2/H) \Sigma_n g_n f_n \Phi_{n3}(n) = 0 \end{aligned}$$

or

$$\begin{aligned} &\Sigma_n g_n f_n \Phi_{n1}(n) = H/R^2 \\ &\Sigma_n g_n f_n \Phi_{n2}(n) = 0 \\ &\Sigma_n g_n f_n \Phi_{n3}(n) = 0 \end{aligned} \qquad (2.9)$$

or

$$\Phi^T V = W$$

where

$$V = \begin{pmatrix} g_1(23) f_1(1) \\ g_2(31) f_2(2) \\ g_3(12) f_3(3) \end{pmatrix} \qquad W = \begin{pmatrix} H/R^2 \\ 0 \\ 0 \end{pmatrix}$$

which is the same as in our simple separation work except for the factor R^2. If we now invert the equation as we did in the simple separation analysis and we find from the top row of $V = [\Phi^T]^{-1} W$ that

$$g_n f_n = M_n H/(SR^2) \qquad (2.10)$$

We can then combine this with (2.2)

$$(H/h_n^2) = g_n f_n R^2 \qquad \Rightarrow g_n f_n = (H/R^2)(1/h_n^2)$$

to get this result

$$M_n H/(SR^2) = (H/R^2)(1/h_n^2)$$

$$M_n / S = (1/h_n^2) \qquad (2.11)$$

which agrees with our *simple* separation analysis. Now go back to (2.10) which says

$$H/(SR^2) = g_n f_n/M_n$$

We can now repeat our simple separation analysis that led us to the Robertson condition there. That is, we could impose this same condition.

Here is a quick review of this last thread. We processed the Helmholtz equation into (2.4) which everybody agrees with. We then wrote out the expected form of our ODE separated equation (same as in the simple case) using the same Φ elements, and when combined with the processed Helmholtz equation this led us to (2.6) which I repeat here,

$$\Sigma_n [K_1^2\Phi_{n1}(n)+k_2^2\Phi_{n2}(n)+k_3^2\Phi_{n3}(n)]/h_n^2+\Sigma_n (1/[h_n^2 f_n R]) \partial_n\{f_n(\partial_n R)\} -K_1^2 = 0 \qquad (2.6)$$

where we are supposed to have three "free" constants, so the coefficients of each of these constants needs to be 0. I observed that if the sum in red does not vanish, we are dead in the water because we cannot possibly solve an equation of this form

$$K_1^2 \text{ (stuff)} + k_2^2\text{(stuff')} + k_3^2\text{(stuff")} = -\Sigma_n (1/[h_n^2 f_n R]) \partial_n\{f_n(\partial_n R)\}$$

At this point, I split off from the M&F presentation just to pursue my own private path for a while. I asked "what if" the R(123) we were forced to use from (2.2) *happens* to result in

$$\Sigma_n (1/[h_n^2 f_n R]) \partial_n\{f_n(\partial_n R)\} = 0$$

Now remember that everything in this equation is "forced" to be whatever it is by our CL system, so it is extremely unlikely that this sum would happen to come out being 0. But if it just happened to be 0 for some particular curvilinear system, then we end up with two equations which are the same as in the simple separation world, namely,

$$M_n / S = (1/h_n^2)$$
$$H/S = f_1(1)\, f_2(2)\, f_3(3)$$

where this second condition would be something we impose on our Φ elements as in the simple case.

OK, since the sum shown above is NOT going to be 0, we can now forget this little side path.

Resume main flow. Since all the objects in our R sum above are determined by the CL system, the R sum is going to just be some result which we cannot control. We might as well remove the R from the LHS denominator. Then let's assume that the resulting sum has this form

$$\Sigma_n (1/[h_n^2 f_n R]) \partial_n\{f_n(\partial_n R)\} = -k_1^2/Q(123) \qquad (2.7)$$

so what we are saying is that this sum *determines* function Q(123) which we define in a slightly strange way by including the other two factors shown. Once we settle on a scale for Q(123), k_1^2 will have some specific value. It seems we could have just set it to 1, but leave it as it is. Notice that (2.7) is in no way a restriction of anything, it is the definition of our second function Q which we now carry along with R.

Here then is what (2.6) becomes

$$\Sigma_n [K_1^2\Phi_{n1}(n)+k_2^2\Phi_{n2}(n)+k_3^2\Phi_{n3}(n)]/h_n^2 - k_1^2/Q - K_1^2 = 0$$

$$\Sigma_n [K_1^2\Phi_{n1}(n)+k_2^2\Phi_{n2}(n)+k_3^2\Phi_{n3}(n)]/h_n^2 - K_1^2 = k_1^2/Q$$

$$K_1^2 \text{ (stuff)} + k_2^2\text{(stuff')} + k_3^2\text{(stuff'')} = k_1^2/Q$$

where again the three LHS constants are supposed to be free. I claim that this equation cannot be solved. For example, if we set all three constants to 0, there is clearly no solution since k_1^2 is determined. This *may be* the reason that you cannot solve Helmholtz with R separation. It would appear that even if you were to set $K_1^2 = 0$ that there is still no solution , but I will ignore that fact for the moment.

