Fredholm Determinant, Round 2 PhL 3.28.09

Before reading this document, one should read "Subdeterminants of a matrix.doc" which explains details about the notations used here.

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Motivation. I wanted to learn a little about this "Fredholm stuff" because it was omitted from Stakgold's integral equations chapter. When the Fredholm integral equations are put on a lattice, they are reduced to N x N matrix problems. One of these equations is "the eigenvalue problem" and the eigenvalues can be determined from a certain determinant (The Fredholm Determinant), and the general inhomogeneous equation can then be solved by inverting a certain matrix which as usual can be done using cofactors which invoke minors (Fredholm's First Minor). One can do this for any N one wants. In the case that the kernel K of the integral equation set is "small" so it is written λK, it is possible to find expansions in powers of λ for both the Fredholm Determinant and the Fredholm First Minor, and it is really these expansions that have the official Fredholm names. If one then takes the limit N→∞ to move back off the lattice, these expansions are still well defined, but each has an infinite number of terms. Again, if λ is small, one usually deals with just the first few terms of these N=∞ expansions. The present Round 2 document deals with the Fredholm Determinant, the next Round 2 document with the Minor.

1. Preliminaries from Integral Equation Theory

Our Neumann form integral equation is this

$$\varphi = f + \lambda K \varphi$$

and the solution can be expressed in this manner:

$$\varphi = (1 - \lambda K)^{-1} f. = (1 + \lambda K + \lambda^2 K^2 + ...)\ f = f + \Sigma_{m=1} \lambda^m K^m f = (1 + \Gamma)\ f$$

All we did was use the high school algebra formula for 1/(1-x) as a series, and then defined this object

$$\Gamma = \Sigma_{m=1} \lambda^m K^m \quad = \textbf{"the resolvent"}$$

The first few terms here are $\Gamma \approx \lambda K + \lambda^2 K^2$ so if $\lambda << 1$, we have $\Gamma \approx \lambda K$ so Γ is of order λ in general.

In all this, I like to think of K as a finite dimensional matrix and take the ∞ limit later on. So the object K^m is just the product of K matrices multiplied together. Looking at the above, we of course have

$$\varphi = f + \Gamma f$$

$$(1 - \lambda K)^{-1} = 1 + \Gamma$$

from which last we quickly conclude that

$$\Gamma = \lambda K + \lambda K \Gamma = \lambda K + \lambda \Gamma K$$

We also know the following fact from very elementary matrix theory,

$$(1 - \lambda K)^{-1} = [\ \mathrm{cof}\ (1-\lambda K)\ ]^T / \det(\ 1 - \lambda K) = 1 + \Gamma$$

1A. Preliminaries Restated in terms of Γ' = Γ/λ.

Other people like Mikhlin use a different Γ which I will call Γ':

$$\Gamma' = [\Gamma/\lambda\ ] = \Sigma_{m=1} \lambda^{m-1} K^m$$

I in fact like this because then $\Gamma' \sim \lambda^0$, so this Γ' is more the "natural Γ" to use.

We can rewrite various equations above using Γ' :

$$\varphi = f + \lambda K \varphi$$

$$\varphi = (1 - \lambda K)^{-1} f. = (1 + \lambda K + \lambda^2 K^2 + ...)\ f = f + \Sigma_{m=1} \lambda^m K^m f = (1 + \lambda \Gamma')\ f$$

$$\Gamma' = \Sigma_{m=1} \lambda^{m-1} K^m$$

$$\varphi = f + \lambda \Gamma' f \qquad (1 - \lambda K)^{-1} = 1 + \lambda \Gamma' \qquad \Gamma' = K + \lambda K \Gamma' = K + \lambda \Gamma' K$$

Here are three little theorems that appear in **my Mikhlin notes** which I will now prove:

Theorem 1: $\mathrm{tr}(\Gamma') = \Sigma_{m=1}^{\infty} \lambda^{m-1} T_m$ where $T_m = \mathrm{tr}(K^m)$

or $\int dx\ \Gamma'(x,x;\lambda) = \Sigma_{m=1}^{\infty} \lambda^{m-1} \int dx\ k_m(x,x;\lambda)$ // continuum notation

Proof: $\Gamma'_{ii} = \Sigma_{m=1} \lambda^{m-1} (K^m)_{ii}$ => $\mathrm{tr}(\Gamma') = \Sigma_{m=1} \lambda^{m-1}\ T_m$ QED

In terms of Γ we would have: $\mathrm{tr}(\Gamma) = \Sigma_{m=1} \lambda^m\ T_m$

Theorem 2: $\Gamma'^2 = \partial_\lambda \Gamma'$ or $\int dt\, \Gamma'(x,t;\lambda)\, \Gamma'(t,y;\lambda) = \partial_\lambda \Gamma'(x,y;\lambda)$

Proof: $\Gamma'^2 = (\Sigma_{p=1} \lambda^{p-1} K^p)(\Sigma_{q=1} \lambda^{q-1} K^q) = \Sigma_{p=1} \Sigma_{q=1}\ \lambda^{p+q-2} K^{p+q}$

On of my math notes "double summation theorems" says this:

$$\Sigma_{p=1,2..^\infty} \Sigma_{q=1,2..^\infty}\ G_{p,q} = \Sigma_{s=0,1..^\infty}\ \Sigma_{d=s,s-2...-s}\ \ G_{(s-d)/2+1,\ (s+d)/2+1}$$

which we can apply here with $G_{p,q} = \lambda^{p+q-2} K^{p+q} = f_{p+q}$ where $f_m = \lambda^{m-2} K^m$

