Theorem M&S page 78: "No cylindrical system supports R separation "

Parabolic cylinder does, but R = 1. We want to prove the assertion that you cannot do R separation with a cylindrical system. Therefore we shall follow the four steps of our doc #5

We know that $h_3 = 1$. I don't think they are assuming conformal. So which equation cannot be satisfied? I guess here is what we know

$h_1 = h_1(12)$
$h_2 = h_2(12)$
$h_3 = 1$ $\quad H = h_1h_2 \quad H/h_1^2 = h_2/h_1 \quad H/h_2^2 = h_1/h_2 \quad H/h_3^2 = h_1h_2$

(1) Is there a problem with (3.5) ?

$(H/[R^2h_n^2]) = f_n(n)g_n(\neq n)$

$(h_2/h_1) = f_1(1)\, g_1(23)R^2 \quad => \quad g_1(23) = G_1(2)$
$(h_1/h_2) = f_2(2)\, g_2(13)R^2 \quad => \quad g_2(13) = G_2(1)$
$(h_1h_2) = f_3(3)\, g_3(12)R^2 \quad => \quad f_3(3) = 1$

Since the h's don't depend on ξ_3, we can define G_1 and G_2 as shown. Recall that none of the f_n or g_n can vanish. Multiply the first two lines to get

$1 = f_1(1)\, g_1(23)R^2 * f_2(2)\, g_2(31)R^2 = f_1g_1f_2g_2R^4$

Therefore

$R^{-4} = f_1g_1f_2g_2 = f_1(1)G_1(2)f_2(2)G_2(1) = [f_1(1)G_2(1)][G_1(2)f_2(2)] \equiv J_1(1)\, J_2(2)$

so we know that R = R(12). Note that $R^{-2} = \sqrt{J_1(1)\, J_2(2)}$

We can rewrite the third line above as

$(h_1h_2) = g_3(12)R^2$

so that

$g_3(12) = (h_1h_2)/R^2 = (h_1h_2)\sqrt{J_1(1)\, J_2(2)}$

which has this functional form

$g_3(12) = (h_1(12)h_2(12)\sqrt{J_1(1)\, J_2(2)}$

(2) We are supposed to compute Q using (3.9) which says

$$\Sigma_n\,(1/[h_n^2 f_n R])\,[\partial_n\{f_n(\partial_n R)\}] \;= -\,k_1^2/Q(123) \qquad (3.9)$$

Since $\partial_3 R = 0$, there will be only two terms in the sum

$$-\,k_1^2/Q(123) = (1/[h_1^2 f_1 R])\,[\partial_1\{f_1(\partial_1 R)\}] + (1/[h_2^2 f_2 R])\,[\partial_2\{f_2(\partial_2 R)\}]$$

$$= (1/[h_1^2 f_1 R])\,[\partial_1\{f_1(\partial_1 R)\}] + (1/[h_2^2 f_2 R])\,[\partial_2\{f_2(\partial_2 R)\}]$$

If I put these two terms into Maple as T1 and T2 and have $R^{-4} = f_1 g_1 f_2 g_2$, we just get this mess

```
T1 := 1/(h1(x1,x2)^2*f1(x1)*R(x1,x2))*diff(f1(x1)*diff(R(x1,x2),x1),x1):simplify(%);
```

$$\frac{1}{16}\,\frac{\left(\frac{\partial}{\partial x1}\mathrm{f1}(x1)\right)^2 \mathrm{G2}(x1)^2 - 2\left(\frac{\partial}{\partial x1}\mathrm{f1}(x1)\right)\mathrm{G2}(x1)\,\mathrm{f1}(x1)\left(\frac{\partial}{\partial x1}\mathrm{G2}(x1)\right) + 5\,\mathrm{f1}(x1)^2\left(\frac{\partial}{\partial x1}\mathrm{G2}(x1)\right)^2 - 4\,\mathrm{f1}(x1)\,\mathrm{G2}(x1)^2\left(\frac{\partial^2}{\partial x1^2}\mathrm{f1}(x1)\right) - 4\,\mathrm{f1}(x1)^2\,\mathrm{G2}(x1)\left(\frac{\partial^2}{\partial x1^2}\mathrm{G2}(x1)\right)}{\mathrm{f1}(x1)^2\,\mathrm{G2}(x1)^2\,\mathrm{h1}(x1,x2)^2}$$

```
T2 := 1/(h2(x1,x2)^2*f2(x2)*R(x1,x2))*diff(f2(x2)*diff(R(x1,x2),x2),x2):simplify(%);
```

