The det(1+A) determinant expansion and the Fredholm Determinant PhL 3.23.09

These are "Round 1" notes written when I was still confused about many things. The whole subject is now better presented in my "Round 2" notes in a separate document. Still, there are some things in these notes that are worth keeping, so I won't delete this document. A reader interested in learning about this subject should

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0. Overview of This Discussion.

I wanted to learn a little about this "Fredholm stuff" because it was omitted from Stakgold's integral equations chapter. When the Fredholm integral equations are put on a lattice, they are reduced to N x N matrix problems. One of these equations is "the eigenvalue problem" and the eigenvalues can be determined from a certain determinant (The Fredholm Determinant), and the general inhomogeneous equation can then be solved by inverting a certain matrix which as usual can be done using cofactors which invoke minors (Fredholm's First Minor). One can do this for any N one wants. In the case that the kernel K of the integral equation set is "small" so it is written λK, it is possible to find expansions in powers of λ for both the Fredholm Determinant and the Fredholm First Minor, and it is really these expansions that have the official Fredholm names. If one then takes the limit $N\to\infty$ to move back off the lattice, these expansions are still well defined, but each has an infinite number of terms. Again, if λ is small, one usually deals with just the first few terms of these $N=\infty$ expansions.

Various buzzwords occur in this development that I will mention here.

First of all, the inhomogeneous integral equation in the "Neumann form" can be written $\varphi = f + \lambda K\varphi$ where I show the smallness parameter λ. The formal solution is $\varphi = (1-\lambda K)^{-1}f$. So it is the matrix $1-\lambda K$ that we want to "invert" to solve our problem, and this brings up the Fredholm determinant $\det(1-\lambda K)$ and this also brings up the minors since we know that $(1 - \lambda K)^{-1} = [\ \mathrm{cof}\ (1-\lambda K)\ ]^{T} / \det(1-\lambda K)$.

Now, every high school student knows how to write $(1-\lambda K)^{-1}$ as a series $1 + \Sigma_{m=1}\lambda^m K^m$ where we break out the first term which is 1. The so-called resolvent Γ is defined by $1 + \Gamma = (1-\lambda K)^{-1}$ so we can equate the resolvent with the formal series solution $\Gamma = \Sigma_{m=1}\lambda^m K^m$. One can then write various integral equations which involve the resolvent, such as $\Gamma = \lambda K + \lambda K\Gamma = \lambda K + \lambda\Gamma K$. And our solution can then be written $\varphi = f + \Gamma f$, so finding the resolvent solves the problem.

I will now outline each of the sections below, by number.

1. In the first section below, I define certain determinants such as $\det_{456}$ which are "diagonal" subdeterminants of a larger matrix. I use the word diagonal for this reason: if you assemble the 3x3 array of numbers of which $\det_{456}$ is the determinant, you will find that all the diagonal elements of your array came from the diagonal of the original full matrix. So these subdeterminants like $\det_{456}$ are a certain small subset of all possible subdeterminants of a matrix. They are those that you get by crossing out the same set of rows and columns (in this case 1,2 and 3). It turns out that these are the only subdeterminants we have to worry about to derive the expansion formula for the Fredholm determinant. A general diagonal subdeterminant of a matrix can be written in the following somewhat obscure manner:

$$\det_{I1\ I2\ I3\ldots In} = \varepsilon^{i1\ i2\ i3\ldots.in} A_{I1\ i1} A_{I2\ i2} ..... A_{In\ in} \qquad \text{// n factors of A}$$

$$\text{where i1 i2 i3....in} \in \{I1,I2,I3...I_n\}\ ,\ \varepsilon^{i1\ i2\ i3\ldots.in} \text{ anti-symmetric, } \varepsilon^{I1\ I2\ I3\ldots.In} = +1$$

where the matrix A is NxN and $2 \le n \le N$, and where all the indices i_1 through i_n are implicitly summed.

2. Using this general formula, and setting A = 1+B, I am able to show that:

$$\det(1+B) = 1 + \mathrm{tr}(B) + (1/2!)\ \Sigma_{ij} \det_{ij} + (1/3!)\ \Sigma_{ijk} \det_{ijk} + (1/4!)\ \Sigma_{ijk\ell} \det_{ijk\ell} + ... + \det(B)$$

which is basically the Fredholm determinant expansion. For finite N, starting with the third term the dimensionality of the subdeterminants increases by 1 with each term until it reaches the full N. In the case N=5, all terms are displayed above and we see there are 6 terms, so in general there are N+1 terms in the expansion, where the first is 1 and the last is det(B). Each summation index has full range 1 to N, and of course any $\det_{ij\ldots}$ object is zero if any two subscripts are the same, as follows from the ε expression above. The factorials appearing in the expansion are overcounting corrections which allow free range of the summation indices. An alternate formula can be written where there is no overcounting:

$$\det(1+B) = 1 + \mathrm{tr}(B) + \Sigma_{i<j} \det_{ij} + \Sigma_{i<j<k} \det_{ijk} + \Sigma_{i<j<k<\ell} \det_{ijk\ell} + ... + \det(B)$$

3. For application with Fredholm integral equations, as was just shown above, we want B = -λK. With this replacement, the Fredholm determinant expansion takes on a more familiar form,

$$\det(1-\lambda K) = 1 - \lambda \mathrm{tr}(K) + (\lambda^2/2)\ \Sigma_{ij} \det_{ij} - (\lambda^3/3!)\ \Sigma_{ijk} \det_{ijk} + (\lambda^4/4!)\ \Sigma_{ijk\ell} \det_{ijk\ell} + ... + (-\lambda)^n \det(K)$$

where the various $\det_{ijk..}$ objects are subdeterminants of the matrix K. Now we see that the expansion is in fact a power series in λ, which is the main idea since λ is a smallness parameter. Also, we pick up alternating signs. The above is often presented in a more graphical but equivalent notation

$$\det(1-\lambda K) = 1 - \lambda \Sigma_i K_{ii} + (\lambda^2/2)\Sigma_{ij} \begin{vmatrix} Kii & Kij \\ Kji & Kjj \end{vmatrix} - (\lambda^3/3!)\Sigma_{ijk} \begin{vmatrix} Kii & Kij & Kik \\ Kji & Kjj & Kjk \\ Kki & Kkj & Kkk \end{vmatrix} + ...$$

where now you "see" the various subdeterminants involved.

To take the limit N→∞, we make these changes: (1) replace things like K_{xy} by k(x,y). (3) replace summation indices with integrations. When this is done, the Fredholm Determinant takes on its traditional appearance which is this:

$$\det(1-\lambda K) = 1 - \lambda \int dx\, k(x,x) + (\lambda^2/2) \int dx\, dy \begin{vmatrix} k(x,x) & k(x,y) \\ k(y,x) & k(y,y) \end{vmatrix}$$

$$- (\lambda^3/3!) \int dx dy dz \begin{vmatrix} k(x,x) & k(x,y) & k(x,z) \\ k(y,x) & k(y,y) & k(y,z) \\ k(z,x) & k(z,y) & k(z,z) \end{vmatrix} + \text{.... (goes on forever)}$$

Although the series never ends, for small λ we might do well keeping only the first few terms in the series if we are doing approximation work.

4. I derived a formula for Fredholm's First Minor in a separate document. Although I believe this proof (it is based on 1970's notes I had, probably from the Mikhlin book), I don't much like the proof, and I wanted to find a derivation similar to the one above for Fredholm's Determinant. However, in this case I need general formulas for a different class of subdeterminants: those obtained from a full matrix by crossing out row p and column q. Unless p = q, these subdeterminants are not "diagonal ones" as described above, so a new formula was needed.

It turns out that the formula for a general subdeterminant can be written in a form very similar to that shown above for a diagonal one. Here is the general result:

$$\det_{I1\ I2\ I3...In;\ a1\ a2\ a3...an;} = \varepsilon^{i1\ i2\ i3....in} A_{I1\ i1} A_{I2\ i2} ..... A_{In\ in} \qquad \text{// n factors of A}$$

where i1 i2 i3....in ∈ {a1 a2 a3...an} , $\varepsilon^{i1\ i2\ i3....in}$ anti-symmetric, $\varepsilon^{I1\ I2\ I3....In} = +1$

The new feature is that we now have two groups of integer labels on our det object. The first controls the row indices of the A factors, while the second group affects the column indices of the A factors. For a diagonal subdeterminant, these two sets of integer labels are the same, so $\det_{456} = \det_{456;456}$ in our fancier more general notation. In our new more general case, the first index set denotes the labels of the rows of the original matrix which are contributing to our submatrix, and the second set denotes the labels of the columns which are contributing to our submatrix. Indices not shown in these sets represent rows and columns that are "crossed out" of the full matrix to expose the elements of the subdeterminant.

As a special case of the above general formula, we can set n = N-1 and think about the subdeterminant obtained. Unfortunately, I have not been able to find an elegant way to express the det objects for this special case, other than the above formula. For example, if N = 4 and we cross out row p=1 and column q=3, we get

$$\det_{234;124} = A_{21}A_{32}A_{44} + \text{signed permutations} = \det_{\overline{1};\overline{3}}$$

where "signed permutations" means other terms obtained by permuting the second (column) indices on the A factors, such that an odd number of index swaps implies a minus sign, otherwise a plus sign. For example

$$\det_{234;124} = A_{21}A_{32}A_{44} - A_{21}A_{34}A_{42} + \text{other signed permutations} = \det_{\bar{1};\bar{3}}$$

where I show a permutation obtained by swapping one pair of indices. This is just the ε object doing its work, but I am suppressing the messy ε factor by just saying " + signed permutations".

