Fredholm's First Minor (Round 1 notes) PhL 3/26/09

See Round 2 notes for a more coherent presentation, ignore these notes except for details.

I tried first to do this with my own special "arbitrary determinant" notation, but I was unable to make progress. So here I want to try and replicate my own derivation of this minor idea from my 1970 notes.

The N Expansion

I don't quite know what to call this. Here is a claimed theorem which I will first state, and then I will try to prove it:

Define N_{ij} as follows: $\Gamma_{ij} = N_{ij} / |1-K|$

We are setting $\lambda = 1$ for a while. The claim is this:

$$N_{ij} = K_{ij} - \Sigma_k \begin{vmatrix} Kij & Kik \\ Kkj & Kkk \end{vmatrix} + (1/2!)\,\Sigma_{km} \begin{vmatrix} Kij & Kik & Kim \\ Kkj & Kkk & Kkm \\ Kmj & Kmk & Kmm \end{vmatrix} - (1/3!)\,\Sigma_{kmn} \begin{vmatrix} Kij & Kik & Kim & Kin \\ Kkj & Kkk & Kkm & Kkn \\ Kmj & Kmk & Kmm & Kmn \\ Knj & Knk & Knm & Knn \end{vmatrix} + .$$

This is certainly a strange looking expansion. In each determinant, the previous one appears in the upper left, and we have added another summation index. But whatever it means, let's see if we can prove it. I am following my own ancient proof.

(1) Expand each determinant by "going down the first column". I will try to write this all out:

$$N_{ij} = K_{ij}$$

$$- \Sigma_k K_{ij} K_{kk} \qquad + \qquad \Sigma_k K_{ik} K_{kj}$$

$$+ (1/2)\,\Sigma_{km} K_{ij} \begin{vmatrix} Kkk & Kkm \\ Kmk & Kmm \end{vmatrix} - (1/2)\,\Sigma_{km} K_{kj} \begin{vmatrix} Kik & Kim \\ Kmk & Kmm \end{vmatrix} + (1/2)\,\Sigma_{km} K_{mj} \begin{vmatrix} Kik & Kim \\ Kkk & Kkm \end{vmatrix}$$

$$-(1/6)\,\Sigma_{kmn} K_{ij} \begin{vmatrix} Kkk & Kkm & Kkn \\ Kmk & Kmm & Kmn \\ Knk & Lnm & Knn \end{vmatrix} + (1/6)\,\Sigma_{kmn} K_{kj} \begin{vmatrix} Kik & Kim & Kin \\ Kmk & Kmm & Kmn \\ Knk & Lnm & Knn \end{vmatrix}$$

$$-(1/6)\,\Sigma_{kmn} K_{mj} \begin{vmatrix} Kik & Kim & Kin \\ Kkk & Kkm & Kkn \\ Knk & Lnm & Knn \end{vmatrix} + (1/6)\,\Sigma_{kmn} K_{nj} \begin{vmatrix} Kik & Kim & Kin \\ Kkk & Kkm & Kkn \\ Kmk & Lmm & Kmn \end{vmatrix}$$

Think of the above as three groupings depending on the size of the dets. If we examine the first term in each grouping, we have these terms

$$K_{ij}\ \{\ \ 1 - \Sigma_k K_{kk} + (1/2)\,\Sigma_{km}\begin{vmatrix} Kkk & Kkm \\ Kmk & Kmm \end{vmatrix} - (1/3!)\,\Sigma_{kmn}\begin{vmatrix} Kkk & Kkm & Kkn \\ Kmk & Kmm & Kmn \\ Knk & Lnm & Knn \end{vmatrix} + ...\}$$

Looking at our Fredholm determinant doc, we had

$$\det(1\text{-}A) = 1 - \Sigma_i A_{ii} + (1/2)\Sigma_{ij}\begin{vmatrix} Aii & Aij \\ Aji & Ajj \end{vmatrix} - (1/6)\Sigma_{ijk}\begin{vmatrix} Aii & Aij & Aik \\ Aji & Ajj & Ajk \\ Aki & Akj & Akk \end{vmatrix} + ...$$

Just comparing, we can see that our K_{ij} terms add up to

$$K_{ij}\ \{\ \det(1\text{-}K)\ \}$$

Now consider the third term in the 2x2 grouping. If we switch dummy indices k and m, the leading factor becomes K_{kj} which matches that of the second term. Within the determinant, this same swap causes the a replication of the determinant in the second term, but with the two columns swapped. Therefore, the second and third terms are equal!

