Appendix A: A small digression on inverting a matrix

From linear algebra, if we have a matrix equation in E^n

$$\mathbf{X} = A\mathbf{Y}$$

we know that if detA $\neq 0$ we can invert to get

$$\mathbf{Y} = A^{-1}\mathbf{Y}$$

where

$$[A^{-1}]_{pq} = \text{cof}(A_{qp})/\det(A)$$

Notice that the subscripts are switched on the two sides, and

$$\text{cof}(A_{qp}) = (-1)^{p+q}\,\text{minor}(A_{qp})$$

where minor(A_{qp}) is the determinant of the (n-1)x(n-1) matrix obtained by crossing out row q and column p of the matrix A. Now suppose we are given this problem instead

$$\mathbf{X} = B^T\mathbf{Y}$$

where T means transpose. If $\det(B^T) \neq 0$, we can invert to get

$$\mathbf{Y} = (B^T)^{-1}\mathbf{X}$$

where, noting that $\det(B^T) = \det(B)$,

$$(B^T)^{-1}_{pq} = \text{cof}(B^T{}_{qp})/\det(B^T)$$

or

$$(B^T)^{-1}_{pq} = \text{cof}(B_{pq})/\det(B)$$

where now the indices are in the *same* order on both sides. To clarify notation, we might then define a matrix C where

$$C_{pq} \equiv \text{cof}(B_{pq}) \qquad \text{" the cofactor matrix"}$$

so we then have

$$(B^T)^{-1}_{pq} = [1/\det(B)]C_{pq}$$

so that our inversion is given by

$$\mathbf{Y} = [1/\det(B)]\ C\ \mathbf{X} \qquad \text{inversion of } \mathbf{X} = B^T\mathbf{Y}$$