5. I then consider the case N = 5, set B = 1+A as with the determinant calculation, so no λ floating around here. I consider that we cross out row p and column q, so we are faced with a 4x4 subdeterminant. I write this first as

$$\det_{I1,I2,I3,I4\ ;\ a1,a2,a3,a4} =$$

$$(\delta_{I1,a1} + B_{I1,a1})\,(\delta_{I2,a2} + B_{I2,a2})\,(\delta_{I3,a3} + B_{I3,a3})\,(\delta_{I4,a4} + B_{I4,a4}) \quad + \text{signed perms}$$

I then analyze those terms with no B factors, and then those terms with one B factor. These cases were plenty hard to do, and I did not go on to do the more general cases of three or more B factors.

I found these results for these first two cases (after much work) :

no B factors: $\delta_{p,q}$

one B factor: $\mathrm{tr}(B)\,\delta_{p,q} - (-1)^{p+q} B_{q,p}$

6. I then defined a new matrix N = det(1+B)Γ where Γ is the resolvent discussed earlier. Using our usual method of simply "inverting a matrix", one finds that

$$N_{pq} = -\det(1+B)\delta_{p,q} + (-1)^{p+q}\ \text{minor}\ [(1+B)_{qp}]$$

and one sees here the need for our special subdeterminant with a row and column crossed out, since that is what the word "minor" means. The indices are swapped since the matrix inversion formula involves the transpose of the cofactor matrix, as outlined above.

If we insert into this formula our Fredholm Determinant results for det(1+B) through first order in B, we get

$$N_{pq} = -[\ 1 + \mathrm{tr}(B)\ ]\ \delta_{p,q} + (-1)^{p+q}\ \text{minor}\ [(1+B)_{qp}] \qquad \text{// through order } B^1$$

If we then insert our results above which say (no B and single B terms only)

$$\text{minor}\ [(1+B)_{qp}] = \delta_{p,q} + \mathrm{tr}(B)\,\delta_{p,q} - (-1)^{p+q} B_{p,q}$$

we then find that

$$N_{pq} = -B_{pq}$$

The good news is that this is in fact correct for the B^0 and B^1 terms of N_{pq}. If we set B = -K, the correct formula for N is given by (this is the Fredholm First Minor expansio∂n)

$$N_{ij} = K_{ij} - \Sigma_k \begin{vmatrix} Kij & Kik \\ Kkj & Kkk \end{vmatrix} + (1/2!)\,\Sigma_{km} \begin{vmatrix} Kij & Kik & Kim \\ Kkj & Kkk & Kkm \\ Kmj & Kmk & Kmm \end{vmatrix} - (1/3!)\,\Sigma_{kmn} \begin{vmatrix} Kij & Kik & Kim & Kin \\ Kkj & Kkk & Kkm & Kkn \\ Kmj & Kmk & Kmm & Kmn \\ Knj & Knk & Knm & Knn \end{vmatrix} + .$$

Although I did what seemed "a lot of work", all I got was the first term showing in this expansion!

7. I took a quick look at that B^2 terms, and then "gave up" on this approach, at least in the way I am doing it right now. I feel there is some trick that I am missing, but I do feel that my approach using direct formulas for the "minor" type subdeterminants would completely validate the above formula, if I had enough time and energy to follow through.

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1. Preliminary Facts concerning the Determinants of Certain Submatrices of a Matrix.

(0) The determinant of a 3x3 matrix can be written in this way

$$\det = \varepsilon^{ijk} A_{1i} A_{2j} A_{3k} = A_{11}A_{22}A_{33} + 5 \text{ other terms}$$

We know there are 6 terms in all because ε^{ijk} has 3*2*1 = 6 non-zero values such as $\varepsilon^{123} = 1$. Remember that ε is a totally antisymmetric tensor and so changes sign when any index pair is swapped and of course therefore must be 0 if any two indices are the same. The other terms can be generated "one at a time" by just swapping alternating pairs of second indices on the previous term, and adding a minus sign which that swap creates due to the ε factor. Thus

$$= A_{11}A_{22}A_{33} - A_{12}A_{21}A_{33} + A_{12}A_{23}A_{31} - A_{13}A_{22}A_{31} + A_{13}A_{21}A_{32} - A_{11}A_{23}A_{32}$$

where the underlined indices show the swap we are about to do to make the term to the right. A 4x4 determinant of course has 4*3*2*1 = 24 terms and the terms could be "generated" in a similar (though not identical) fashion. Basically you just set the set of second indices to all possible permutations of 123...n , and if it took an odd number of swaps to reach a certain permutation, that term gets a minus sign.

Now, let A be a 6x6 matrix. Later we shall generalize to an N x N matrix.

(1) Consider:

$$\det_{456} = \varepsilon^{123ijk} A_{4i} A_{5j} A_{6k} = A_{44}A_{55}A_{66} + \text{other terms}$$

This is the determinant of the 3x3 matrix obtained by crossing our rows and columns 1,2,3 of the matrix A. This fact should be clear based on our item (0) above -- the only difference is that the indices now take values4,5,6 instead of 1,2,3. Notice that the indices 456 on $\det_{456}$ are in the "natural order" corresponding to the natural order in which the rows and columns of A are numbered, 1,2,3,4,5,6. That is to say, 4,5,6 are in increasing order.

We now want to have $\det_{546}$ be the determinant of this same 3x3 matrix but which has rows *and* columns 4 and 5 swapped. We know that if we swap a pair of rows AND a pair of columns, the

determinant does not change, so we know that $\det_{546} = \det_{456}$. The question is how to write $\det_{546}$ using the ε symbol. Here is the answer

$$\det_{546} = \varepsilon^{123ijk} A_{5j}A_{4i}A_{6k} = A_{55}A_{44}A_{66} + \text{other terms}$$

The rule is that we start with $\det_{456}$ and make the change $A_{4i}A_{5j} \rightarrow A_{5j}A_{4i}$, as I shall prove immediately below. But we see that this is really "no change" so $\det_{546} = \det_{456}$ as expected.

Proof: Here is my proof that this is the right result for swapping both rows and columns 4 and 5. This proof is indirect, there is probably a faster way to do it, but at least this works.

Start with $\det_{456}$ and make the change $A_{4i}A_{5j} \rightarrow A_{5i}A_{4j}$. This accomplishes the swapping of *rows* 4 and 5 (row index = first index on A). We then have this intermediate result, having started with $\det_{456}$,

$$\det_{456} = \varepsilon^{123ijk} A_{4i}A_{5j}A_{6k} \rightarrow \varepsilon^{123ijk} A_{5i}A_{4j}A_{6k}$$

We then rewrite this intermediate result using the fact that the determinant of a matrix is the same as that of its transpose, so we get

$$\varepsilon^{123ijk} A_{5i}A_{4j}A_{6k} = \varepsilon^{123ijk} A_{i5}A_{j4}A_{k6}$$

Now we swap *columns* 5 and 4 (second indices on A) so make the change $A_{i5}A_{j4} \rightarrow A_{i4}A_{j5}$. So now after doing both the row and column swaps we have

$$\varepsilon^{123ijk} A_{i5}A_{j4}A_{k6} \rightarrow \varepsilon^{123ijk} A_{i4}A_{j5}A_{k6}$$

Finally, we use our transpose rule a second time to write this last as

$$\varepsilon^{123ijk} A_{i4}A_{j5}A_{k6} = \varepsilon^{123ijk} A_{4i}A_{5j}A_{6k}$$

So far, then, we have

$$\det_{456} = \varepsilon^{123ijk} A_{4i}A_{5j}A_{6k} \rightarrow (\text{swap row and cols } 4,5) \rightarrow \varepsilon^{123ijk} A_{4i}A_{5j}A_{6k}$$

Notice that we never mess with the ε indices in all the steps. Rewrite one more time:

$$\det_{546} = \varepsilon^{123ijk} A_{4i}A_{5j}A_{6k} = \varepsilon^{123ijk} A_{5j}A_{4i}A_{6k} = \text{as claimed above, } \mathbf{QED.}$$

(2) In general, any permutation of the indices on $\det_{456}$ yields the same result. We can think of a general permutation being accomplished by a sequence of row/column swaps of the type shown in (1). Thus we know that

$$d_{456} = \det_{465} = \det_{546} = \det_{564} = \det_{645} = \det_{654}$$

(3) We now seek a way to write a general formula for d_{IJK} . Let's start going back to d_{456}:

$$\det_{456} = \varepsilon^{123ijk} A_{4i} A_{5j} A_{6k} = A_{44}A_{55}A_{66} + \text{other terms}$$

This full notation is a little clumsy, because we have to put the ε indices in the right place, for example,

$$\det_{136} = \varepsilon^{i2j45k} A_{4i} A_{5j} A_{6k} = A_{11}A_{33}A_{66} + \text{other terms}$$

So let's rewrite $\det_{456}$ in a slightly different notation:

$$\det_{456} = \varepsilon^{ijk} A_{4i} A_{5j} A_{6k} \quad \text{where ijk} \in \{4,5,6\},\ \varepsilon^{ijk} = \text{anti-symmetric, and } \varepsilon^{456} = +1.$$

Notice that ε^{ijk} is basically "the usual" rank 3 totally antisymmetric tensor, but its indices take values 4,5,6 instead of 1,2,3 AND we are going to set a certain specific element of this tensor to +1 and this element may not be the one you think it should be. All other elements can be obtained from the one we set by doing permutations with minus signs as appropriate.