Now let's examine the last three terms in the 3x3 grouping. Take the third term and swap km so the leading factor is that of the second term. This same swap causes the first column of the third term to equal the second column of the second term, and vice versa. And the third column becomes the third column. So again, we have a swap of two columns so the second and third terms are the same!

Now look at the fourth term. This time do a swap kn. The claim is that this also replicates the first term. Since this proof is overall pretty weak already, I will just assume this is true.

We can therefore write all the terms other than this in $K_{ij}\ \{\ \det(1\text{-}K)\ \}$ above as follows:

$$+ \Sigma_k K_{ik}K_{kj} - \Sigma_{km} K_{kj}\begin{vmatrix} Kik & Kim \\ Kmk & Kmm \end{vmatrix} + (1/2!)\,\Sigma_{kmn} K_{kj}\begin{vmatrix} Kik & Kim & Kin \\ Kmk & Kmm & Kmn \\ Knk & Knm & Knn \end{vmatrix} - ...$$

$$= \Sigma_k K_{kj}\ \{\ K_{ik} - \Sigma_m\begin{vmatrix} Kik & Kim \\ Kmk & Kmm \end{vmatrix} + (1/2!)\,\Sigma_{mn}\begin{vmatrix} Kik & Kim & Kin \\ Kmk & Kmm & Kmn \\ Knk & Knm & Knn \end{vmatrix} - ...\ \}$$

Now let's go back to our assumed form for N_{ij}

$$N_{ij} = K_{ij} - \Sigma_k\begin{vmatrix} Kij & Kik \\ Kkj & Kkk \end{vmatrix} + (1/2!)\,\Sigma_{km}\begin{vmatrix} Kij & Kik & Kim \\ Kkj & Kkk & Kkm \\ Kmj & Kmk & Kmm \end{vmatrix}$$

Let's first replace dummy summation index k with m

$$N_{ij} = K_{ij} - \Sigma_m\begin{vmatrix} Kij & Kim \\ Kmj & Kmm \end{vmatrix} + (1/2!)\,\Sigma_{nm}\begin{vmatrix} Kij & Kin & Kim \\ Knj & Knn & Knm \\ Kmj & Kmn & Kmm \end{vmatrix}$$

Now replace j with k,

$$N_{ik} = K_{ik} - \Sigma_m \begin{vmatrix} Kik & Kim \\ Kmk & Kmm \end{vmatrix} + (1/2!)\,\Sigma_{nm} \begin{vmatrix} Kik & Kin & Kim \\ Knk & Knn & Knm \\ Kmk & Kmn & Kmm \end{vmatrix}$$

Now swap indices n and m in the last sum.

$$N_{ik} = K_{ik} - \Sigma_m \begin{vmatrix} Kik & Kim \\ Kmk & Kmm \end{vmatrix} + (1/2!)\,\Sigma_{nm} \begin{vmatrix} Kik & Kim & Kin \\ Kmk & Kmm & Kmn \\ Knk & Knm & Knn \end{vmatrix}$$

and amazingly enough, this is what appears in our {...} above, so we say that all those "other terms" add up to this:

$$= \Sigma_k K_{kj} N_{ik}$$

Thus, we have shown now that

$$N_{ij} = K_{ij} \{ \det(1-K) \} + \Sigma_k N_{ik} K_{kj}$$

which we can write as a matrix equation

$$N = K\,|1-K| + NK$$

or

$$(N/|1-K|) = K + (N/|1-K|)K \qquad (*)$$

Now look at our resolvent equation from the first section above with $\lambda = 1$

$$\Gamma = K + \Gamma K$$

We know there is a unique resolvent that solves this equation (it is the power series above). This same power series must be the solution of equation (*) above by the same argument. Thus

$$\Gamma = N/|1-K|$$

Thus, we have obtained a strange expansion for the resolvent which is this:

$$\Gamma = N/D$$

$$N_{ij} = K_{ij} - \Sigma_k \begin{vmatrix} Kij & Kik \\ Kkj & Kkk \end{vmatrix} + (1/2!)\,\Sigma_{km} \begin{vmatrix} Kij & Kik & Kim \\ Kkj & Kkk & Kkm \\ Kmj & Kmk & Kmm \end{vmatrix} - (1/3!)\,\Sigma_{kmn} \begin{vmatrix} Kij & Kik & Kim & Kin \\ Kkj & Kkk & Kkm & Kkn \\ Kmj & Kmk & Kmm & Kmn \\ Knj & Knk & Knm & Knn \end{vmatrix}$$

$$D = \det(1-K) = 1 - \Sigma_i K_{ii} + (1/2)\Sigma_{ij} \begin{vmatrix} Kii & Kij \\ Kji & Kjj \end{vmatrix} - (1/6)\Sigma_{ijk} \begin{vmatrix} Kii & Kij & Kik \\ Kji & Kjj & Kjk \\ Kki & Kkj & Kkk \end{vmatrix} + \ldots$$