$$-\frac{1}{16}\,\frac{2\left(\frac{\partial}{\partial x2}\mathrm{G1}(x2)\right)\mathrm{f2}(x2)\,\mathrm{G1}(x2)\left(\frac{\partial}{\partial x2}\mathrm{f2}(x2)\right) - \mathrm{G1}(x2)^2\left(\frac{\partial}{\partial x2}\mathrm{f2}(x2)\right)^2 - 5\left(\frac{\partial}{\partial x2}\mathrm{G1}(x2)\right)^2\mathrm{f2}(x2)^2 + 4\,\mathrm{f2}(x2)^2\,\mathrm{G1}(x2)\left(\frac{\partial^2}{\partial x2^2}\mathrm{G1}(x2)\right) + 4\,\mathrm{f2}(x2)\,\mathrm{G1}(x2)^2\left(\frac{\partial^2}{\partial x2^2}\mathrm{f2}(x2)\right)}{\mathrm{f2}(x2)^2\,\mathrm{G1}(x2)^2\,\mathrm{h2}(x1,x2)^2}$$

where the first line has lots of ∂_1's and the second lots of ∂_2's. There is really nothing more I can do without knowing more about the various functions. But one thing I notice. Because R = R(12), you might think that T1 and T2 would each have complicated (12) dependence. But what we seeing is this

$$T1 = (\,A1(1)/B1(1)) / h_1^2$$
$$T2 = (\,A2(2)/B2(2)) / h_2^2$$

So our computation is telling us that

$$-\,k_1^2/Q(12) = (\,A1(1)/B1(1)) / h_1^2 + (\,A2(2)/B2(2)) / h_2^2$$

so let's leave it at that form the moment.

(3) We now look at equation (7.5) which reads

$$\begin{aligned} g_1(23) &= E(2)I(3)\text{-}F(2)H(3) \\ -\,g_2(31) &= B(1)I(3)\text{-}C(1)H(3) \\ g_3(12) &= B(1)F(2)\text{-}C(1)E(2) \end{aligned} \qquad (7.5)$$

and in our current situation this says

$$\begin{aligned} G_1(2) &= E(2)I(3)\text{-}F(2)H(3) \\ -\,G_2(1) &= B(1)I(3)\text{-}C(1)H(3) \\ g_3(12) &= B(1)F(2)\text{-}C(1)E(2) \end{aligned}$$

Let's try H(3) = 0 and F(2) = 0 since that worked in our cylinder example, then

$$\begin{aligned} G_1(2) &= E(2)I(3) \\ -\,G_2(1) &= B(1)I(3) \\ g_3(12) &= -C(1)E(2) \end{aligned}$$

We must then have $E(2) = k_2G_1(2)$ and $B(1) = k_1G_2(1)$, apart from constants, so we get

$$\begin{aligned} G_1(2) &= k_2G_1(2)I(3) \\ -\,G_2(1) &= k_1G_2(1)I(3) \\ g_3(12) &= -C(1)\,k_2G_1(2) \end{aligned}$$

or

$$\begin{aligned} 1 &= k_2I(3) \\ -\,1 &= k_1I(3) \\ g_3(12) &= -C(1)\,k_2G_1(2) \end{aligned}$$

Adding the first two equations tells us $k_2 = -k_1$ so we select $k_2 = 1$ and $k_1 = -1$ to get

$$\begin{aligned} 1 &= I(3) \\ -\,1 &= -I(3) \\ g_3(12) &= -C(1)\,G_1(2) \end{aligned}$$

So I(3) = 1 and we are left with the single equation

$$g_3(12) = -C(1)\,G_1(2)$$

This equation implies that $g_3(12)$ factors as shown. Assuming it does, we get

$$-C(1) = g_3(12)/\,G_1(2)$$

so our only requirement so far is that $g_3(12)$ must factor, which it might in some cylindrical system. So in step (3) we are pretty much still alive with some $R(12) \neq 1$.

Let's look again at our factorization requirement

$$g_3(12) = -C(1)\,G_1(2)$$

We learned earlier that

$$g_3(12) = (h_1(12)h_2(12)\sqrt{J_1(1)\,J_2(2)}$$

so we are now requiring that the product of our two h_i be a factorizable function. This is certainly the case for cylindrical coordinates, but not for elliptic cylinder coordinates.

(4) Now we consider

$$1/Q = a(1)\,(1/h_1^2) + d(2)\,(1/h_2^2) + g(3)\,(1/h_3^2)$$

or

$$1/Q(12) = a(1)\,(1/h_1^2) + d(2)\,(1/h_2^2) + g(3)$$

or

$$1/Q(12) = a(1)\,(1/h_1^2) + d(2)\,(1/h_2^2)$$

where we remove the last term since there is no 3-dependence in the LHS. So now the question is whether this last equation is compatible with our Q(12) calculation! Above we got

$$-k_1^2/Q(12) = (A1(1)/B1(1))/h_1^2 + (A2(2)/B2(2))/h_2^2$$

So we could solve this by saying

$$k_1^2 = -1$$

$$a(1) = (A1(1)/B1(1))$$

$$d(2) = (A2(2)/B2(2))$$

so in theory we have no problem at all with step (4) !!!