Here then is our claimed generalized formula:

$$\det_{IJK} = \varepsilon^{ijk} A_{Ii} A_{Jj} A_{Kk} \quad \text{where ijk} \in \{I,J,K\},\ \varepsilon^{ijk} = \text{anti-symmetric, and } \varepsilon^{IJK} = +1.$$

For example, we get (in our first example, our set value is $\varepsilon^{546} = 1$)

$$\det_{546} = \varepsilon^{ijk} A_{5i} A_{4j} A_{6k} = \varepsilon^{546} A_{55} A_{44} A_{66} + ... = A_{55} A_{44} A_{66} + ...$$

$$\det_{215} = \varepsilon^{ijk} A_{2i} A_{1j} A_{5k} = \varepsilon^{215} A_{22} A_{11} A_{55} + ... = A_{22} A_{11} A_{55} + ...$$

There are 6 terms in one of these 3x3 determinants, and it is our goal to avoid having to write them out! That is why we are using this slightly strange compressed ε notation. It is clumsy to write out the 6 terms in a linear fashion, and it is also clumsy to write it out as a 3x3 matrix surrounded by vertical lines, but we could of course write things out in either of these forms it we really wanted to, for example

$$\det_{ijk} = \begin{vmatrix} Aii & Aij & Aik \\ Aji & Ajj & Ajk \\ Aki & Akj & Akk \end{vmatrix} = A_{ii}A_{jj}A_{kk} + \text{other terms}$$

(4) Let's go back to our general formula above,

$$\det_{456} = \varepsilon^{123ijk} A_{4i} A_{5j} A_{6k}$$

where ε here is the true antisymmetric tensor in our 6D space. If the matrix A has the same numbers in rows 4 and 5, the RHS above = 0 because ε is antisymmetric on ij while $A_{4i}A_{5j} = A_{4i}A_{4j}$ is symmetric on ij. Let's define this as the meaning of $\det_{446}$. This is still the determinant of a 3x3 matrix, but two rows are the same, so we know $\det_{446} = 0$. Similarly, we will define $\det_{IJK} = 0$ if any of the two indices are the same. We can just define this to be true with no justification needed, or we can think of it in terms

of two rows having the same numbers. And of course it would also be zero if all three indices are the same.

(5) Because all the permutations are the same, we can write

$\det_{456} = 1/6\ \Sigma_{IJK \in \{4,5,6\}} \det_{IJK}$ and $1/6 = 1/3!$

We have in mind of course that we are summing over all possible sets IJK where IJK are different integers taken from the set {4,5,6} . But we can also think of this as including *all* choices IJK $\in$ {4,5,6} since we know that $\det_{IJK} = 0$ when any two indices are the same.

(6) Suppose we were to encounter the following sum:

$d_{123} + d_{124} + d_{125} + d_{126} + d_{134} + d_{135} + d_{136} + d_{145} + d_{146} + d_{156}$

$+ \quad d_{234} + d_{235} + d_{236} + d_{245} + d_{246} + d_{256}$

$+ \quad d_{345} + d_{346} + d_{356} \quad + \quad d_{456}$

Here we include all sets of indices which are in natural order. We could write this sum as

$\Sigma_{i<j<k}\, d_{ijk}$

We know that there are 3! = 6 permutation of each term in the sum. Thus, we could write this as

$\Sigma_{i<j<k}\, d_{ijk} = 1/3!\ \Sigma_{ijk}\, d_{ijk}$

where we can regard the Σ_{ijk} as being over all sets of ijk that are different integers, *or* as being over all sets of ijk whether or not two indices are the same, since we know that $d_{ijk} = 0$ when two indices are the same.

(7) In sections 1-6 above, we focused on the determinants of 3x3 submatrices of a matrix A. In similar fashion, we could think about 2x2 submatrices. Here are all the above results cast into this form:

[1] $\det_{45} = \varepsilon^{123ij6} A_{4i}A_{5j} = A_{44}A_{55} - A_{45}A_{54}$

$\det_{54} = \varepsilon^{123ij6} A_{5j}A_{4i} = A_{55}A_{44} - A_{54}A_{45}$

[2] $\det_{45} = \det_{54}$ // only 2 = 2! terms

[3] $\det_{IJ} = \varepsilon^{ij} A_{Ii}A_{Jj}$ where ij $\in$ {I,J}, ε^{ij} = anti-symmetric, and $\varepsilon^{IJ} = +1$.

$= \varepsilon^{IJ} A_{II}A_{JJ} + \ldots = A_{II}A_{JJ} + \ldots$ // = $A_{II}A_{JJ} - A_{IJ}A_{JI}$ in this simple case

$\det_{13} = \varepsilon^{ij} A_{1i}A_{3j} = A_{11}A_{33} - A_{13}A_{31}$

$$\det_{ij} = \begin{vmatrix} Aii & Aij \\ Aji & Ajj \end{vmatrix}$$

[4] $\det_{44} = 0$ and in general $\det_{II} = 0$, two rows equal numbers

[5] $\det_{45} = 1/2 \ \Sigma_{IJ \in \{4,5\}} \det_{IJ}$ and 1/2 = 1/2!

[6] $\Sigma_{i<j} d_{ij} = 1/2! \ \Sigma_{ij} d_{ij}$

(8) Now let's do it all again for 4x4 submatrices:

[1] $\det_{3456} = \varepsilon^{12ijk\ell} A_{3i}A_{4j}A_{5k}A_{6\ell} = A_{33}A_{44}A_{55}A_{66}$ + 23 other terms
= det of 4 x 4 submatrix obtained by crossing out row/columns 1 and 2

$\det_{4356} = \varepsilon^{12ijk\ell} A_{4j}A_{3i}A_{5k}A_{6\ell} = A_{44}A_{33}A_{55}A_{66}$ + 23 other terms

[2] $\det_{3456}$ = 23 other permutations, all are equal, total permutations = 4*3*2*1 = 24 = 4!

[3] $\det_{IJKL} = \varepsilon^{ijk\ell} A_{Ii}A_{Jj}A_{Kk}A_{L\ell}$ where ijk$\ell \in$ {I,J,K.L}, $\varepsilon^{ijk\ell}$ = anti-symmetric, and $\varepsilon^{IJKL} = +1$.

$= \varepsilon^{IJKL} A_{II}A_{JJ}A_{KK}A_{LL} + ... = A_{II}A_{JJ}A_{KK}A_{LL} + ...$

$\det_{1256} = \varepsilon^{ijk\ell} A_{1i}A_{2j}A_{5k}A_{6\ell} = \varepsilon^{1256} A_{11}A_{22}A_{55}A_{66} + ... = A_{11}A_{22}A_{55}A_{66} + ...$

[4] $\det_{1156} = 0$, etc for any two indices the same, two rows of 4x4 have same numbers

[5] $\det_{3456} = 1/24 \ \Sigma_{IJKL \in \{3,4,5,6\}} \det_{IJKL}$ and 1/24 = 1/4!

[6] $\Sigma_{i<j<k<\ell} d_{ijk\ell} = 1/4! \ \Sigma_{ijk\ell} d_{ijk\ell}$

(9) Now we shall try to express our main results for general n x n submatrix determinants, now that we have studied the cases n = 3,2 and 4. We do this using the above notation which conceals the fact that we started with A = 6 x 6 matrix. These results apply to A = N x N matrix, as long as n ≤ N.

[2] $\det_{I1\ I2\ I3...In}$ has n! permutations, all of which are equal and which can be interpreted in terms of sets of row/column swaps. This thing is the determinant of an n x n submatrix of A where we have crossed out the rows and columns corresponding to the integers which do *not* appear in $_{I1\ I2\ I3...In}$.

[3] $\det_{I1\ I2\ I3...In} = \varepsilon^{i1\ i2\ i3....in} A_{I1\ i1} A_{I2\ i2} ..... A_{In\ in}$ // n factors of A

where i1 i2 i3....in $\in$ {I1,I2,I3...I_n} , $\varepsilon^{i1\ i2\ i3....in}$ anti-symmetric, $\varepsilon^{I1\ I2\ I3....In} = +1$

$det_{I1\ I2\ I3...In} = A_{I1,I1}\, A_{I2,I2}$ ..... $A_{In,In}$ + other terms found from above ε notation

[4] $det_{I1\ I2\ I3...In} = 0$ if any two (or more) indices are the same

[6] $\Sigma_{i1<i2<i3\ ....<in}\ det_{i1\ i2\ i3...in} = (1/n!)\ \Sigma_{i1,i2,i3...\ in}\ det_{i1\ i2\ i3...in}$

2. Application: computation of det(1+A)

(1) Consider the general determinant formula applied to a matrix 1+A in the case n = 4:

$det(1+A) = (1+A)_{1i}(1+A)_{2j}(1+A)_{3k}(1+A)_{4\ell}\ \varepsilon^{ijk\ell}$ // implied sum on all letter indices

I give some proofs of this general formula for a determinant in my notes, so let's accept it for now [We have been using the formula all along.] Write out the above as:

$(\delta_{1i} + A_{1i})\ (\delta_{2j} + A_{2j})\ (\delta_{3k} + A_{3k})\ (\delta_{4\ell} + A_{4\ell})\ \varepsilon^{ijk\ell}$

(2) If we take the left side of each factor we get just $\varepsilon^{1234} = 1$, so det(1+A) = 1 + other stuff.

(3) If we take the first three δ's and the last $A_{4\ell}$ we get $A_{4\ell}\ \varepsilon^{123\ell} = A_{44}$. There are three similar terms and when we add them we get tr(A). Thus, we know that det(1+A) = 1 + tr(A) + other stuff.

(4) Now let's select all terms which have *two* factors of A. There are (4,2) = 6 ways to do this. Each of these 6 ways has the first indices of the A factors in the "natural" increasing order. For example, if we take the last two A factors, the term is

$\delta_{1i}\ \delta_{2j}\ A_{3k}\ A_{4\ell}\ \varepsilon^{ijk\ell} = A_{3k}\ A_{4\ell}\ \varepsilon^{12k\ell} = A_{33}A_{44} - A_{34}A_{43}$

This is exactly the object det_{34} discussed in section (7) of our Preliminaries above. In the det(1+A) above we are obtaining all 6 terms det_{ij} where i,j are in natural increasing order. Thus, we can say

the two-A terms = $\Sigma_{i<j}\ det_{ij} = (1/2!)\ \Sigma_{ij}\ det_{ij}$

where on the right we use our result [6] to rewrite as a general sum including i = j values, where 2! is our overcount correction factor since each term has 2! permutations.