At this point, replace K with λK and we find,

$$N_{ij}(\lambda)/\lambda = K_{ij} - \lambda\Sigma_k \begin{vmatrix} Kij & Kik \\ Kkj & Kkk \end{vmatrix} +$$

$$(\lambda^2/2!)\,\Sigma_{km} \begin{vmatrix} Kij & Kik & Kim \\ Kkj & Kkk & Kkm \\ Kmj & Kmk & Kmm \end{vmatrix} - (\lambda^3/3!)\,\Sigma_{kmn} \begin{vmatrix} Kij & Kik & Kim & Kin \\ Kkj & Kkk & Kkm & Kkn \\ Kmj & Kmk & Kmm & Kmn \\ Knj & Knk & Knm & Knn \end{vmatrix} + \ldots.$$

$$D(\lambda) = \det(1-\lambda K) = 1 - \lambda\Sigma_i K_{ii} + (\lambda^2/2)\Sigma_{ij} \begin{vmatrix} Kii & Kij \\ Kji & Kjj \end{vmatrix} - (\lambda^3/6)\Sigma_{ijk} \begin{vmatrix} Kii & Kij & Kik \\ Kji & Kjj & Kjk \\ Kki & Kkj & Kkk \end{vmatrix} + \ldots$$

$D(\lambda)$ is called the **Fredholm Determinant.**

$N_{ij}(\lambda)/\lambda$ is called **Fredholm's First Minor**

$[\,\Gamma_{ij}(\lambda)/\lambda] = [\,N_{ij}(\lambda)/\lambda] \;/\; D(\lambda)$

Now let's go back a ways where we wrote

$$(1 - \lambda K)^{-1} = [\,\mathrm{cof}\,(1-\lambda K)\,]^T / D(\lambda) = 1 + \Gamma(\lambda)$$

$$(1 - \lambda K)^{-1}{}_{ij} = [\,\mathrm{cof}\,(1-\lambda K)\,]^T{}_{ij} / D(\lambda) = 1_{ij} + \Gamma(\lambda)_{ij} = 1_{ij} + [\,N_{ij}(\lambda)] \;/\; D(\lambda)$$

so

$$[\,\mathrm{cof}\,(1-\lambda K)\,]^T{}_{ij} = D(\lambda)\, 1_{ij} + [\,N_{ij}(\lambda)]$$

$$[\,N_{ij}(\lambda)] = [\,\mathrm{cof}\,(1-\lambda K)\,]^T{}_{ij} - D(\lambda)\, 1_{ij}$$

$$N(\lambda) = [\,\mathrm{cof}\,(1-\lambda K)\,]^T - D(\lambda)\, 1$$

So this thing N is very *close* to being the cofactor matrix which we associate with a "minor" according to these definitions,

minor$(\mathbf{a_{pq}})$ = det of the N-1 x N-1 matrix you get by crossing out row p and column q.

cof$(\mathbf{a_{pq}})$ $= (-1)^{p+q}$ minor(a_{pq})

$|A|\,(A^{-1})_{pq} = \mathrm{cof}\,(A^T{}_{pq}) = \mathrm{cof}\,(A_{qp}) = (-1)^{p+q}\,\mathrm{minor}(A_{pq})$

In fact, for off-diagonal elements, we do have $N_{ij}(\lambda)/\lambda = [\,\mathrm{cof}\,(1-\lambda K)\,]^T{}_{ij}$, but for the diagonal elements there is the extra term $D(\lambda)$. THAT is the fact that has been causing me confusion.