We then get $G_{(s-d)/2+1,\ (s+d)/2+1} = f_{s+2} = \lambda^s K^{s+2}$. Thus our double sum theorem says

$$\Sigma_{p=1} \Sigma_{q=1}\ \lambda^{p+q-2} K^{p+q} = \Sigma_{p=1} \Sigma_{q=1}\ f_{p+q} = \Sigma_{s=0,1..^\infty}\ \Sigma_{d=s,s-2...-s}\ \lambda^s K^{s+2}$$

$$= \Sigma_{s=0,1..^\infty}\ \lambda^s K^{s+2}\ (\Sigma_{d=s,s-2...-s}) = \Sigma_{s=0,1..^\infty}\ \lambda^s K^{s+2}\ (s+1)$$

$$= \Sigma_{r=1,2..}\ r\, \lambda^{r-1} K^{r+1} \qquad \text{// where } r = s+1$$

So we have now shown that $\Gamma'^2 = \Sigma_{r=1,2..}\ r\, \lambda^{r-1} K^{r+1}$. But consider:

$$\Gamma' = \Sigma_{m=1} \lambda^{m-1} K^m \quad \Rightarrow \quad \partial_\lambda\, \Gamma' = \Sigma_{m=1} (m-1) \lambda^{m-2} K^m = \Sigma_{m=2} (m-1) \lambda^{m-2} K^m$$

Let r = m-1 so this last sum is then $\partial_\lambda\, \Gamma' = \Sigma_{r=1}\ r\, \lambda^{r-1} K^{r+1}$. But this is the same as our result just shown above for Γ'^2. Thus we have shown that $\partial_\lambda\, \Gamma' = \Gamma'^2$. QED.

In terms of Γ we would have $\partial_\lambda\, (\lambda\Gamma) = (\lambda\Gamma)^2$ or $\lambda\, \partial_\lambda \Gamma + \Gamma = \lambda^2 \Gamma^2$, not quite as simple.

This one is pretty marginal, but I will leave it here.

Theorem 3: $\det(1-\lambda K) = \exp[(-\int_0^\lambda d\lambda'\, \delta(\lambda')\ ]$ where $\delta(\lambda) \equiv \text{tr}(\Gamma')$

Proof: Jumping ahead a little, from Section 8 below we know that

$$\det(1-\lambda K) = \exp\ [-\Sigma_{n=1}^\infty\, \lambda^n / n\ \ \text{tr}(K^n)]\ \ = e^{-f(\lambda)} \qquad \text{where } f(\lambda) = \Sigma_{n=1}^\infty\, \lambda^n / n\ \ \text{tr}(K^n)$$

Notice that $\partial_\lambda f\ = \Sigma_{n=1}^\infty\, \lambda^{n-1}\ \text{tr}(K^n)\ = \Sigma_{m=1}^\infty\, \lambda^{m-1} T_m\ = \text{tr}(\Gamma')$ from Theorem 1 above. Thus we find that:

$$\det(1-\lambda K) = e^{-f(\lambda)} \qquad \text{where } \partial_\lambda f = \text{tr}(\Gamma') = \int dx\, \Gamma'(x,x;\lambda)\ = \delta(\lambda).$$

Therefore it seems to me that we should have

$f(\lambda) = \int_0^{\lambda} d\lambda' \, \delta(\lambda')$ so that then $\det(1-\lambda K) = e^{-f(\lambda)} = \exp[- \int_0^{\lambda} d\lambda' \, \delta(\lambda')]$ QED

Comments: However, Mikhlin seems to be claiming in my M notes that $\det(1-\lambda K) = \exp[-\oint d\lambda' \, \delta(\lambda')]$ where this is some unspecified contour integral. I don't see how he gets that result. We have

$$\delta(\lambda) = \mathrm{tr}(\Gamma') = \int dx \, \Gamma'(x,x;\lambda) = \Sigma_{m=1}^{\infty} \lambda^{m-1} T_m$$

Of course this power series only converges around $\lambda'=0$ so cannot account for poles in $\delta(\lambda)$. From the fact that $(1 - \lambda K)^{-1} = 1 + \lambda\Gamma'$ I know that Γ' will have poles where $\det(1-\lambda K)$ has zeros, which are the eigenvalues of K. This also follows from $\Gamma'(x,y;\lambda) = D_1(x,y;\lambda)/ D(\lambda)$: Γ' has poles at the eigenvalues of the Fredholm determinant, so you would think that $\delta(\lambda)$ would have these same poles. But according to the above, if $\det(1-\lambda K)$ has a zero, we would expect $f(\lambda) = +\infty$. But if $f = \oint d\lambda' \, \delta(\lambda')$, it would have to mean that the residue of the pole in $\delta(\lambda)$ was infinite. So this is all a bit confusing, and has to do with the subject of : λ-plane analysis of Fredholm stuff which I have really not studied. I suspect that the contour implied is a great circle which would then pick up all the eigenvalue poles maybe.

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2. The determinant of 1+A.

Using the above general ε formula for the determinant of a matrix, we know that

$$\det(1+A) = \Sigma_{i1=1}^{N} \Sigma_{i2=1}^{N} ..... \Sigma_{iN=1}^{N}$$

$$(\delta_{1,i1} + A_{1,i1}) (\delta_{2,i2} + A_{2,i2}) (\delta_{3,i3} + A_{3,i3}) ..... (\delta_{N,iN} + A_{N,iN}). \varepsilon^{i1\ i2\ i3...iN}$$

If we examine those terms which have n factors of A, it turns out that the following is true (we shall prove this in a moment)

$$\text{total of all terms with n factors of A} = \Sigma_{r1<r2<r3...<rn} \det_{r1\ r2\ r3...rn}$$

where the individual indices in the summation range 1 to N subject to the restriction shown. The object here denoted by $\det_{r1\ r2\ r3...rn}$ is an n x n diagonal subdeterminant of the matrix A which has all rows and columns crossed out other than those of the group $r_1, r_2 ... r_n$. The subject of subdeterminants in general and diagonal subdeterminants in particular is discussed in detail in the following separate document:

" Subdeterminants of a matrix.doc"

Notice in the language of that document that all the terms in our total sum above are in natural order. We could have written, also in the language of this external document,

$$\text{total of all terms with n factors of A} = \Sigma'_{rows(n)} \det_{rows(n)}$$

where the sum Σ' includes only groups of n rows which are in natural order. Here rows(n) just means a group "rows" which contains exactly n row numbers. For example, assuming $N \geq 5$, one set of row numbers included in the set rows(4) would be 1245. The set 1254 would not be included because it is not in natural order (ie, not monotonically increasing).