Conclusion: I don't see any reason why many cylindrical systems could allow R separation! Any system for which $g_3(12)$ factors will be R separable! I am mystified by the M&S comment.

But let's go back to my conclusion above

we are now requiring that the product of our two h_i be a factorizable function.

So in order to get R separation, we need to have

$$h_1(12) = a_1(1)b_1(2)$$
$$h_2(12) = a_2(1)b_2(2)$$

and in M&S that means g_{11} and g_{22} have to factor. In his table starting on page 79, we can just look at these g_{ii} factors and see if any of his systems meet my requirement.

Page 79: no
Page 80,81: regular cylindricals do separate and our condition is met. E3C at first looks like a problem until we see ρ = a mess there. So nothing on these pages.
Page 82.83: no
Page 84,85: no

So all the cylindrical systems in the M&S table (their 21 systems have $h_1 = h_2$ AND h_1 does not factor.

Question: Why is it, then, that elliptical cylinder coordinates don't have factoring h's, but still are separable?

Go back to this point

$$G_1(2) = E(2)I(3)-F(2)H(3)$$
$$-G_2(1) = B(1)I(3)-C(1)H(3)$$
$$g_3(12) = B(1)F(2)-C(1)E(2)$$

What conditions are thus imposed on the scale factors? We can perhaps regard the first two equations here as requiring that H(3) = 0 or I(3) = 0, so lets try the first: H(3) = 0 I think "without loss of generality". We then get

$$G_1(2) = E(2)I(3)$$
$$-G_2(1) = B(1)I(3)$$
$$g_3(12) = B(1)F(2)-C(1)E(2)$$

Now pick $E(2) = k_2 G_1(2)$ and $I(3) = 1/k_2$ so the first line is OK. The next two lines are then

$$-G_2(1) = B(1)1/k_2$$
$$g_3(12) = B(1)F(2)-C(1)\, k_2 G_1(2)$$

In the top line here we must select $B(1) = -k_2 G_2(1)$ so our third line is then

$$g_3(12) = -k_2 G_2(1) F(2)-C(1)\, k_2 G_1(2)$$

But we can absorb our constants into F(2) and C(1) in general (or set $k_2 = -1$) to get

$$g_3(12) = G_2(1) F(2)- C(1) G_1(2)$$

and THIS then is our requirement on $g_3(12)$, and it is pretty specific.

Let's examine this relating to elliptic-cylinder coordinates M&S page 17. We have there

$$h_1^2 = h_2^2 = a^2[\, ch^2(\xi_1) - \cos^2(\xi_2)]$$

Our equations (3.5) become

$$(h_2/h_1) = f_1(1)\, g_1(23)R^2 \quad \Rightarrow \quad g_1(23) = G_1(2)$$
$$(h_1/h_2) = f_2(2)\, g_2(13)R^2 \quad \Rightarrow \quad g_2(13) = G_2(1)$$
$$(h_1h_2) = f_3(3)\, g_3(12)R^2 \quad \Rightarrow \quad f_3(3) = 1$$

$$R^{-4} = f_1g_1f_2g_2 = f_1(1)G_1(2)f_2(2)G_2(1) = [f_1(1)G_2(1)][G_1(2)f_2(2)] \equiv J_1(1)\, J_2(2)$$

$$1 = f_1(1)\, g_1(23)R^2 \quad \Rightarrow \quad g_1(23) = G_1(2)$$

$1 = f_2(2)\, g_2(13) R^2 \;\Rightarrow\; g_2(13) = G_2(1)$
$h_1^2 = f_3(3)\, g_3(12) R^2 \;\Rightarrow\; f_3(3) = 1$

$1 = f_1(1)\, G_1(2) R^2$
$1 = f_2(2)\, G_2(1) R^2$
$h_1^2 = g_3(12) R^2$

$R^{-4} = f_1 g_1 f_2 g_2 = f_1(1) G_1(2) f_2(2) G_2(1)$

or $R^{-2} = f_1(1)\, G_1(2) = f_2(2)\, G_2(1) = h_1^2 / g_3(12)$

One of these equations says

$f_1(1)\, G_1(2) = h_1^2 / g_3(12)$

$\Rightarrow g_3(12) = h_1^2 / [f_1(1)\, G_1(2)]$

But we know h_1^2 for this system so we get

$g_3(12) = a^2[\, ch^2(\xi_1) - \cos^2(\xi_2)] / [f_1(1)\, G_1(2)]$

$= a^2[\, ch^2(\xi_1) / [f_1(1)\, G_1(2)] - a^2 \cos^2(\xi_2) / [f_1(1)\, G_1(2)]$

and this fits our mold which was

$g_3(12) = G_2(1)\, F(2) - C(1)\, G_1(2)$

so that is WHY this system is R separable. Now how can we compute R or show it to be constant? If we try M&S suggestion that $f_1 = f_2 = 1$ we get this