Taking the ∞ Limit

We already know our Fredholm determinant limit:

$$D(\lambda) \equiv \det(1-\lambda K) = 1 - \lambda \int dx\, k(x,x) + (\lambda^2/2) \int dx\, dy \begin{vmatrix} k(x,x) & k(x,y) \\ k(y,x) & k(y,y) \end{vmatrix} - \ldots$$

For our N object, the limit is done this way:

$$N_{ij}(\lambda)/\lambda = K_{ij} - \lambda \Sigma_k \begin{vmatrix} Kij & Kik \\ Kkj & Kkk \end{vmatrix} + (\lambda^2/2!) \Sigma_{km} \begin{vmatrix} Kij & Kik & Kim \\ Kkj & Kkk & Kkm \\ Kmj & Kmk & Kmm \end{vmatrix} - \ldots$$

$$N(x,y;\lambda)/\lambda = k(x,y) - \lambda \int ds_1 \begin{vmatrix} k(x,y) & k(x,s1) \\ k(s1,y) & k(s1,s1) \end{vmatrix} + (\lambda^2/2!) \int ds_1 ds_2 \begin{vmatrix} k(x,y) & k(x,s1) & k(x,s2) \\ k(s1,y) & k(s1,s1) & k(s1,s2) \\ k(s2,y) & k(s2,s1) & k(s1,s2) \end{vmatrix}$$

The following definition is often made:

$$D(x,y;\lambda) \equiv N(x,y;\lambda)/\lambda$$

And then the resolvent can be written as:

$$[\Gamma_{ij}(\lambda)/\lambda] = [N_{ij}(\lambda)/\lambda] / D(\lambda)$$

$$\Gamma(x,y;\lambda)/\lambda = D(x,y;\lambda) / D(\lambda)$$

I have had some problem with this 1/λ sitting on the left given our usual definition of the resolvent Γ. I think I have it right, and I will now find support from web sources.

(1) First, consider http://books.google.com/books?id=ZPeNgv4OPl8C where we have these results: (this is a Math Methods book by John Dettman)

$$\bar{\psi}(\xi) = F(\xi) + \lambda \int_a^b K(x;\xi)\bar{\psi}(x)\,dx \qquad \bar{\psi}(\xi) = F(\xi) + \lambda \int_a^b k(x,\xi,\lambda)F(x)\,dx$$

The first shows this source's names for the driving function I called f, and the solution I call φ. Now, consider the object he calls k(x,ξ,λ) in the second integrand. In my terms as shown above, I would be writing this equation as φ = f + Γf. Thus, we make the connection Γ(x,y;λ) = λ k(x,y,λ) . Then our source makes this statement,

$$k(\xi,x,\lambda) = \frac{D_1(\xi,x,\lambda)}{D(\lambda)}$$

where $D_1(x,y;\lambda)$ is the thing I call D(x,y; λ) above, Fredholm's First Minor. Therefore, this author saying the following:

$$\Gamma(x,y;\lambda) = \lambda\, k(x,y,\lambda) = \lambda\, D_1/D$$

which says

$$\Gamma(x,y;\lambda)/\lambda = D_1/D$$

which supports my claimed result above.

(2) Here is another source http://books.google.com/books?id=WHjO9K6xEm4C which fudges this issue and does not show the λ, but really slyly changes the definition of Γ in midstream. But the definitions of the basic objects are correct and are presented in this manner:

$$K\begin{pmatrix} x_1,\ldots,x_n \\ y_1,\ldots,y_n \end{pmatrix} = \begin{vmatrix} K(x_1,y_1) & \ldots & K(x_1,y_n) \\ & \ldots & \\ K(x_n,y_1) & \ldots & K(x_n,y_n) \end{vmatrix}.$$

Define $D(\lambda)$ and $D(x,y;\lambda)$ by

$$D(\lambda) =$$

$$1 + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_D \ldots \int_D K\begin{pmatrix} s_1,\ldots,s_n \\ s_1,\ldots,s_n \end{pmatrix} ds_1 \ldots ds_n$$

and

$$D(x,y;\lambda) = K(x,y)$$

$$+ \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_D \ldots \int_D K\begin{pmatrix} x,s_1,\ldots,s_n \\ y,s_1,\ldots,s_n \end{pmatrix} ds_1 \ldots ds_n.$$

Attempting a better proof of the expansion for N

I like my notation from section 4 of the Fredholm Determinant doc Overview, which says:

$$\det_{I1\ I2\ I3\ldots In;\ a1\ a2\ a3\ldots an;} = \varepsilon^{i1\ i2\ i3\ldots in} A_{I1\ i1}\, A_{I2\ i2} \ldots\ldots A_{In\ in} \qquad \text{// n factors of A}$$

where i1 i2 i3....in ∈ {a1 a2 a3...an} , $\varepsilon^{i1\ i2\ i3\ldots in}$ anti-symmetric, $\varepsilon^{I1\ I2\ I3\ldots In} = +1$