Instead, we return now to (2.4), and just resume from there, so

$$\Sigma_n (1/[h_n^2 f_n X_n]) \partial_n\{f_n(\partial_n X_n)\} = \Sigma_n (1/[h_n^2 f_n R]) \partial_n\{f_n(\partial_n R)\} - K_1^2 \qquad (2.4)$$

and we replace the R-sum using (2.7) to get (I stubbornly retain $K_1^2 \neq 0$ for now)

$$\Sigma_n (1/[h_n^2 f_n X_n]) \partial_n\{f_n(\partial_n X_n)\} = - k_1^2/Q - K_1^2 \qquad (2.12)$$

Now we introduce the Φ matrix for the first time and write our separated equation this way

$$\mathcal{L}_n X_n = (1/f_n)\partial_n[f_n(\partial_n X_n)] +[\kappa_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0 \qquad (2.13)$$

where we use κ_1^2 to avoid confusion. We make our usual assumption that k_2^2 and k_3^2 are separation constants that can take any values, and κ_1^2 is TBD. So now we have some Φ components to worry about and to try to compute.

Now at this point suppose out of the blue we impose this condition on Φ

$$M_n(\Phi)/S(\Phi) = Q/h_n^2 \qquad (2.14)$$

which differs form our simple theory in that Q is present. We are of course allowed to impose whatever condition we like on Φ, the question is "can we solve for Φ" with the conditions we impose?

Let's now multiply (2.12) by Q

$$\Sigma_n (Q/[h_n^2 f_n X_n]) \partial_n\{f_n(\partial_n X_n)\} + k_1^2 - K_1^2 Q$$

and then make use of (2.14) to write

$$\Sigma_n (M_n(\Phi)/[S(\Phi) f_n X_n]) \partial_n\{f_n(\partial_n X_n)\} + k_1^2 = K_1^2 Q \qquad (2.15)$$

Now we pause and go back to our simple-separation analysis and we note that we could write the ODE there for X_n in the following manner

$$\Sigma_n \ (M_n(\Phi)/[\ S(\Phi)f_nX_n]) \ \partial_n\{f_n(\partial_nX_n)\} + k_1^2 = 0 \qquad \text{(a)}$$

and we also had there that

$$\mathcal{L}_nX_n = (1/f_n)\partial_n[f_n(\partial_nX_n)] + [\ k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0 \qquad \text{(b)}$$

These two equations go together. The first (a) is the processed Helmholtz equation with parameter k_1^2 and (b) is the separated ODE that we obtain, in which k_1^2 appears.

Now, if (the big if) we set $K_1^2 = 0$, then 2.15 looks just like (a), and we know the "solution" ODE for (a) is (b). Therefore, in this case, it must be that the κ_1^2 appearing in (2.13) is k_1^2, which remember is the thing appearing in our R sum (2.7).

What happened to the Robertson Condition? I have never used it or mentioned it. What role does it play in the simple separation analysis? We wrote it there as (2.4) [in the Stackel matrix business doc] and here is what we did, I will quote equations from that doc:

$$g_n(\neq n)f_n(n) = H/h_n^2 \qquad n = 1,2,3 \qquad \text{SS(2.1)}$$
$$1/h_n^2 = M_n(\neq n)/S \qquad \text{SS(2.3)}$$
$$H/S = f_1(1)\ f_2(2)\ f_3(3) \qquad \text{SS(2.4)}$$

$$\Rightarrow\ M_1(\neq 1) = S/h_1^2 = (S/H)\ g_1(\neq 1)f_1(1) = g_1(\neq 1)f_1(1)/[\ f_1(1)\ f_2(2)\ f_3(3)\ ]$$
$$= g_1(\neq 1)/\ [f_2(2)\ f_3(3)\ ] \qquad \text{SS(2.5)}$$

Notice that we used all three quoted equations to get this result! Notice also that the result tells us the value of M_1 from data that we know! And we also know S from (2.4). So knowing S and the M_n, we are able to solve for the entire matrix Φ.