Our final result is then going to be

$$\det(1+A) = \Sigma_{n=0}^{N} \{ \Sigma'_{\text{rows(n)}} \det_{\text{rows(n)}} \}$$

about which we will have much more to say below. But first we want to prove our claim that

$$\text{total of all terms with n factors of A} = \Sigma_{r1<r2<r3\ldots<rn} \det_{r1\ r2\ r3\ldots rn}$$

The reader is reminded that a formula for this subdeterminant is given in the abovementioned document,

$$\det_{r1\ r2..rn} = \Sigma_{i1,i2,..in \in \{r1\ r2\ ..rn\}} A_{r1,i1} A_{r2,i2} A_{r3,i3} \ldots A_{rn,in} \varepsilon^{i1\ i2\ ..in}$$

$$\text{with } \varepsilon^{r1\ r2\ ..rn} = 1$$

Theorem 1: total of all terms with n factors of A = $\Sigma_{r1<r2<r3\ldots<rn}$ $\det_{r1\ r2\ r3\ldots rn}$

Proof:

(A) To see why the above claim is true, let's start by rewriting the above det(1+A) by separating the first n sums from the last N-n sums and doing similar segregation in the δ factors and ε indices:

$$\det(1+A) = (\Sigma_{i1=1}^{N} \Sigma_{i2=1}^{N} \ldots\ldots \Sigma_{in=1}^{N})(\Sigma_{in+1=1}^{N} \Sigma_{in+2=1}^{N} \ldots\ldots \Sigma_{iN=1}^{N})$$

$$(\delta_{1,i1} + A_{1,i1}) (\delta_{2,i2} + A_{2,i2}) (\delta_{3,i3} + A_{3,i3}) \ldots\ldots (\delta_{n,in} + A_{n,in})$$
$$(\delta_{n+1,in+1} + A_{n+1,in+1}) (\delta_{n+2,in+2} + A_{n+2,in+2}) \ldots (\delta_{N,iN} + A_{N,iN})$$

$$\varepsilon^{i1\ i2\ i3\ldots in\ ,\ in+1\ in+2\ \ldots\ldots iN}$$

The reader understands that by in+1 we really mean i_{n+1} but our word processor has trouble showing this in something that is already a subscript or superscript and we don't want to slow down to correct this cosmetic issue, though it is correctable. In the ε index set, we put a cosmetic comma just to show the break in the two groups.

Now let's look at the terms in the above det(1+A) where we select the A factors from the leftmost n parentheses, and the δ factors from the rightmost N-n parentheses. When we do this, the rightmost set of summations hits against the δ factors causing those indices to "pin" to the values n+1,n+2...N. That is to say, we get $i_{n+1} = n+1$, $i_{n+2} = n+2$, ... $i_N = N$. We then get

$$\text{contrib to det(1+A) from choosing leftmost n A factors} =$$

$$\Sigma_{i1=1}^{N} \Sigma_{i2=1}^{N} ..... \Sigma_{in=1}^{N}$$

$$(A_{1,i1}) (A_{2,i2}) (A_{3,i3}) ..... (A_{n,in})$$

$$\varepsilon^{i1\ i2\ i3...in,\ n+1\ n+2\ .....N}$$

Now, since the last N-n indices of the ε are pinned to their natural order values as shown, we know that the ε factor will be zero if any of the indices i_1 through i_n hits any of these high values. Thus, we can replace all the sums going to N with sums going to n causing no change, so we then have:

$$= \Sigma_{i1=1}^{n} \Sigma_{i2=1}^{n} ..... \Sigma_{in=1}^{n} (A_{1,i1}) (A_{2,i2}) (A_{3,i3}) ..... (A_{n,in})\ \varepsilon^{i1\ i2\ i3...in,\ n+1\ n+2\ .....N}$$

But now we can simply replace the N-dimensional ε tensor with an n-dimensional one since the last values of the N-dimensional one are pinned to their natural order values n+1 through N, so we have:

$$= \Sigma_{i1=1}^{n} \Sigma_{i2=1}^{n} ..... \Sigma_{in=1}^{n} (A_{1,i1}) (A_{2,i2}) (A_{3,i3}) ..... (A_{n,in})\ \varepsilon^{i1\ i2\ i3...in}$$

$$= \Sigma_{i1,i2,..in\ \in\ \{1\ 2\ ..\ n\}} A_{1,i1}\ A_{2,i2}\ A_{3,i3}..... A_{n,in}\ \varepsilon^{i1\ i2\ i3...in}$$

From our subdeterminant formula quoted at the end of the previous section, this summation is recognized to be just $\det_{123..n}$. So we have shown that

contrib to det(1+A) from choosing leftmost n A factors $= \det_{123..n}$

which is the determinant of an n x n submatrix of A. This submatrix happens to be in the upper left corner of the matrix A.