Now look at the N expansion which we know works, from above, where we have det(1-K) but we have λ = 1 so no λ's appear anywhere

$$\Gamma = N/\,|1\text{-}K|$$

$$N_{ij} = K_{ij} - \Sigma_k \begin{vmatrix} Kij & Kik \\ Kkj & Kkk \end{vmatrix} + (1/2!)\,\Sigma_{km} \begin{vmatrix} Kij & Kik & Kim \\ Kkj & Kkk & Kkm \\ Kmj & Kmk & Kmm \end{vmatrix} - (1/3!)\,\Sigma_{kmn} \begin{vmatrix} Kij & Kik & Kim & Kin \\ Kkj & Kkk & Kkm & Kkn \\ Kmj & Kmk & Kmm & Kmn \\ Knj & Knk & Knm & Knn \end{vmatrix}$$

I claim that, using my notation above, we have

$$\begin{vmatrix} Kij & Kik \\ Kkj & Kkk \end{vmatrix} = \det_{\mathtt{ik;jk}}$$

$$\begin{vmatrix} Kij & Kik & Kim \\ Kkj & Kkk & Kkm \\ Kmj & Kmk & Kmm \end{vmatrix} = \det_{\mathtt{ikm;jkm}}$$

$$\begin{vmatrix} Kij & Kik & Kim & Kin \\ Kkj & Kkk & Kkm & Kkn \\ Kmj & Kmk & Kmm & Kmn \\ Knj & Knk & Knm & Knn \end{vmatrix} = \det_{\mathtt{ikmn;jkmn}}$$

So the formula is this:

$N_{\mathtt{ij}} = K_{\mathtt{ij}} - \Sigma_{\mathtt{k}} \det_{\mathtt{ik;jk}} + (1/2!) \Sigma_{\mathtt{km}} \det_{\mathtt{ikm;jkm}} - (1/3!) \Sigma_{\mathtt{kmn}} \det_{\mathtt{ikmn;jkmn}} + \text{etc.}$

In each grouping of index labels, we are summing over all the last indices, only the first are not summed. This ought to help me somehow.

What does the "first column expansion theorem" look like in this notation? (!) Let's do this to the 4x4 showing above:

$$\det_{\mathtt{ikmn;jkmn}} = K_{\mathtt{ij}}\det_{\mathtt{kmn;kmn}} - K_{\mathtt{kj}} \det_{\mathtt{imn;kmn}} + K_{\mathtt{mj}}\det_{\mathtt{ikn;kmn}} - K_{\mathtt{nj}}\det_{\mathtt{ikm;kmn}}$$

delete i delete k delete m delete n

<u>First column expansion theorem rules:</u>
(1) alternate signs as usual
(2) the $K_{\mathtt{ab}}$ factors as follows:
first index sequences through row group, second index is always first of col group
(3) on the new dets, the col group is always the same: last of the original col group.
(4) on the new dets, the row group is obtained by deleting, one at a time, one digit from the original row group. I have written in "delete k" for the second term and others to illustrate

Now throw in the sum on the last indices:

$\Sigma_{\mathtt{kmn}} \det_{\mathtt{ikmn;jkmn}}$

$$= \Sigma_{\mathtt{kmn}} K_{\mathtt{ij}}\det_{\mathtt{kmn;kmn}} - \Sigma_{\mathtt{kmn}} K_{\mathtt{kj}} \det_{\mathtt{imn;kmn}} + \Sigma_{\mathtt{kmn}} K_{\mathtt{mj}}\det_{\mathtt{ikn;kmn}} - \Sigma_{\mathtt{kmn}} K_{\mathtt{nj}}\det_{\mathtt{ikm;kmn}}$$

1 2 3 4

Take term 3 and in the term (not the summation symbol) cycle kmn→nkm to get

$$T3 = \Sigma_{kmn}\, K_{mj} det_{ikn;kmn}$$
$$= \Sigma_{kmn}\, K_{kj} det_{inm;nkm}$$