$1 = G_1(2) R^2$
$1 = G_2(1) R^2$
$h_1^2 = g_3(12) R^2$

$R^{-2} = G_1(2) = G_2(1)$

This last line says R = constant, so let's set it to 1 and we get

$1 = G_1(2)$
$1 = G_2(1)$
$h_1^2 = g_3(12)$

This in turn says G's are one, so we get this solution:

$f_1 = f_2 = f_3 = 1 \qquad g_1 = g_2 = 1 \qquad g_3 = h_1^2 = a^2[\, ch^2(\xi_1) - a^2 \cos^2(\xi_2)] \qquad R = 1$

Our cofactors are

$M_1 = g_1 / f_2 f_3 = 1 = M_2$
$M_3 = g_3 / f_1 f_2 = g_3 = a^2[\, ch^2(\xi_1) - a^2 \cos^2(\xi_2)]$ // agree M&S p 17

****

Suppose we now limit ourselves to $h_1 = h_1$ which means the 2D is conformal. Then what?

$h_1 = h_2 = h_1(12)$
$h_3 = 1 \qquad H = h_1^2 \qquad H/h_1^2 = 1 \qquad H/h_2^2 = 1 \qquad H/h_3^2 = h_1^2$

(1) Is there a problem with (3.5) ?

$(H/[R^2 h_n^2]) = f_n(n) g_n(\neq n)$

$(h_2/h_1) = f_1(1)\, g_1(23) R^2 \quad \Rightarrow \quad g_1(23) = G_1(2)$
$(h_1/h_2) = f_2(2)\, g_2(13) R^2 \quad \Rightarrow \quad g_2(13) = G_2(1)$
$(h_1 h_2) = f_3(3)\, g_3(12) R^2 \quad \Rightarrow \quad f_3(3) = 1$

where we set $f_3(3) = 1$ since below we will find that R = R(12)

$1 = f_1(1)\, g_1(23) R^2$
$1 = f_2(2)\, g_2(13) R^2$
$h_1^2 = f_3(3)\, g_3(12) R^2$

$1 = f_1(1)\, G_1(2) R^2$
$1 = f_2(2)\, G_2(1)) R^2$
$h_1^2 = g_3(12) R^2$

$R^{-2} = f_1(1)\, G_1(2) = f_2(2)\, G_2(1)) = g_3(12)/h_1^2$

Our solution of (7.5) condition is that

$g_3(12) = -C(1)\, G_1(2) = h_1^2\, f_1(1)\, G_1(2)$

$\Rightarrow\ -C(1) = h_1^2\, f_1(1)$

which says h_1 must be $h_1(1)$! If this were the case, what happens to $h_1 = h_2$? We conclude that

$h_2 = h_2(1)$ as well.

Meanwhile we then have

$R^{-2} = g_3(12)/h_1^2 = h_1^2\, f_1(1)\, G_1(2)/h_1^2 = f_1(1)\, G_1(2)$

Our three (3.5) equations then say

$1 = f_1(1)\, G_1(2)/\, f_1(1)\, G_1(2)$	OK
$1 = f_2(2)\, G_2(1))\, /\, f_1(1)\, G_1(2)$	already knew this
$h_1^2 = g_3(12)\, /\, f_1(1)\, G_1(2)$	already know this too

So nothing much happening here.

$(h_1 h_2) = g_3(12)R^2 = [W(2)G_2(1) + V(1)G_1(2)]\, R^2$

$= [W(2)G_2(1) + V(1)G_1(2)]\, /\sqrt{[f_1(1)G_2(1)][G_1(2)f_2(2)]}$

$= W(2)G_2(1)/\sqrt{[f_1(1)G_2(1)][G_1(2)f_2(2)]} + V(1)G_1(2)/\sqrt{[f_1(1)G_2(1)][G_1(2)f_2(2)]}$

$= W'(2)\, G_2(1)/\sqrt{f_1(1)G_2(1)} + V'(1)\, G_1(2)\, /\sqrt{f_s(s)G_1(2)}$

$= W'(2)\sqrt{G_2(1)/f_1(1)} + V'(1)\sqrt{G_1(2)/f_2(2)}$

$= W'(2)\, T_1(1) + V'(1)\, T_2(2)$