(B) Now, what would happen if we were to swap one of our A factors with one of our δ factors. For the sake of argument, suppose we swap our $(A_{2,i2})$ factor with our $\delta_{n+2,in+2}$ factor. Our result would be

$$\Sigma_{i1=1}^{N} \cancel{\Sigma_{i2=1}^{N}} ..... \Sigma_{in=1}^{N} \Sigma_{in+2=1}^{N}$$

$$(A_{1,i1}) \cancel{(A_{2,i2})} (\delta_{2,i2}) (A_{3,i3}) ..... (A_{n,in}) (A_{n+2,in+2})$$

$$\varepsilon^{i1\ \cancel{i2}\ 2\ i3...in,\ n+1\ \cancel{n+2}\ in+2\ n+3\ .....N}$$

We show in red previous items crossed out and new items added. Notice that the product of A factors is always "naturally ordered" (monotonic increase) left to right with respect to the first (row) index of A_{ab} (before and after our change). No matter what pair of A and δ we swap, this would always be true.

In this new result above, there are still n summation indices and the ε has its other N-n indices "pinned" to hard values. Those pin values are now { 2, n+1,n+3.....N} . The remaining summation indices can therefore only take values in the *complement* of this set, which is {1,3,4,5...n,n+2}, otherwise we just get ε = 0. We can thus rewrite the above as

$$\Sigma_{i1,i3,i4..in,in+2 \in \{1,3,4,5...n,n+2\}} (A_{1,i1})(A_{3,i3})(A_{4,i4})...(A_{n,in})(A_{n+2,in+2})\,\varepsilon^{i1\ 2\ i3...in,\ n+1\ in+2\ n+3\ .....N}$$

Since the pinned values on the ε tensor are all in their "natural positions", we can replace the above with

$$= \Sigma_{i1,i3,i4..in,in+2 \in \{1,3,4,5...n,n+2\}} A_{1,i1}A_{3,i3}A_{4,i4}...A_{n,in}A_{n+2,in+2}\,\varepsilon^{i1\ i3...in,\ in+2}$$

where again we have reduced from the N-dimensional ε tensor to an n-dimensional one. Since the summation indices are all dummies, we can rename them as follows

$$i1,i3,i4..in,in+2 \rightarrow i1,i2,i3..in-1\ in$$

so the above becomes

$$\Sigma_{i1,i2,i3..in \in \{1,3,4,5...n,n+2\}} A_{1,i1}A_{3,i2}A_{4,i3}...A_{n,in-1}A_{n+2,in}\ \varepsilon^{i1\ i2...in-1,\ in}$$

and looking at our quoted formula above, we see this thing is just another diagonal subdeterminant,

$$\det_{134..n\ n+2}$$

(C) In parts (A) and (B) of our proof, we have shown that:

contrib to det(1+A) from choosing leftmost n A factors $= \det_{123..n}$

contrib to det(1+A) from choosing leftmost n A factors
but swap $(A_{2,i2})$ with $(\delta_{n+2,in+2})$ $= \det_{134...n\ n+2}$

The main point really is that we have not picked up any "minus signs" or other strange factors in doing our little swap, we have just re-ordered the rows and cols groups on our det result. In both cases above, we can see that on $\det_{rows}$ we have

rows = the sequence of first (row) indices on the set of A factors we selected

This is the key point. In our first example, we picked

$(A_{1,i1})(A_{2,i2})(A_{3,i3})\ .....\ (A_{n,in})$ => 123...n

In our second example we instead selected this set of A factors

$(A_{1,i1})(A_{3,i3})(A_{4,i4})...(A_{n,in})(A_{n+2,in+2})$ => 134..n,n+2

As we select different sets of n A factors, the row indices are always in monotonically increasing numerical order. This should be obvious from our original expression of det(1+A) which has these factors

$(\delta_{1,i1} + A_{1,i1})\,(\delta_{2,i2} + A_{2,i2})\,(\delta_{3,i3} + A_{3,i3})\,.....\,(\delta_{N,iN} + A_{N,iN})$

Obviously every monotonically increasing set of first indices of A factors is going to occur when we select all possible sets of n A factors. We then get the result we claimed above:

total of all terms with n factors of A $= \Sigma_{i1<i2<i3...<in}\ \det_{i1\ i2\ i3...in}$

where the individual indices in the summation range 1 to N subject to the restriction shown. We thus have a sum of a set of n x n diagonal subdeterminants of the matrix A

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3. Overcounting.

We now resume our development of a formula for det(1+A). Our next little topic is how removing the restriction on the summation indices causes overcounting which we compensate for by a factorial. Consider the case N = 4 and n = 3 where we then have

total of all terms with 3 factors of A $= \Sigma_{i1<i2<i3}\ \det_{i1\ i2\ i3}$

$= \det_{123} + \det_{124} + \det_{134} + \det_{234}$

If we remove the restriction i1<i2<i3 on the sum, we include new terms that were not there before. In fact, each term is replicated 3! times. For example, in an unrestricted sum we get

$\det_{123} + \det_{132} + \det_{312} + \det_{321} + \det_{231} + \det_{213}$

Since these are diagonal subdeterminants, these six items are all the same (see external doc). This is because swapping two rows and two columns does not change a determinant:

$\det_{123} \equiv \det_{123;123} = + \det_{213;213} = \det_{213}$

Thus

$\det_{123} + \det_{132} + \det_{312} + \det_{321} + \det_{231} + \det_{213} = 3!\ \det_{123}$

Also, by removing our restriction i1<i2<i3 we allow for terms like $\det_{112}$ but these are all zero since having two rows the same makes a determinant vanish. Thus we have shown that

total of all terms with 3 factors of A $= \Sigma_{i1<i2<i3}\ \det_{i1\ i2\ i3}$

total of all terms with 3 factors of A $= (1/3!)\ \Sigma_{i1,i2,i3}\ \det_{i1\ i2\ i3}$

and we can see that more generally

total of all terms with n factors of A = $\Sigma_{i1<i2<i3...<in}\ \det_{i1\ i2\ i3...in}$

total of all terms with n factors of A = $(1/n!)\ \Sigma_{i1,i2,i3...\ in}\ \det_{i1\ i2\ i3...in}$

We can now assemble our final result for det(1+A):

$\det(1+A) = \Sigma_{n=0}^{N}$ { total of all terms with n factors of A }

$= \Sigma_{n=0}^{N}\ (1/n!)\ \Sigma_{i1,i2,i3...\ in=1}^{N}\ \det_{i1\ i2\ i3...in}$

where the det object is an nxn subdeterminant of matrix A. Thus in words we can say

det(1+A) = the sum with weighting factor (1/n!) of ALL diagonal subdeterminants of A
where n is the dimensionality of the subdeterminant.