(5) Now let's select all terms which have *three* factors of A. There are (4,3) = 4 ways to do this. Each of these 4 ways has the first indices of the A factors in the "natural" increasing order. For example, if we take the last three A factors, the term is

$\delta_{1i}\ A_{2j}\ A_{3k}\ A_{4\ell}\ \varepsilon^{ijk\ell} = A_{2j}\ A_{3k}\ A_{4\ell}\ \varepsilon^{1jk\ell} = A_{22}A_{33}A_{44}$ + other terms

This is exactly the object $\det_{\mathbf{234}}$ discussed in our Preliminaries above. We get all the natural order terms, so we have:

the three-A terms = $\Sigma_{\mathbf{i<j<k}} \det_{\mathbf{ijk}} = (1/3!)\ \Sigma_{\mathbf{ijk}} \det_{\mathbf{ijk}}$

(6) We can see now the pattern. The next group will be

the four-A terms = $\Sigma_{\mathbf{i<j<k<\ell}} \det_{\mathbf{ijk\ell}} = (1/4!)\ \Sigma_{\mathbf{ijk\ell}} \det_{\mathbf{ijk\ell}}$

If we started with A = 4x4 matrix, then we can simplify this "last term" because there is only one 4x4 submatrix of A, so all 24 of the $\det_{\mathbf{ijk\ell}}$ are exactly the same and are just detA. Another way to say this is that $\Sigma_{\mathbf{i<j<k<\ell}} \det_{\mathbf{ijk\ell}}$ contains just one term which is $\det_{\mathbf{1234}} = \det(A)$.

(7) Here then is the general result for an NxN matrix A:

$\det(1+A) = 1 + \mathrm{tr}(A) + (1/2!)\ \Sigma_{\mathbf{ij}} \det_{\mathbf{ij}} + (1/3!)\ \Sigma_{\mathbf{ijk}} \det_{\mathbf{ijk}} + (1/4!)\ \Sigma_{\mathbf{ijk\ell}} \det_{\mathbf{ijk\ell}} + \ldots + \det(A)$

where

$\det_{\mathbf{I1\ I2\ I3\ldots In}} = \varepsilon^{\mathbf{i1\ i2\ i3\ldots in}} A_{\mathbf{I1\ i1}} A_{\mathbf{I2\ i2}} \ldots\ldots A_{\mathbf{In\ in}}$

so that in particular

$\det_{\mathbf{IJ}} = \varepsilon^{\mathbf{ij}} A_{\mathbf{Ii}} A_{\mathbf{Jj}}$ where ij $\in$ {I,J}, $\varepsilon^{\mathbf{ij}}$ = anti-symmetric, and $\varepsilon^{\mathbf{IJ}} = +1$.

$\det_{\mathbf{IJK}} = \varepsilon^{\mathbf{ijk}} A_{\mathbf{Ii}} A_{\mathbf{Jj}} A_{\mathbf{Kk}}$ where ijk $\in$ {I,J,K}, $\varepsilon^{\mathbf{ijk}}$ = anti-symmetric, and $\varepsilon^{\mathbf{IJK}} = +1$.

$\det_{\mathbf{IJKL}} = \varepsilon^{\mathbf{ijk\ell}} A_{\mathbf{Ii}} A_{\mathbf{Jj}} A_{\mathbf{Kk}} A_{\mathbf{L\ell}}$ where ijkℓ $\in$ {I,J,K.L}, $\varepsilon^{\mathbf{ijk\ell}}$ = anti-symmetric, and $\varepsilon^{\mathbf{IJKL}} = +1$.

etc

Notice what is appearing in the above expression:

tr(A) is the sum of all diagonal elements, or the sum of all 1x1 diagonal subdeterminants
$\Sigma_{\mathbf{ij}} \det_{\mathbf{ij}}$ is the sum of all 2x2 diagonal subdeterminants
$\Sigma_{\mathbf{ijk}} \det_{\mathbf{ijk}}$ is the sum of all 3x3 diagonal subdeterminants
and so on

The following more graphical formula states the same result:

$$\det(1+A) = 1 + \frac{1}{1!}\ \Sigma_{\mathbf{i}}\, A_{\mathbf{ii}} + \frac{1}{2!} \Sigma_{\mathbf{ij}} \begin{vmatrix} Aii & Aij \\ Aji & Ajj \end{vmatrix} + \frac{1}{3!} \Sigma_{\mathbf{ijk}} \begin{vmatrix} Aii & Aij & Aik \\ Aji & Ajj & Ajk \\ Aki & Akj & Akk \end{vmatrix} + \ldots + \det(A)$$

Another way to write our expansion would be this:

$$\det(1+A) = 1 + \mathrm{tr}(A) + \Sigma_{i<j} \det_{ij} + \Sigma_{i<j<k} \det_{ijk} + \Sigma_{i<j<k<\ell} \det_{ijk\ell} + ... + \det(A)$$

where now we don't do the overcounting.

(8) If A has elements all >> 1, then the last term is the dominant term and $\det(1+A) \approx \det(A)$.

(9) If A is a "small matrix" we might write it as $A \rightarrow \lambda A$. Then our theorem is:

$$\det(1+\lambda A) = 1 + \lambda \mathrm{tr}(A) + (\lambda^2/2)\, \Sigma_{ij} \det_{ij} + (\lambda^3/3!)\ \Sigma_{ijk} \det_{ijk} + (\lambda^4/4!)\ \Sigma_{ijk\ell} \det_{ijk\ell} + ... + \lambda^n \det(A)$$

where we now have a power series expansion for $\det(1+\lambda A)$ and we might keep just the first few terms in the expansion if we are doing some kind of approximation work. If we keep only the trace term, we are then making "the trace approximation".

(10) Suppose $A_{ij} = a_i b_j$, a matrix which is "factorizable". Then

$$\det_{I1\ I2\ I3...In} = \varepsilon^{i1\ i2\ i3....in} A_{I1\ i1} A_{I2\ i2} ..... A_{In\ in}$$

$$= \varepsilon^{i1\ i2\ i3....in} a_{I1}\ b_{i1}\, a_{I2}\, b_{i2} ..... a_{In}\ b_{in}$$

$$= a_{I1}\, a_{I2}.... a_{In} \ * \ \varepsilon^{i1\ i2\ i3....in}\, b_{i1}\, b_{i2}.... b_{in}$$

For $n \geq 2$, we have a contraction between an antisymmetric and a symmetric tensor, so det = 0.
Thus, our entire result boils down to this:

$$\det(1+\lambda A) = 1 + \lambda \mathrm{tr}(A)$$

where we can say

$$\mathrm{tr}(A) = \Sigma_i A_{ii} = \Sigma_i\, a_i b_i\ .$$

So in the factorizable case, the trace approximation is exact.

(11) There is yet another way to write our expansion. We know this general matrix theorem:

$\det(M) = e^{tr[\ ln(M)\ ]}$ which is really just $\det(e^A) = e^{tr(A)}$ with A = ln(M)

Now set M = 1+A. We then have

$$\det(1+A) = \exp[\ \mathrm{tr}(\ \{\ \ln(1+A)\}\ ]$$

But we know that

$$\ln(1+A) = A - A^2/2 + A^3/3 - ... = -\Sigma_{n=1}^{\infty} (-1)^n A^n/n$$

Then we can say

$$\mathrm{tr}(\{\ln(1+A)\} = -\mathrm{tr}\{\Sigma_{n=1}^{\infty}(-1)^n A^n/n\} = -\Sigma_{n=1}^{\infty}(-1)^n/n * \mathrm{tr}(A^n)$$

Then we have shown that

$$\det(1+A) = \exp[-\Sigma_{n=1}^{\infty}(-1)^n/n * \mathrm{tr}(A^n)]$$

or

$$\det(1-A) = \exp[-\Sigma_{n=1}^{\infty}\mathrm{tr}(A^n)/n]$$

or

$$\det(1-\lambda A) = \exp[-\Sigma_{n=1}^{\infty}\lambda^n\ \mathrm{tr}(A^n)/n]$$

which agrees with page 16 of my Mikhlin notes.

3. Application: The Fredholm Determinant

(1) Consider the eigenvalue problem: $Au = \mu u$ or $\lambda Au = u$ where $\lambda = 1/u$. We can write this last as:

$$(\lambda A-1)u = 0 \qquad \text{or} \qquad (1-\lambda A)u = 0.$$

This eigenvalue equation will have solutions u only if $\det(1-\lambda A) = 0$. But we have an expansion for this determinant from above

$$\det(1-\lambda A) = 1 - \lambda\mathrm{tr}(A) + (\lambda^2/2)\Sigma_{ij}\det_{ij} - (\lambda^3/3!)\Sigma_{ijk}\det_{ijk} + (\lambda^4/4!)\Sigma_{ijk\ell}\det_{ijk\ell} + \ldots + (-\lambda)^n\det(A)$$

where now we have a minus sign so we just take $\lambda \rightarrow -\lambda$ in our previous λ result.

(2) If A is a finite dimensional matrix of degree N, then we have written $\det(1-\lambda A)$ as a polynomial of degree N in λ. If A is symmetric, this is going to have N real roots, and these are going to be the eigenvalues of our eigenvalue equation.

(3) In general you would never write this polynomial using the formula above unless perhaps N was somewhat large and λ was small, so you would keep only the first several terms and in this way you would find the "dominant" eigenvalues.

(4) If we are dealing with an integral equation so A = K is an integral operator with kernel k(x,y), we can approximate the equation on an N x N lattice and then the secular equation $\det(1-\lambda K) = 0$ would give us an approximation to the some of the eigenvalues. I suspect these might be the smallest eigenvalues in λ and the largest in μ, but I am not sure, have to reread my Stak notes on this.

(5) We can also consider the above expansion in the limit that $n \rightarrow \infty$. The first two terms would be:

$$\det(1-\lambda K) = \det(1 - \lambda k_{x,y}) = 1 - \lambda\, tr(k) = 1 - \lambda \int dx\, k(x,x)$$

Notice that for $\lambda = 0$, you would say $\det(1) = 1$ = the product of an infinite number of 1's on the diagonal of an ∞ x ∞ matrix.