Now we want to do the analogous thing here in our R separation analysis. So far we already know from our (2.2) that

$$(H/h_1^2) = g_1f_1R^2 \qquad \Rightarrow \qquad g_n(\neq n)f_n(n) = (H/h_n^2)(1/R^2)$$

Also, from (2.14) we have (we had to do this to make things work above)

$$M_n(\Phi)/S(\Phi) = Q/h_n^2 \qquad \Rightarrow \quad 1/h_n^2 = (M_n(\neq n)/S)(1/Q)$$

Now suppose we impose a new Robertson condition of the form

$$H/S(\Phi) = f_1\ f_2\ f_3\ R^2/Q$$

We will then have

$$g_n(\neq n)f_n(n) = (H/h_n^2)\ (1/R^2) \qquad n = 1,2,3 \qquad \text{SS(2.1)'}$$
$$1/h_n^2 = (M_n(\neq n)/S\ )(1/Q) \qquad \text{SS(2.3)'}$$

$$H/S = f_1(1)\, f_2(2)\, f_3(3)(R^2Q) \qquad \text{SS(2.4)'}$$

and if we do our little calculation of M_1 we now get

$$M_1 = (S/h_1^2)\{Q\} = (S/H)(H/h_1^2)\{Q\} = (S/H)\, g_1f_1\, \{QR^2\} = g_1f_1\, \{QR^2\}\, Q/[\, f_1(1)\, f_2(2)\, f_3(3)\, R^2Q]$$

$$= g_1/\,[f_2(2)\, f_3(3)] \qquad \text{SS(2.5)'}$$

Thus we see that if we assume the modified Robertson condition shown in (2.4'), then our expressions for the cofactors M_n are *exactly the same* as they were in the simple separation theory.

Conclusion: Here are the steps for doing an R-separation of a CL system

(1) Compute the 7 objects g_n, f_n and R(123) from (2.2): $(H/h_1^2) = g_1f_1R^2$

(2) Compute Q and a value for k_1^2 from this expression:

$$\Sigma_n\, (1/[h_n^2 f_n R])\, \partial_n\{f_n(\partial_n R)\} = -\, k_1^2/Q(123)$$

(3) Compute the M_1 from $M_1 = g_1/\,[f_2(2)\, f_3(3)]$ and similarly for M_2 and M_3.

(4) Compute S from $1/h_n^2 = (M_n/S)(1/Q)$, using any n.

(5) You now know the M_i and S from which you can deduce a Stackel matrix Φ.

You then have your separated ODE equations as

$$(1/f_n)\partial_n[f_n(\partial_n X_n)] + [\, k_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)]X_n = 0$$

(6) If you maintain $K_1^2 \neq 0$ as in (2.15), this method does not work. Thus, we conclude that we do not know how to separate the Helmholtz equation in this R-separation model, although perhaps it is somehow possible. This method works only for the Laplace equation.

M&S state on page 96: "No case is known in which the Helmholtz equation is R-separable."

After all, 2.15 says

$$\Sigma_n\, (M_n(\Phi)/[\, S(\Phi)f_nX_n])\, \partial_n\{f_n(\partial_n X_n)\} + k_1^2 = K_1^2Q(123) \qquad (2.15)$$

and in general Q(123) tangles everything together so you can't separate. If for some system you could perhaps write $Q(123) = \Sigma_n\, q_n(n)$, then you would have

$$\Sigma_n\, [\; (M_n(\Phi)/[\, S(\Phi)f_nX_n])\, \partial_n\{f_n(\partial_n X_n)\} - K_1^2\, q_n(n)\;] + k_1^2 = 0$$

Then you could separate in this mechanical way

$$(M_1(\Phi)/[\, S(\Phi) f_1 X_1]) \, \partial_1\{f_1(\partial_1 X_1)\} \; - K_1^2 \, q_1(1) \;\; = \; c_1^2$$
$$(M_2(\Phi)/[\, S(\Phi) f_2 X_2]) \, \partial_2\{f_2(\partial_2 X_2)\} \; - K_1^2 \, q_2(2) \;\; = \; c_2^2$$
$$(M_3(\Phi)/[\, S(\Phi) f_3 X_3]) \, \partial_3\{f_3(\partial_3 X_3)\} \; - K_1^2 \, q_3(3) \;\; = \; c_3^2$$

$$\Sigma_n \, [\;\; (M_n(\Phi)/[\, S(\Phi) f_n X_n]) \, \partial_n\{f_n(\partial_n X_n)\} \; - K_1^2 \, q_n(n) \;\;] \; + k_1^2$$

$$= c_1^2 + c_2^2 \;\; + c_3^2 \; + k_1^2 = 0$$

So you would have two free separation constants c_2^2 and c_3^2, and the third is $c_1^2 = -k_1^2$.

But I suspect $Q(123) = \Sigma_n \, q_n(n)$ never happens for any known system.

Question: is there some way to identify something like k_2^2 as a separation constant in the above mechanical way in the simple separation theory? Do we get $k_2^2 = c_2^2$?

Problem: try this plan for toroidals and compare to M&S p 97.