In our earlier notation we could then write (where now all rows sets are allowed)

$\det(1+A) = \Sigma_{n=0}^{N}\ (1/n!)\ \Sigma_{rows(n)}\ \det_{rows(n)}$

We can now look at some particular terms in the series. We know that the n=0 term is just 1 because it corresponds to picking all the δ functions in

$\det(1+A) = \Sigma_{i1=1}^{N}\ \Sigma_{i2=1}^{N}\ .....\ \Sigma_{iN=1}^{N}$

$(\delta_{1,i1} + A_{1,i1})\ (\delta_{2,i2} + A_{2,i2})\ (\delta_{3,i3} + A_{3,i3})\ .....\ (\delta_{N,iN} + A_{N,iN}).\ \varepsilon^{i1\ i2\ i3...iN}$

which of course just gives us $\varepsilon^{123...N} = +1$. To make our formula above give this result, we could extend our definition of $\det_{rows,cols}$ such that $\det_{\varphi,\varphi} = 1$, where φ means a null set. To avoid this clumsy confusion, let's restate our result exposing this first term:

$\det(1+A) = 1 + \Sigma_{n=1}^{N}\ (1/n!)\ \Sigma_{i1,i2,i3...\ in=1}^{N}\ \det_{i1\ i2\ i3...in}$

The n = 1 term is $\Sigma_{i1}\ \det_{i1}$. But $\det_{i1} \equiv \det_{i1;i1}$ is the determinant of a 1 x 1 matrix whose element is simply $A_{i1,i1}$ so the n = 1 term is $\Sigma_{i1}\ A_{i1,i1}$ = trace(A) = tr(A).

The n=N term is 1/N! times a sum which contains the full det(A) N! times. So this term is just det(A). So we could write the above sum once again as

$\det(1+A) = 1 + tr(A) + \Sigma_{n=2}^{N-1}\ (1/n!)\ \Sigma_{i1,i2,i3...\ in=1}^{N}\ \det_{i1\ i2..in} + \det(A)$

There are N+1 terms in the sum.

4. The Fredholm Determinant

In our section 1 above on Integral Equation Preliminaries, we encountered the operator $(1-\lambda K)^{-1}$ which we can think of as an NxN matrix (and later take N→∞). This inverse is given by

$$(1 - \lambda K)^{-1} = [\ \text{cof}\,(1-\lambda K)\ ]^{T} / \det(1 - \lambda K) \quad = 1 + \Gamma$$

which involves the object det(1 - λK) which is known as **the Fredholm Determinant**, though this can be written in various ways. Using our result for det(1+A) above with A = -λK we can say

$$\det(\ 1 - \lambda K) \ = 1 + \Sigma_{n=1}^{N}\ (-\lambda)^{n}/n!\ \ \Sigma_{i1,i2,i3\ldots\ in=1}^{N}\ \det_{i1\ i2\ i3\ldots in}$$

where the dets are now diagonal subdeterminants of the matrix K. We pick up a new factor $(-\lambda)^{n}$ from the obvious source

$$AAA...A \ = \ (-\lambda K)\ (-\lambda K)\ (-\lambda K)....\ (-\lambda K) \ = (-\lambda)^{n}\ KKK..K$$

where we look at some arbitrary term in the determinant in the nth term in our first sum.

We now state a key point: this expansion for det(1-λK) is a power series in λ, and if λ is small, keeping only the first several terms of this series is probably appropriate. This series is the only such power series in λ, there are no others, it is unique, and that is why we want to know about it. As shown elsewhere, the series converges when $\lambda\ \|K\| \le 1$ where ||K|| is the (assumed to exist) bound on the integral operator K. When K is a symmetric (ie, Hermitian) matrix, ||K|| is equal to the largest in magnitude eigenvalue of K.

The formula above can be written in various other ways. Here is a way which draws pictures of the subdeterminants in question:

$$\det(1-\lambda K) = 1 - (\lambda)\ \Sigma_{i}\ K_{ii} + (\lambda^{2}/2)\Sigma_{ij}\ \begin{vmatrix} Kii & Kij \\ Kji & Kjj \end{vmatrix} - (\lambda^{3}/3!)\Sigma_{ijk}\ \begin{vmatrix} Kii & Kij & Kik \\ Kji & Kjj & Kjk \\ Kki & Kkj & Kkk \end{vmatrix} + \ldots$$

$$\det(1-\lambda K) = 1 + \Sigma_{n=1}^{N}\ (-\lambda)^{n}/n!\ \ \Sigma_{i1,i2,i3\ldots\ in=1}^{N}\ \det_{i1\ i2\ i3\ldots in}$$

$$\det(1-\lambda K) = 1 + (-\lambda)^{1}\ \ \Sigma_{i1}\ \det_{i1} + (-\lambda)^{2}/2!\ \Sigma_{i1,i2}\ \det_{i1\ i2} + (-\lambda)^{3}/3!\ \Sigma_{i1,i2,i3}\ \det_{i1\ i2\ i3} + ..$$

$$\det(1-\lambda K) = 1 + (-\lambda)^{1}\ \ \Sigma_{i}\ \det_{i} + (-\lambda)^{2}/2!\ \Sigma_{ij}\ \det_{ij} + (-\lambda)^{3}/3!\ \Sigma_{ijk}\ \det_{ijk} + ..$$

where we juxtapose our formula several times for comparison. In other words, we have

$$\det_{ij} = \begin{vmatrix} Kii & Kij \\ Kji & Kjj \end{vmatrix} \qquad \det_{ijk} = \begin{vmatrix} Kii & Kij & Kik \\ Kji & Kjj & Kjk \\ Kki & Kkj & Kkk \end{vmatrix} \qquad \text{and so on}$$