(6) What would the next term be?

$\det_{IJ} = \varepsilon^{ij} A_{Ii} A_{Jj}$ where $ij \in \{I,J\}$, ε^{ij} = anti-symmetric, and $\varepsilon^{IJ} = +1$.

$$= A_{II}A_{JJ} - A_{IJ}A_{JI}$$

$$+(\lambda^2/2) \Sigma_{ij} \det_{ij} = (\lambda^2/2) \int dx\, dy\, [k(x,x)\, k(y,y) - k(x,y)\, k(y,x)] \qquad (*)$$

$$= (\lambda^2/2) \int dx\, dy \begin{vmatrix} k(x,x) & k(x,y) \\ k(y,x) & k(y,y) \end{vmatrix}$$

and I think the higher terms generalize similarly. Of course in this case the series goes on forever, but for small λ or perhaps small λ $\|K\|$ we could keep the first few terms as an approximation, but this is a different kind of approximation from the lattice one described above. Again, we set $\det(1-\lambda K) = 0$ to find some eigenvalues λ.

(7) **Expressing Fredholm terms in terms of traces.**

Let's rewrite the above again for n = N:

$$(\lambda^2/2) \Sigma_{IJ} \det_{IJ} = (\lambda^2/2) \Sigma_{IJ} \{ \varepsilon^{ij} A_{Ii} A_{Jj} \} = (\lambda^2/2) \Sigma_{IJ} \{ A_{II}A_{JJ} - A_{IJ}A_{JI} \}$$

$$= (\lambda^2/2) \{ \Sigma_I A_{II} \Sigma_J A_{JJ} - \Sigma_{IJ} A_{IJ}A_{JI} \} = (\lambda^2/2) \{ [tr(A)]^2 - tr(A^2) \}$$

$$= (\lambda^2/2) \{ T_1^2 - T_2 \}$$

so this replicates our n=∞ result (*) above, where we define $T_n = tr(A^n)$.

Let's try the next term in terms of traces:

$$-(\lambda^3/3!) \Sigma_{IJK} \det_{IJK} = -(\lambda^3/3!) \Sigma_{IJK} \{ \varepsilon^{ijk} A_{Ii} A_{Jj} A_{Kk} \}$$

We have no choice at this point but to write out all the terms:

$$\varepsilon^{ijk} A_{Ii} A_{Jj} A_{Kk}$$

$$= A_{II}A_{JJ}A_{KK} - A_{IJ}A_{JI}A_{KK} + A_{IJ}A_{JK}A_{KI} - A_{IK}A_{JJ}A_{KI} + A_{IK}A_{JI}A_{KJ} - A_{II}A_{JK}A_{KJ}$$

where I exchange one pair for each term. Now we can look at our sum

$$\Sigma_{IJK}\{A_{II}A_{JJ}A_{KK} - A_{IJ}A_{JI}A_{KK} + A_{IJ}A_{JK}A_{KI} - A_{IK}A_{JJ}A_{KI} + A_{IK}A_{JI}A_{KJ} - A_{II}A_{JK}A_{KJ}\}$$

$$= [\mathrm{tr}(A)]^3 - \mathrm{tr}(A)\,\mathrm{tr}(A^2) + \mathrm{tr}(A^3) - \mathrm{tr}(A)\mathrm{tr}(A^2) + \mathrm{tr}(A^3) - \mathrm{tr}(A)\,\mathrm{tr}(A^2)$$

$$= [\mathrm{tr}(A)]^3 - 3\,\mathrm{tr}(A)\,\mathrm{tr}(A^2) + 2\,\mathrm{tr}(A^3)$$

So our term is then

$$-(\lambda^3/3!)\{[\mathrm{tr}(A)]^3 - 3\,\mathrm{tr}(A)\,\mathrm{tr}(A^2) + 2\,\mathrm{tr}(A^3)\}$$

$$= (\lambda^3/3!)\{(\int dx\, k(x,x))^3 - 3\int dx\, k(x,x)\int dy\, k_2(y,y) + 2\int dx\, k_3(x,x)\}$$

$$= (\lambda^3/3!)\{(\int dxdydz\, k(x,x)\,k(y,y)\,k(z,z)$$

$$- 3\int dxdydz\, k(x,x)\,k(x,y)\,k(y,z)$$

$$+ 2\int dxdydz\, k(x,y)\,k(y,z)\,k(z,x)$$

$$= (\lambda^3/3!)\int dxdydz\{k(x,x)\,k(y,y)\,k(z,z) - 3\,k(x,x)\,k(x,y)\,k(y,z) + 2\,k(x,y)\,k(y,z)\,k(z,x)\}$$

and of course this is going to have the expected form in terms of |...|. That is to say, start over and say

$$-(\lambda^3/3!)\,\Sigma_{ijk}\begin{vmatrix} Aii & Aij & Aik \\ Aji & Ajj & Ajk \\ Aki & Akj & Akk \end{vmatrix}$$

and then replace Σ_i with $\int dx$ and so on, so we get

$$-(\lambda^3/3!)\int dxdydz\begin{vmatrix} k(x,x) & k(x,y) & k(x,z) \\ k(y,x) & k(y,y) & k(y,z) \\ k(z,x) & k(z,y) & k(z,z) \end{vmatrix}$$

But ultimately, the *simplest* way to write this thing is:

$$-(\lambda^3/3!)\{[\mathrm{tr}(A)]^3 - 3\,\mathrm{tr}(A)\,\mathrm{tr}(A^2) + 2\,\mathrm{tr}(A^3)\}$$

$$- (\lambda^3/3!)\ \{ T_1^3 - 3\, T_1T_2 + 2T_3\}$$

where $T_n = tr(T^n) = tr(TT..T) = \int dx\, k_n(x,x)$.

(8) It is in this $n\to\infty$ situation that we refer to $det(1-\lambda K)$ as **The Fredholm Determinant**.

(9) **Compare results of (7) to the trace expansion (11) below.**

Let's now examine our alternate form given in (11) above,

$$det(1-\lambda A) = \exp\left[- \Sigma_{n=1}^{\infty} \lambda^n\ tr(A^n)/n\right]$$

For an NxN matrix A we can expand $e^{-x} = 1 - x + x^2/2! - x^3/3! + .. = \Sigma_{i=0}^{\infty} (-x)^i/i!$ Thus we get

$$det(1-\lambda A) = \exp\left[- \Sigma_{n=1}^{\infty} \lambda^n\ tr(A^n)/n\right] \qquad x = \Sigma_{n=1}^{\infty} \lambda^n\ tr(A^n)/n$$

$$= \Sigma_{i=0}^{\infty} (-\{\Sigma_{n=1}^{\infty} \lambda^n\ tr(A^n)/n\ \})^i/i!$$

$$= \Sigma_{i=0}^{\infty} (-1)^i/\ i!\ *\ \{\ \Sigma_{n=1}^{\infty} \lambda^n\ tr(A^n)/n\}^i$$

I can only keep the first few terms in the n sum to see what is going on here,

$$= \Sigma_{i=0}^{\infty} (-1)^i/\ i!\ *\ \{\lambda tr(A) + \lambda^2 tr(A^2)/2\ + \lambda^3 tr(A^3)/3 + ...\}^i$$

Let's define these traces such that $T_3 = tr(A^3)$ etc, so we then have

$$= \Sigma_{i=0}^{\infty} (-1)^i/\ i!\ *\ \{\lambda T_1 + \lambda^2 T_2/2\ + \lambda^3 T_3/3 + ...\}^i$$

$$= 1 - \{\lambda T_1 + \lambda^2 T_2/2\ + \lambda^3 T_3/3 + ...\}\ + \{\lambda T_1 + \lambda^2 T_2/2\ + \lambda^3 T_3/3 + ...\}^2/2!$$

$$- \{\lambda T_1 + \lambda^2 T_2/2\ + \lambda^3 T_3/3 + ...\}^3/3!\ + ...$$

We can now gather up terms of a given power of λ, where λ^0 term is just the 1 and the λ^1 term is $-\lambda T_1$. Then

λ^2: $\quad - \lambda^2 T_2/2\ + \lambda^2 T_1^2/2 = (\lambda^2/2)\ (\ T_1^2 - T_2)$ // agrees with above

λ^3: $\quad - \lambda^3 T_3/3 + \lambda^3 (T_1T_2)/2 - \lambda^3 T_1^3/3! = -(\lambda^3/3!)\ \{\ T_1^3 - 3T_1T_2 + 2\ T_3\ \}$

which also agrees with our result above. So we have shown directly through order λ^3 that our two expansions give the same result.