5. The limit N→∞.

In order to take the limit N→∞ to get results appropriate for an integral equation, we replace the summations on the discrete indices like i1 i2 with integrals over continuous variables s_1 s_2. Our formula then becomes

$$\det(1-\lambda K) = 1 + \Sigma_{n=1}^{N} (-\lambda)^n /n! \int ds_1 ds_2 .... ds_n \det_{s1\ s2\ s3 \ldots sn}$$

The determinants are still nxn of course. As we take the limit N→∞, the matrix element $K_{i1,i2}$ becomes the continuous $K_{s1,s2}$ which is usually written $k(s_1\ s_2)$ where the function k is now the "kernel" of the integral operator K:

$\varphi = f + \lambda K\varphi$ // operator equation

$\varphi_i = f_i + \lambda \Sigma_j K_{ij}\varphi_j$ // matrix equation

$\varphi(x) = f(x) + \lambda \int dy\ k(x,y)\varphi(y)$ // integral equation

We have then, for example,

$$\det_{s1\ s2} \equiv \det_{s1\ s2;\ s1\ s2} = \begin{vmatrix} k(s1,s1) & k(s1,s2) \\ k(s2,s1) & k(s2,s2) \end{vmatrix} \qquad \det_{rows,cols}$$

Notice that the rows group provides the left arguments for the k functions, and the cols group provides the right arguments, just as if it were $K_{s1,s2}$. This is made a little clearer if we were to write out a non-diagonal determinant

$$\det_{s1\ s2;\ s3\ s5} = \begin{vmatrix} k(s1,s3) & k(s1,s5) \\ k(s2,s3) & k(s2,s5) \end{vmatrix}$$

The Fredholm determinant involves only diagonal subdeterminants, but the Fredholm Minors involve off diagonal ones as well, as we shall see below.

The Fredholm determinant can be written then in graphical form in this way:

$$\det(1-\lambda K) = 1 - \lambda \int dx\ k(x,x) + (\lambda^2/2) \int dx\ dy \begin{vmatrix} k(x,x) & k(x,y) \\ k(y,x) & k(y,y) \end{vmatrix}$$
$$- (\lambda^3/3!) \int dx dy dz \begin{vmatrix} k(x,x) & k(x,y) & k(x,z) \\ k(y,x) & k(y,y) & k(y,z) \\ k(z,x) & k(z,y) & k(z,z) \end{vmatrix} + \ldots. \text{ (goes on forever)}$$

Notice that just because we have taken the limit N→ ∞ the various terms don't necessarily diverge. For "reasonable" kernels, we expect each of these fancy integrals above to be finite, and we really do have a power series in λ with finite coefficients. Notice that $tr(K) = \int dx\ k(x,x)$, the "sum of the diagonal elements" of the "matrix K".

Here is a common notation that appears in the literature:

$$K\begin{pmatrix} x_1,\ldots,x_n \\ y_1,\ldots,y_n \end{pmatrix} = \begin{vmatrix} K(x_1,y_1) & \ldots & K(x_1,y_n) \\ & \ldots & \\ K(x_n,y_1) & \ldots & K(x_n,y_n) \end{vmatrix}$$

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_D \ldots \int_D K\begin{pmatrix} s_1,\ldots,s_n \\ s_1,\ldots,s_n \end{pmatrix} ds_1 \ldots ds_n$$

We would write the first object as

$\det_{\texttt{x1 x2 x3...xn; y1 y2 y3..yn}} = \det_{\texttt{rows;cols}}$

and D(λ) = det(1-λK). As we know, only the diagonal det's appear in D(λ). The subscripts on the integrals refer to the interval on which the integral equation is defined, perhaps D = (0,1) . What we write as k(x,y) for the kernel appears as K(x,y) in this author's notation.

6. The Case of Factorizable K and the Trace Approximation

Recall from above (stated just before the Theorem and its proof, we now insert K in place of A)

$$\det_{\texttt{r1 r2..rn}} = \Sigma_{\texttt{i1,i2,..in} \in \texttt{\{r1 r2 ..rn\}}} K_{\texttt{r1,i1}} K_{\texttt{r2,i2}} K_{\texttt{r3,i3}} \ldots K_{\texttt{rn,in}} \varepsilon^{\texttt{i1 i2 ..in}}$$

with $\varepsilon^{\texttt{r1 r2 ..rn}} = 1$

If the matrix K "factorizes" so if can be written $K_{ij} = f_i g_j$, then we get

$$\det_{\texttt{r1 r2..rn}} = \Sigma_{\texttt{i1,i2,..in} \in \texttt{\{r1 r2 ..rn\}}} K_{\texttt{r1,i1}} K_{\texttt{r2,i2}} K_{\texttt{r3,i3}} \ldots K_{\texttt{rn,in}} \varepsilon^{\texttt{i1 i2 ..in}}$$

$$= \Sigma_{\texttt{i1,i2,..in} \in \texttt{\{r1 r2 ..rn\}}} f_{\texttt{r1}} g_{\texttt{i1}} f_{\texttt{r2}} g_{\texttt{i2}} f_{\texttt{r3}} g_{\texttt{i3}} \ldots f_{\texttt{rn}} g_{\texttt{in}} \varepsilon^{\texttt{i1 i2 ..in}}$$

$$= f_{\texttt{r1}} f_{\texttt{r2}} f_{\texttt{r3}} f_{\texttt{rn}} \left[\Sigma_{\texttt{i1,i2,..in} \in \texttt{\{r1 r2 ..rn\}}} g_{\texttt{i1}} g_{\texttt{i2}} g_{\texttt{i3}} \ldots g_{\texttt{in}} \varepsilon^{\texttt{i1 i2 ..in}} \right]$$

where we are interested in n ≥ 2. In the [..] we have the contraction of a symmetric tensor against an antisymmetric tensor which gives 0. Thus we conclude that