(10) So we can now **summarize** the above several results:

$$\det(1-\lambda K) = 1 - \lambda T_1 + (\lambda^2/2)\{T_1^2 - T_2\} - (\lambda^3/3!)\{T_1^3 - 3T_1T_2 + 2T_3\} + O(\lambda^4)$$

$$T_n = \mathrm{tr}(K^n) = \mathrm{tr}(KKK..K) = \int dx\, k_n(x,x)$$

$$= \int dx_1 dx_2 dx_3 .... dx_n \,.\, k(x_1,x_2)\, k(x_2,x_3)\, k(x_3,x_4) .... k(x_{n-1},x_n)$$

$$\det(1-\lambda K) = 1 - \lambda \int dx\, k(x,x) + (\lambda^2/2) \int dx\, dy \begin{vmatrix} k(x,x) & k(x,y) \\ k(y,x) & k(y,y) \end{vmatrix}$$

$$- (\lambda^3/3!) \int dx dy dz \begin{vmatrix} k(x,x) & k(x,y) & k(x,z) \\ k(y,x) & k(y,y) & k(y,z) \\ k(z,x) & k(z,y) & k(z,z) \end{vmatrix} + ... \text{ (goes on forever)}$$

$$\det(1-\lambda A) = 1 - \frac{\lambda}{1!} \Sigma_i A_{ii} + \frac{\lambda^2}{2!} \Sigma_{ij} \begin{vmatrix} Aii & Aij \\ Aji & Ajj \end{vmatrix} - \frac{\lambda^3}{3!} \Sigma_{ijk} \begin{vmatrix} Aii & Aij & Aik \\ Aji & Ajj & Ajk \\ Aki & Akj & Akk \end{vmatrix} + ... + (-\lambda)^n \det(A)$$

$$= 1 - \lambda T_1 + (\lambda^2/2)\{T_1^2 - T_2\} - (\lambda^3/3!)\{T_1^3 - 3T_1T_2 + 2T_3\} + O(\lambda^4)$$

where $T_n = \mathrm{tr}(A^n)$

$$\det(1-\lambda A) = \exp[-\Sigma_{n=1}^{\infty} \lambda^n \mathrm{tr}(A^n)/n] = e^{\mathrm{tr}[\ln(1-\lambda A)]}$$

$$= \Sigma_{i=0}^{\infty} (-1)^i / i! * \{\Sigma_{n=1}^{\infty} \lambda^n \mathrm{tr}(A^n)/n\}^i$$

(11) It is not easy to find these results on the web, but I have found them in a google scanned book which is this one:

http://books.google.com/books?id=WHjO9K6xEm4C

and the result is presented in this way:

$$K\begin{pmatrix} x_1, \ldots, x_n \\ y_1, \ldots, y_n \end{pmatrix} = \begin{vmatrix} K(x_1, y_1) & \ldots & K(x_1, y_n) \\ & \ldots & \\ K(x_n, y_1) & \ldots & K(x_n, y_n) \end{vmatrix}.$$

Define $D(\lambda)$ and $D(x, y; \lambda)$ by

$$D(\lambda) =$$

$$1 + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_D \ldots \int_D K\begin{pmatrix} s_1, \ldots, s_n \\ s_1, \ldots, s_n \end{pmatrix} ds_1 \ldots ds_n$$

and

$$D(x, y; \lambda) = K(x, y)$$

$$+ \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_D \ldots \int_D K\begin{pmatrix} x, s_1, \ldots, s_n \\ y, s_1, \ldots, s_n \end{pmatrix} ds_1 \ldots ds_n.$$

where my result is the first object above so that det(1-λK) = D(λ). The other object will be commented on soon.

4. Preliminary Facts concerning the Determinants of Certain Other Submatrices of a Matrix.

(1) In our above preliminary notes, we learned how to write formulas for submatrices of a matrix which were centered on the diagonal of the main matrix, for example

$\det_{456} = \varepsilon^{123ijk} A_{4i} A_{5j} A_{6k} = A_{44} A_{55} A_{66}$ + other terms

$\det_{456} = \varepsilon^{ijk} A_{4i} A_{5j} A_{6k}$ where ijk ∈ {4,5,6}, ε^{ijk} = anti-symmetric, and $\varepsilon^{456} = +1$.

These were the only "objects" we needed to develop the results for the Fredholm Determinant.

(2) But **how would we write a formula for an *arbitrary* submatrix of a matrix?**

Theorem: Start with some NxN matrix A. We are interested in the determinant of an arbitrary nxn submatrix where n ≤ M. Let the set of n increasing integers IJK.... label the rows of the full matrix which are part of the submatrix. And let the set of n increasing numbers abc... label the columns of the full matrix which are part of the submatrix. Then the subdeterminant is given by:

$\det_{IJK..\,;\,abc...} = \varepsilon^{ijk\cdots} A_{Ii} A_{Jj} A_{Kk} \ldots$ where ijk... ∈ {abc...} and $\varepsilon^{abc\cdots} = 1$.

Using the transpose determinant formula, we could also write this as

$\det_{IJK..\,;\,abc...} = \varepsilon^{ijk\cdots} A_{ia} A_{jb} A_{kc} \ldots$ where ijk... ∈ {IJK...} and $\varepsilon^{IJK\cdots} = 1$.

Proof: The only real issue here is whether we have the sign right. We know that we are correctly accessing the right rows and columns. We imagine that we "assemble" the n^2 numbers of interest into a contiguous n x n matrix without changing ordering of columns or rows, and it is the determinant of that contiguous "ancillary" matrix that we are interested in. By requiring that our two sets of indices be in "natural order", we make sure that no such shuffles of columns or rows occur. The first term in our result is always going to be this, with a plus sign:

$$+ A_{Ia}A_{Jb}A_{Kc} \ldots$$

so I think our theorem is basically proved.

(3) Let's try a fancier way to write the first result above:

$$\det_{I1,I2,I3..In\ ;\ a1,a2,a3...an} = \varepsilon^{i1\ i2..in} A_{I1,i1}A_{I2,i2}A_{I3,i3} \ldots A_{In,in}$$

where i1,i2,...in... $\in$ { a1,a2,a3...an} and $\varepsilon^{a1\ a2\ a3..an} = 1$.

Thus we can say this:

$$\det_{I1,I2,I3..In\ ;\ a1,a2,a3...an} = A_{I1,a1}A_{I2,a2}A_{I3,a3} \ldots A_{In,an}$$

+ signed permutations of pairs of second indices

and we could write

$$A_{I1,i1}A_{I2,i2}A_{I3,i3} \ldots A_{In,in} = \prod_{s=1}^{n} A_{Is,is}$$

The full NxN determinant would be written:

$$|A| = \det_{1,2,3..N\ ;\ 1,2,3..N} =$$

$$\Sigma_{i1,i2..iN}\ \varepsilon^{i1\ i2..iN} A_{1,i1}A_{2,i2}A_{3,i3} \ldots A_{N,iN}$$

$$= \Sigma_{i1,i2..iN\ \in\ \{1,2,3...N\}}\ \varepsilon^{i1\ i2..iN} \prod_{s=1}^{N} A_{s,is}$$

Here there are N indices on the ε tensor, and there are N factors of A.

(4) Let's try to make hay with this non-ε version of our formula:

$$\det_{I1,I2,I3..In\ ;\ a1,a2,a3...an} = A_{I1,a1}A_{I2,a2}A_{I3,a3} \ldots A_{In,an}$$

+ signed permutations of pairs of second indices

and we could write

$$A_{I1,a1}A_{I2,a2}A_{I3,a3} \ldots A_{In,an} = \prod_{s=1}^{n} A_{Is,as}$$

Then we have for our arbitrary nxn subdeterminant of an NxN matrix A:

$$\det_{I1,I2,I3..In\ ;\ a1,a2,a3...an} = \prod_{s=1}^{n} A_{Is,as} \quad + \text{signed permutations}$$

Now let's knock out row p and column q and have n = N-1 as a special case. Let's just start with an example where N = 4 and we knock out p = 1 and q = 3. Then the above reads

$$\det_{234;124} = A_{21}A_{32}A_{44} + \text{permutations} = \det_{\overline{1};\overline{3}}$$

The full 4x4 determinant would be this

$$\det_{1234;1234} = A_{11}A_{22}A_{33}A_{44} + \text{permutations}$$

So here is my basic problem: I don't know a good algebraic notational way to express the RHS of this equation:

$$\det_{\overline{1};\overline{3}} = A_{21}A_{32}A_{44} + \text{perms}$$

How do I write the RHS in terms of the numbers "1" and "3" ?

5. Figuring out the B^0 and B^1 terms of the Fredholm First Minor expansion

Let's go with the case N=5. Let's take as our general starting point this:

$$\det_{I1,I2,I3..In\ ;\ a1,a2,a3...an} = A_{I1,a1}A_{I2,a2}A_{I3,a3} \ldots A_{In,an} + \text{perms}$$

where we are dealing with an n x n submatrix. I am still interested in the subdeterminant you get when you cross out row,column = p,q. So setting N = 5, that will be a 4x4 subdeterminant.

Write

$$\det_{I1,I2,I3,I4\ ;\ a1,a2,a3,a4} = A_{I1,a1}A_{I2,a2}A_{I3,a3}A_{I4,a4} + \text{perms}$$

where I1,I2,I3,I4 is the set of integers 1,2,3,4,5 with one integer "p" knocked out.
where a1,a2,a3,a4 is the set of integers 1,2,3,4,5 with one integer "q" knocked out.

Now right away, let's go to our A = 1+B form. We avoid the extra baggage of λ and minus signs for the moment, later we can replace K → -λK. So the above becomes

$$\det_{I1,I2,I3,I4\ ;\ a1,a2,a3,a4} =$$

$$(\delta_{I1,a1} + B_{I1,a1})\,(\delta_{I2,a2} + B_{I2,a2})\,(\delta_{I3,a3} + B_{I3,a3})\,(\delta_{I4,a4} + B_{I4,a4}) \quad + \text{perms}$$

Some comments in passing: (1) this has the same number of factors as my N=4 fredholm det situation, that is why I chose N=5 to start with; (2) using the +perms avoids the problem of figuring out what the "sign" is. (3) Remember that in the first term shown above, both index sets are in "natural order". (4) the notation + perms means that you permute the a_i index set and add signs as appropriate. But in all of these terms the I_i indices will remain in natural order.

• Let's study the B^0 terms. We write:

$$(\delta_{I1,a1})\,(\delta_{I2,a2})\,(\delta_{I3,a3})\,(\delta_{I4,a4}) \quad + \text{perms } a_i$$

If $p \neq q$, then the two integer sets I_i and a_i are *different*. The first term shown here must be 0 because there is no way that we can make every $\delta = 1$. And this is true for all the perms as well. If you have two sets of integers like boys = 1234 and girls = 2345 there is no way you can pair the boys and girls into four couples such that all four couples have the same number labels. You could do 22,33,44 but then you are left with 15. That is the best you could do in this little example, and of course $\delta_{15} = 0$ so the product of the four deltas = 0.