$\det_{\texttt{r1 r2..rn}} = 0 \qquad n \geq 2 \qquad \text{if } K_{ij} = f_i g_j$

For the case n= 1 we get $tr(K_{ij}) = \Sigma_i K_{ii} = \Sigma_i f_i g_i$ which is not in general zero. Thus, we arrive at this fact:

If K is factorizable, then

$$\det(1-\lambda K) = 1 - (\lambda)\, \Sigma_i\, K_{ii} = 1 - \lambda\, tr(K)$$

Even if K is not factorizable, if λ is very small we might keep just this term in the expansion, and this is then called "the trace approximation".

In the continuum world, factorizable means that k(x,y) = f(x)g(y) and we then get

$$\det(1-\lambda K) = 1 - \lambda \int dx\, k(x,x)$$

If K is only "separable" in the sense that $K = \Sigma_s\, f_i^{(s)} g_j^{(s)}$ [or $k(x,y) = \Sigma_s\, f^{(s)}(x) g^{(s)}(y)$] , we don't get the simplification that we get when K is really "factorizable". This is because det(1-λK) is not linear in K, as the Fredholm series pretty clearly shows. In our little tensor contraction analysis above we would get terms of the form

$$g_{i1}^{(s1)}\, g_{i2}^{(s2)}\, g_{i3}^{(s3)} \ldots g_{in}^{(sn)}\, \varepsilon^{i1\ i2\ ..in}$$

and the product of the g's is no longer a symmetric tensor so the dets for n≥2 no longer vanish.

7. Writing the Fredholm determinant entirely in terms of traces.

Define:

$$T_n \equiv tr(A^n) \qquad // \quad T_0 = tr(A^0) = tr(1) = 1$$

Then recall our det(1+A) formula from above

$$\det(1+A) = 1 + tr(A) + \Sigma_{n=2}^{N-1}\ (1/n!)\, \Sigma_{i1,i2,i3\ldots\ in=1}^{N}\ \det_{i1\ i2..in} + \det(A)$$

Let's now consider a few terms.

Ne n=0 term is already the trace T_0, and the n = 1 term is already the trace T_1.

The n=2 term is this:

$$(1/2)\, \Sigma_{i1,i2}\, \det_{i1\ i2} = (1/2) \Sigma_{IJ}\, \det_{IJ} = (1/2)\, \Sigma_{IJ}\, \{\, \varepsilon^{ij}\, A_{Ii} A_{Jj}\}$$

$$= (1/2)\, \Sigma_{IJ}\, \{\, A_{II} A_{JJ} - A_{IJ} A_{JI}\} = (1/2)\, \{\ \Sigma_I A_{II}\ \Sigma_J A_{JJ}\ - \Sigma_{IJ}\, A_{IJ} A_{JI}\ \}$$

$$= (1/2)\, \{\, [\, tr(A)]^2 - tr(A^2)\, \} = (1/2)\, \{\, T_1^2 - T_2\, \}$$

The n=3 term (times 3!) is this:

$$\Sigma_{i1,i2,i3}\, det_{i1\ i2\ i3} = \Sigma_{IJK} det_{IJK} = det_{IJK} \qquad \text{// triple sum present but suppressed}$$

$$= \varepsilon^{ijk} A_{Ii} A_{Jj} A_{Kk}$$

$$= A_{II}A_{JJ}A_{KK} - A_{IJ}A_{JI}A_{KK} + A_{IJ}A_{JK}A_{KI} - A_{IK}A_{JJ}A_{KI} + A_{IK}A_{JI}A_{KJ} - A_{II}A_{JK}A_{KJ}$$

where I exchange one pair to create each new term. Now we can look at our sum

$$\Sigma_{IJK} \{ A_{II}A_{JJ}A_{KK} - A_{IJ}A_{JI}A_{KK} + A_{IJ}A_{JK}A_{KI} - A_{IK}A_{JJ}A_{KI} + A_{IK}A_{JI}A_{KJ} - A_{II}A_{JK}A_{KJ}\}$$

$$= [\, tr(A)]^3 - tr(A)\, tr(A^2) + tr(A^3) - tr(A)tr(A^2) + tr(A^3) - tr(A)\, tr(A^2)$$

$$= [\, tr(A)]^3 - 3\, tr(A)\, tr(A^2) + 2\, tr(A^3) = T_1^3 - 3\, T_1T_2 + 2\, T_3$$

To summarize, we have now displayed the first four Fredholm determinant terms entirely in terms of traces of powers of A:

$$det(1+A) = T_0 + T_1 + (1/2) \{ T_1^2 - T_2 \} + (1/6) \{ T_1^3 - 3\, T_1T_2 + 2\, T_3 \} + ...$$

If we take $A \rightarrow \lambda A$ we of course get

$$det(1+\lambda A) = T_0 + \lambda T_1 + (\lambda^2/2!) \{ T_1^2 - T_2 \} + (\lambda^3/3!) \{ T_1^3 - 3\, T_1T_2 + 2\, T_3 \} + ...$$
$$det(1-\lambda A) = T_0 - \lambda T_1 + (\lambda^2/2!) \{ T_1^2 - T_2 \} - (\lambda^3/3!) \{ T_1^3 - 3\, T_1T_2 + 2\, T_3 \} + ...$$

It seems pretty reasonable to expect that we can express *any* term as a function of the T_n, and there is no doubt a general formula which gives all terms (we might see this come out in the next section).

8. The Trace Expansion

In Section 3 above we obtained this expansion for the Fredholm determinant:

$$det(1+A) = 1 + tr(A) + \Sigma_{n=2}^{N-1} (1/n!) \Sigma_{i1,i2,i3\ldots\ in=1}^{N} det_{i1\ i2..in} + det(A)$$

There is another way to write this expansion. It is based on a very general matrix theorem,

$$det(e^B) = e^{tr(B)}$$

Aside: I have a 1 page proof of this theorem which is based on writing $B = \Sigma B_{ij} e^{(ij)}$ where the basis matrices $e^{(ij)}$ have elements $e^{(ij)}_{mn} = \delta_{im}\delta_{jm}$. This proof is "item 4" in my old matrix notes. After writing the proof, I for some reason thought it was wrong and I crossed it out in red. Then on 11-8-07 I realized it was not wrong. If you think of A as a Lie generator of a special S Lie group like SU(2) or SO(3), then e^B is a rotation and has det 1, so this shows that such generators must be traceless.

In the above theorem, set $e^B = C$ so that $B = \ln C$. We then have

$$\det(C) = e^{tr[\ \ln(C)\ ]}$$

Now set $C = 1+A$ in the above to get.

$$\det(1+A) = \exp[\ tr(\ \{\ \ln(1+A)\}\ ]$$

and this will provide our new expansion for det(1+A). We know that

$$\ln(1+A) = A - A^2/2 + A^3/3 - \ldots = -\Sigma_{n=1}^{\infty} (-1)^n A^n/n$$

Then we can say

$$tr(\ \{\ \ln(1+A)\} = -\ tr\ \{\ \Sigma_{n=1}^{\infty} (-1)^n A^n/n\ \} = -\Sigma_{n=1}^{\infty} (-1)^n/n * tr(A^n)$$

Then we have shown that

$$\det(1+A) = \exp[-\Sigma_{n=1}^{\infty} (-1)^n/n\ \ tr(A^n)]$$
$$\det(1-A) = \exp[-\Sigma_{n=1}^{\infty}\ \ 1/n\ \ tr(A^n)]$$
$$\det(1-\lambda A) = \exp[-\Sigma_{n=1}^{\infty}\ \ \lambda^n/n\ \ tr(A^n)]$$

which agrees with page 16 of my Mikhlin notes. Obviously we can do a second expansion for the exponential and then we get (changing the above sum index to m)

$$\det(1+A) = \Sigma_{n=0}^{\infty} (-1)^n/n! * [\ \Sigma_{m=1}^{\infty}(-1)^m/m * tr(A^m)\ ]^n$$
$$\det(1-A) = \Sigma_{n=0}^{\infty} (-1)^n/n! * [\ \Sigma_{m=1}^{\infty}\ (1/m)\ * tr(A^m)\ ]^n$$

$$\det(1+\lambda A) = \Sigma_{n=0}^{\infty} (-1)^n/n! * [\ \Sigma_{m=1}^{\infty}(-\lambda)^m/m * tr(A^m)\ ]^n$$
$$\det(1-\lambda A) = \Sigma_{n=0}^{\infty} (-1)^n/n! * [\ \Sigma_{m=1}^{\infty}\ (\lambda^m/m)\ * tr(A^m)\ ]^n$$

This must somehow replicate our previous Fredholm formula

$$\det(1-\lambda K) = 1 + \Sigma_{n=1}^{N}\ (-\lambda)^n/n!\ \ \Sigma_{i1,i2,i3\ldots\ in=1}^{N}\ \det_{i1\ i2\ i3\ldots in}$$

Let's look at the first few terms of our first trace expansion, recall $T_m = tr(A^m)$,

$$\det(1-\lambda A) = \Sigma_{n=0}^{\infty} (-1)^n/n! * [\ \Sigma_{m=1}^{\infty} (\lambda^m/m) * T_m\ ]^n$$

$$= 1 - \{\lambda T_1 + \lambda^2 T_2/2 + \lambda^3 T_3/3 + \ldots\} + \{\lambda T_1 + \lambda^2 T_2/2 + \lambda^3 T_3/3 + \ldots\}^2/2!$$

$$- \{\lambda T_1 + \lambda^2 T_2/2 + \lambda^3 T_3/3 + \ldots\}^3/3! + \ldots$$

We can now gather up terms of a given power of λ, where λ^0 term is just the 1 and the λ^1 term is $-\lambda T_1$. Then

λ^0: 1

λ^1: $-\lambda T_1$

λ^2: $-\lambda^2 T_2/2 + \lambda^2 T_1^2/2 = (\lambda^2/2)(T_1^2 - T_2)$

λ^3: $-\lambda^3 T_3/3 + \lambda^3(T_1 T_2)/2 - \lambda^3 T_1^3/3! = -(\lambda^3/3!)\{ T_1^3 - 3T_1T_2 + 2T_3 \}$

In the previous section where we manually rewrote the Fredholm terms in traces, we got

$$\det(1-\lambda A) = T_0 - \lambda T_1 + (\lambda^2/2!)\{ T_1^2 - T_2 \} - (\lambda^3/3!)\{ T_1^3 - 3T_1T_2 + 2T_3 \} + ...$$

and we see then that there is complete agreement for these few terms. As commented earlier, there is probably a general formula for the arbitrary term, but you would have to do a fancy multinomial expansion of the quantity $[\Sigma_{m=1}^{\infty} (\lambda^m/m) * \mathrm{tr}(A^m)]^n = \{\lambda T_1 + \lambda^2 T_2/2 + \lambda^3 T_3/3 + ...\}^n$ to get all the powers of λ grouped together. We shall not attempt this here. (an exercise for the energetic reader?)