If p = q, then the two sets are the same, and there will be one arrangement of couples that gives each $\delta = 1$. For example, if boys = 2345 and girls = 2345 then the pairs would be 22,33,44,55 = 1. Notice that this is in fact the term shown above, not one of the permutations, since 2345 are in natural order for both boys and girls, so we don't have to worry about any permutation sign here.

Therefore we find this very nice and simple result

$$(\delta_{I1,a1})\,(\delta_{I2,a2})\,(\delta_{I3,a3})\,(\delta_{I4,a4}) \quad + \text{perms } a_i \quad = \delta_{p,q}$$

I know this is going to be the 1 matrix of $1 + \Gamma$, somehow.

• Let's study the B^1 terms. We start again with this:

$$(\delta_{I1,a1} + B_{I1,a1})\,(\delta_{I2,a2} + B_{I2,a2})\,(\delta_{I3,a3} + B_{I3,a3})\,(\delta_{I4,a4} + B_{I4,a4}) \quad + \text{perms}$$

There are four terms (ignoring perms). The *one* of these terms with the last B is this:

$$(\delta_{I1,a1})\,(\delta_{I2,a2})\,(\delta_{I3,a3})\,(B_{I4,a4}) \quad + \text{perms } a_i$$

but let's think of all four terms plus their permutations in the following discussion.

If $p \neq q$, reconsider (p=5) boys = 1234 and (q=1) girls = 2345 . In all our terms and their permutations, there is only one surviving term here, the one where the three δ's are 22,33,44. In this example, the B

would be $B_{1,5}$ because that is the left-over pairing. Remember that the first index has to be a boy and the second a girl. That is why it is $B_{1,5}$ and not the other way around. This term would not be in the term shown above, but in the term where the B is all the way to the left. It seems pretty clear that the general result here is this: if $p \neq q$, only one term survives and it is $B_{q,p}$.

But now we have a permutation sign to think about. Our example is really $B_{1,5}\,\delta_{2,2}\,\delta_{3,3}\,\delta_{4,4}$. We put the B first because I1 = 1 and that is where it goes. The I_i indices are in natural order 1234, but the a_i are not! How many pair swaps does it take to get from the order 5234 to order 2345? Three. But how would we express this sign in terms of integers p and q? That is the big problem! We can restate this problem in terms of (p=5) boys = 1234 and (q=1) girls = 2345. The question is: how many swaps to move the 1 of the boys group to the position of the 5 in the girls group. At this point in my writeup, we need to insert a "subroutine". I will make a section below to solve this problem for general p and q, and then I will report back that result here. I do this because that subroutine is probably going to be large.

Result: The sign change produced by the pairwise swaps required to move the q of the boys group to the p position in the girls group = $(-1)^{p+q+1}$.

If p = q, reconsider boys = 2345 and girls = 2345 with p = q = 1. Now there are four different terms contributing, not just one. First, we get 22,33,44 and B_{55}. Second we get 22,33,B_{44},55. Of course permutations like 22,33,45,B_{54} = 0, so lots of the perms are 0. The answer in this case is pretty clear. We get $B_{22} + B_{33} + B_{44} + B_{55}$ which we could write as $\Sigma_{i\neq p}\, B_{ii}$ = tr(B) - B_{pp}. In all four terms, both index sets are in normal order, so there is no "permutation sign" to worry about.

So here is our result for the B^1 terms:

$$B^1 \text{ terms} = \{B_{q,p}\, \text{sign}_{p,q}\}(1 - \delta_{p,q}) + \{\text{tr}(B) - B_{pp}\}\delta_{p,q}$$

$$B^1 \text{ terms} = \{B_{q,p}\,(-1)^{p+q+1}\}(1 - \delta_{p,q}) + \{\text{tr}(B) - B_{pp}\}\delta_{p,q}$$

$$= B_{q,p}\,(-1)^{p+q+1} + \delta_{p,q}\,\{\text{tr}(B) - B_{pp} - B_{p,p}\,(-1)^{p+p+1}\}$$

$$= B_{q,p}\,(-1)^{p+q+1} + \delta_{p,q}\,\{\text{tr}(B) - B_{pp} - B_{p,p}\,(-1)\}$$

$$= B_{q,p}\,(-1)^{p+q+1} + \delta_{p,q}\,\{\text{tr}(B)\}$$

$$= \text{tr}(B)\,\delta_{p,q} - (-1)^{p+q}\,B_{q,p}$$

6. Show that these B^0 and B^1 terms agree with our full Fredholm First Minor Result

From our separate document with λ = 1 and B = -K, we got this result:

$$\Gamma = N/\,|1\text{-}K|$$

$$N_{ij} = K_{ij} - \Sigma_k \begin{vmatrix} Kij & Kik \\ Kkj & Kkk \end{vmatrix} + (1/2!)\,\Sigma_{km} \begin{vmatrix} Kij & Kik & Kim \\ Kkj & Kkk & Kkm \\ Kmj & Kmk & Kmm \end{vmatrix} - (1/3!)\,\Sigma_{kmn} \begin{vmatrix} Kij & Kik & Kim & Kin \\ Kkj & Kkk & Kkm & Kkn \\ Kmj & Kmk & Kmm & Kmn \\ Knj & Knk & Knm & Knn \end{vmatrix}$$

But we also know this fact: (where I set $\lambda = 1$)

$1 + \Gamma = (1-K)^{-1} = \text{cof}^{T}(1-K)/\det(1-K)$ // just from the usual A^{-1} formula

We could write this in finite matrix notation as follows:

$\delta_{ij} + \Gamma_{ij} = \text{cof}\,[(1-K)_{ji}] / \det(1-K)$

If we recall these definitions from matrix theory

$\text{cof}(a_{pq}) = (-1)^{p+q}\,\text{minor}(a_{pq})$. and

$\text{minor}(a_{pq})$ = det of the N-1 x N-1 matrix you get by crossing out row p and column q.

we could write that

$\delta_{ij} + \Gamma_{ij} = (-1)^{j+i}\,\text{minor}\,[(1-K)_{ji}] / \det(1-K)$

Let's now set K = -B and ij = pq to get

$\delta_{p,q} + \Gamma_{pq} = (-1)^{p+q}\,\text{minor}\,[(1+B)_{qp}] / \det(1+B)$

which rewrite again as

$\det(1+B)\delta_{p,q} + \det(1+B)\Gamma_{pq} = (-1)^{p+q}\,\text{minor}\,[(1+B)_{qp}]$ (*)

But from above we have

$\Gamma = N/|1-K| = N/|1+B|$ => $\det(1+B)\,\Gamma = N$ => $\det(1+B)\,\Gamma_{pq} = N_{pq}$

Then (*) becomes

$\det(1+B)\delta_{p,q} + N_{pq} = (-1)^{p+q}\,\text{minor}\,[(1+B)_{qp}]$

=> $N_{pq} = -\det(1+B)\delta_{p,q} + (-1)^{p+q}\,\text{minor}\,[(1+B)_{qp}]$

Now we are in a position to *check* our first "minor" determinant result obtained above. If we keep only terms through B^1 , we can replace $\det(1+B) = 1 + \text{tr}(B)$. Then we have

$N_{pq} = -[\,1 + \text{tr}(B)\,]\,\delta_{p,q} + (-1)^{p+q}\,\text{minor}\,[(1+B)_{qp}]$ // through order B^1

In our work (so far) above, we found that:

$\text{minor}\,[(1+B)_{pq}] = B^0$ piece + B^1 piece

$$= \delta_{p,q} + \text{tr}(B)\,\delta_{p,q} - (-1)^{p+q} B_{q,p} \qquad \text{// through order } B^1$$

This then suggests that

$$N_{pq} = -[\,1 + \text{tr}(B)\,]\,\delta_{p,q} + (-1)^{p+q}\ \text{minor}\,[(1+B)_{qp}]$$

$$= -[\,1 + \text{tr}(B)\,]\,\delta_{p,q} + (-1)^{p+q}\,\{\,\delta_{p,q} + \text{tr}(B)\,\delta_{p,q} - (-1)^{p+q} B_{p,q}\,\}$$

$$= -B_{p,q} + \delta_{p,q}[\,-1 - \text{tr}(B) + 1 + \text{tr}(B)\,]$$

$$-B_{p,q}$$

This agrees (!) with our formula above which says $N_{ij} = K_{ij}$ + higher orders. So yes, we are on the right track! (But it is a long track).

7. Consider the B^2 terms, but then stop this approach for now.

• Let's study the B^2 terms. We start again with this:

$$(\delta_{I1,a1} + B_{I1,a1})\,(\delta_{I2,a2} + B_{I2,a2})\,(\delta_{I3,a3} + B_{I3,a3})\,(\delta_{I4,a4} + B_{I4,a4}) \quad + \text{perms}$$

and there are (4,2) = 12 different contributing terms, each with its permutations. A typical term looks like this:

$$(\delta_{I1,a1})\,(\delta_{I2,a2})\,(B_{I3,a3})\,(B_{I4,a4})$$

If $p \neq q$, reconsider (p=5) boys = 1234 and (q=1) girls = 2345. We have various contributors now:

$B_{14}\delta_{22}\delta_{33}B_{45}$ $\qquad$ $B_{15}\delta_{22}\delta_{33}B_{44}$

$B_{12}B_{25}\delta_{33}\delta_{44}$ $\qquad$ $B_{15}B_{22}\delta_{33}\delta_{44}$

Once again, we have the painful question of getting the sign of each term right. I think the first of the four shown here needs 2 swaps to get the 4 of B_{14} to the right of the two δ's. Maybe rule is $(-1)^{p+q}$. I suspect this is the rule for even number of δ's, and the previous rule I got is for odd number of deltas, but maybe some work to show that to be true. In any event, you can see the 2x2 determinants of B starting to form here, which show up in the N formula, but beyond that the details are pretty hairy.

OK, I agree to stop now on this line of development. This is really not a very good method of proof because you always have det(1+B) as part of the result, and things are all mushed together. I think my original proof in the Fredholm minor paper is the better path, but at least here I have shown that it can *in theory* be done using my formulas for subdeterminants with one row and column crossed out

8.Dealing with the "permutation sign problem" at the B^1 level.

This is the **subroutine referred to from above**.

"(p=5) boys = 1234 and (q=1) girls = 2345. The question is: how many swaps to move the 1 of the boys group to the position of the 5 in the girls group."

Let's start looking at lots of cases:

		boys	girls		swaps
p=1	q=2	2345	1345	move the 2 of boys to 1 position of girls	0
p=1	q=3	2345	1234	move the 3 of boys to 1 position of girls	1 left
p=1	q=4	2345	1235	move the 4 of boys to 1 position of girls	2 left
p=1	q=5	2345	1234	move the 5 of boys to 1 position of girls	3 left

Let's try a situation with larger N

p = 3	q = 6	12456789	12345789 move the 6 of boys to 3 position of girls	2 left
p = 3	q = 7	12456789	12345689 move the 7 of boys to 3 position of girls	3 left

It seems pretty clear now that the number of required swaps is related to the distance q-p-1. A formula that works for this last set of examples would be # swaps = q-p-1 so sign = $(-1)^{q-p-1}$. This also works for the first set of examples. If I repeated all these examples with p↔q, we just switch the boys and girls groups, for example

p = 6	q = 3	12345789	12456789move the 3 of boys to 6 position of girls	2 right
p = 7	q = 3	12345689	12456789move the 3 of boys to 7 position of girls	3 right

The only change is in the direction of the shifts. Here we shift p-q-1 to the right, sign $(-1)^{p-q-1}$. But the original formula still works : $(-1)^{q-p-1} = (-1)^{q-p-1+2p-2q} = (-1)^{p-q-1}$. We can also express this as $(-1)^{p-q-1} = (-1)^{p-q-1+2q} = -(-1)^{p+q}$ which I like, anticipating the "cofactor" work we are about to do.

Theorem:

The sign change produced by the pairwise swaps required to move the q of the boys group to the p position in the girls group = $-(-1)^{p+q}$.

9. Consider the B^2 terms one more time

First, let's look again at our general formula for a subdeterminant (from the Overview above)

"It turns out that the formula for a general subdeterminant can be written in a form very similar to that shown above for a diagonal one. Here is the general result: "

$$\det_{I1\ I2\ I3\ldots In;\ a1\ a2\ a3\ldots an;} = \varepsilon^{i1\ i2\ i3\ldots in} A_{I1\ i1} A_{I2\ i2} \ldots\ldots A_{In\ in} \qquad \text{// n factors of A}$$

where i1 i2 i3....in ∈ {a1 a2 a3...an} , $\varepsilon^{i1\ i2\ i3\ldots in}$ anti-symmetric, $\varepsilon^{I1\ I2\ I3\ldots In} = +1$

For our case with N=5 we write our minor-sized general subdeterminant,

$$\det_{I1,I2,I3,I4;\ a1,a2,a3,a4;} = \Sigma_{i\ =\{a1,a2,a3,a4\}} \varepsilon^{i1\ i2\ i3\ i4} A_{I1,i1} A_{I2,i2} A_{I1,i1} A_{I2,i2}$$

$$= \Sigma_{i1\ =\{a1,a2,a3,a4\}} \Sigma_{i2\ =\{a1,a2,a3,a4\}} \Sigma_{i3\ =\{a1,a2,a3,a4\}} \Sigma_{i4\ =\{a1,a2,a3,a4\}}$$

$$\varepsilon^{i1\ i2\ i3\ i4}\ (\delta_{I1,i1} + B_{I1,i1})\ (\delta_{I2,i2} + B_{I2,i2})\ (\delta_{I3,i3} + B_{I3,i3})\ (\delta_{I4,i4} + B_{I4,i4})$$

$$\begin{aligned}
=\ & \\
& \Sigma_{i1} \Sigma_{i2} \Sigma_{i3} \Sigma_{i4}\ \ \varepsilon^{i1\ i2\ i3\ i4}\ B_{I1,i1}\ B_{I2,i2}\ \delta_{I3,i3}\ \delta_{I4,i4} \\
+\ & \Sigma_{i1} \Sigma_{i2} \Sigma_{i3} \Sigma_{i4}\ \ \varepsilon^{i1\ i2\ i3\ i4}\ B_{I1,i1}\ \delta_{I2,i2}\ B_{I3,i3}\ \delta_{I4,i4} \\
+\ & \Sigma_{i1} \Sigma_{i2} \Sigma_{i3} \Sigma_{i4}\ \ \varepsilon^{i1\ i2\ i3\ i4}\ B_{I1,i1}\ \delta_{I2,i2}\ \delta_{I3,i3}\ B_{I4,i4} \\
& \\
+\ & \Sigma_{i1} \Sigma_{i2} \Sigma_{i3} \Sigma_{i4}\ \ \varepsilon^{i1\ i2\ i3\ i4}\ \delta_{I1,i1}\ B_{I2,i2}\ B_{I3,i3}\ \delta_{I4,i4} \\
+\ & \Sigma_{i1} \Sigma_{i2} \Sigma_{i3} \Sigma_{i4}\ \ \varepsilon^{i1\ i2\ i3\ i4}\ \delta_{I1,i1}\ B_{I2,i2}\ \delta_{I3,i3}\ B_{I4,i4} \\
& \\
+\ & \Sigma_{i1} \Sigma_{i2} \Sigma_{i3} \Sigma_{i4}\ \ \varepsilon^{i1\ i2\ i3\ i4}\ \delta_{I1,i1}\ \delta_{I2,i2}\ B_{I3,i3}\ B_{I4,i4}
\end{aligned}$$

I cannot actually DO any of the sums because we don't know whether or not we get hits with the deltas. But I could relabel the δ associated ε indices replacing i with I as follows:

$$\begin{aligned}
=\ & \Sigma_{i1} \Sigma_{i2} \Sigma_{i3} \Sigma_{i4}\ \ \varepsilon^{i1\ i2\ I3\ I4}\ B_{I1,i1}\ B_{I2,i2}\ \delta_{I3,i3}\ \delta_{I4,i4} \\
+\ & \Sigma_{i1} \Sigma_{i2} \Sigma_{i3} \Sigma_{i4}\ \ \varepsilon^{i1\ I2\ i3\ I4}\ B_{I1,i1}\ \delta_{I2,i2}\ B_{I3,i3}\ \delta_{I4,i4} \\
+\ & \Sigma_{i1} \Sigma_{i2} \Sigma_{i3} \Sigma_{i4}\ \ \varepsilon^{i1\ I2\ I3\ i4}\ B_{I1,i1}\ \delta_{I2,i2}\ \delta_{I3,i3}\ B_{I4,i4} \\
& \\
+\ & \Sigma_{i1} \Sigma_{i2} \Sigma_{i3} \Sigma_{i4}\ \ \varepsilon^{I1\ i2\ I3\ i4}\ \delta_{I1,i1}\ B_{I2,i2}\ B_{I3,i3}\ \delta_{I4,i4} \\
+\ & \Sigma_{i1} \Sigma_{i2} \Sigma_{i3} \Sigma_{i4}\ \ \varepsilon^{I1\ i2\ i3\ I4}\ \delta_{I1,i1}\ B_{I2,i2}\ \delta_{I3,i3}\ B_{I4,i4} \\
& \\
+\ & \Sigma_{i1} \Sigma_{i2} \Sigma_{i3} \Sigma_{i4}\ \ \varepsilon^{i1\ i2\ I3\ I4}\ \delta_{I1,i1}\ \delta_{I2,i2}\ B_{I3,i3}\ B_{I4,i4}
\end{aligned}$$

I could then move certain summations to their deltas:

$$\begin{aligned}
=\ & \Sigma_{i1} \Sigma_{i2}\ \ \varepsilon^{i1\ i2\ I3\ I4}\ B_{I1,i1}\ B_{I2,i2}\ (\Sigma_{i3}\delta_{I3,i3})\ (\Sigma_{i4}\delta_{I4,i4}) \\
+\ & \Sigma_{i1} \Sigma_{i3}\ \ \varepsilon^{i1\ I2\ i3\ I4}\ B_{I1,i1}\ (\Sigma_{i2}\ \delta_{I2,i2})\ B_{I3,i3}\ (\Sigma_{i4}\delta_{I4,i4}) \\
+\ & \Sigma_{i1} \Sigma_{i4}\ \ \varepsilon^{i1\ I2\ I3\ i4}\ B_{I1,i1}\ (\Sigma_{i2}\delta_{I2,i2})\ (\Sigma_{i3}\delta_{I3,i3})\ B_{I4,i4} \\
& \\
+\ & \Sigma_{i2} \Sigma_{i3}\ \ \varepsilon^{I1\ i2\ I3\ i4}\ (\Sigma_{i1}\delta_{I1,i1})\ B_{I2,i2}\ B_{I3,i3}\ (\Sigma_{i4}\delta_{I4,i4}\ ) \\
+\ & \Sigma_{i2} \Sigma_{i4}\ \ \varepsilon^{I1\ i2\ i3\ I4}\ (\Sigma_{i1}\delta_{I1,i1})\ B_{I2,i2}\ (\Sigma_{i3}\delta_{I3,i3})\ B_{I4,i4} \\
& \\
+\ & \Sigma_{i3} \Sigma_{i4}\ \ \varepsilon^{i1\ i2\ I3\ I4}\ (\Sigma_{i1}\delta_{I1,i1})\ (\Sigma_{i2}\delta_{I2,i2})\ B_{I3,i3}\ B_{I4,i4}
\end{aligned}$$

where I am trying to maintain the I1 I2 I3 I4 order of things.

Idea: let's make a simple name for each index set:

R = I1,I2,I3,I4

C = a1,a2,a3,a4;

Then $(\Sigma_{i1}\delta_{I1,i1}) = (I_1 \cap C)$

OK, enough for this little shot, I am not succeeding very well.