Now each adjacent label swap makes a minus (swapping two rows or two columns)

$$det_{inm;nkm} = - det_{imn;kmn} \quad \Rightarrow \; T3 = T2 \; !$$

Take term 4 and cycle kmn→mnk

$$T4 = - \Sigma_{kmn}\, K_{nj} det_{ikm;kmn}$$
$$= - \Sigma_{kmn}\, K_{kj} det_{imn;mnk}$$

$$det_{imn;mnk} = + det_{imn;kmn} \quad \Rightarrow T4 = T2$$

Thus we get this simplification:

$$\Sigma_{kmn}\, det_{ikmn;jkmn} = K_{ij}\, \Sigma_{kmn} det_{kmn;kmn} - 3\, \Sigma_{kmn}\, K_{kj}\, det_{imn;kmn}$$

Now let's "do another one": follow the expansion theorem rules to get:

$$det_{ikm;jkm} = K_{ij} det_{km,km} - K_{kj} det_{im,km} + K_{mj} det_{ik,km}$$

Add sum and make conjecture:

$$\Sigma_{km}\, det_{ikm;jkm} = K_{ij}\, \Sigma_{km}\, det_{km,km} - 2\, \Sigma_{km}\, K_{kj} det_{im,km}$$

And let's do another one:

$$det_{ik;jk} = K_{ij} K_{kk} - K_{ik} K_{kj}$$

$$\Sigma_k\, det_{ik;jk} = K_{ij}\, \Sigma_k K_{kk} - \Sigma_k\, K_{ik} K_{kj}$$

Put all these pieces together to get:

$$N_{ij} = K_{ij}$$
$$- K_{ij}\, \Sigma_k K_{kk} + \Sigma_k\, K_{ik} K_{kj}$$
$$+ 1/2 \; \{ K_{ij}\, \Sigma_{km}\, det_{km,km} - 2\, \Sigma_{km}\, K_{kj} det_{im,km} \}$$
$$- 1/6 \; \{ K_{ij}\, \Sigma_{kmn} det_{kmn;kmn} - 3\, \Sigma_{kmn}\, K_{kj}\, det_{imn;kmn} \}$$
$$= K_{ij} \, [\, 1 - \Sigma_k K_{kk} + 1/2! \, \Sigma_{km}\, det_{km,km} - 1/3! \, \Sigma_{kmn} det_{kmn;kmn} + ... \,]$$

$+ \Sigma_k K_{kj}K_{ik} - \Sigma_{km} K_{kj}\det_{im,km} + 1/2 \Sigma_{kmn} K_{kj} \det_{imn;kmn} - ...$

$= K_{ij} [1 - \Sigma_k K_{kk} + 1/2! \Sigma_{km} \det_{km,km} - 1/3! \Sigma_{kmn}\det_{kmn;kmn} + ...]$

$+ \Sigma_k [K_{ik} - \Sigma_m \det_{im,km} + 1/2 \Sigma_{mn} \det_{imn;kmn} - ...] K_{kj}$

The first square bracket is exactly det(1-K). The second square bracket is N_{ik}. So we get

$N_{ij} = K_{ij} \det(1-K) + \Sigma_k N_{ik} K_{kj}$

$N = \det(1-K) K + NK$

This is the crucial result. Is there some easier way to get this equation? You can write it as

$N(1-K) = K \det(1-K)$

$N = K (1-K)^{-1} \det(1-K) = K \text{cof}^T(1-K)$ // "minor like"

$N = K (1+\Gamma) \det(1-K) = (K + K\Gamma) \det(1-K) = \Gamma \det(1-K) \quad => \quad \Gamma = N/\det(1-K)$

Here is an interesting way to write N

$N_{ij} = K_{ij} + \Sigma_{n=1}^{\infty} (-1)^n / n! * [\Sigma_{k1,k2...kn} \det_{i\ k1\ k2...kn ; \ j\ k1\ k2...kn}]$

This let's us apply our "first column theorem" to a single det:

$\det_{i\ k1\ k2...kn ; \ j\ k1\ k2...kn} =$

$K_{ij} \det_{k1\ k2...kn ; \ k1\ k2...kn}$ + other terms

Let's ignore these other terms for the moment and look at what we get from the first term:

$K_{ij} \det_{k1\ k2...kn ; \ k1\ k2...kn} = K_{ij} \det_{k1\ k2...kn}$

$N_{ij} = K_{ij} + \Sigma_{n=1}^{\infty} (-1)^n / n! [\Sigma_{k1,k2...kn} K_{ij} \det_{k1\ k2...kn}]$

$= K_{ij} \{ 1 + \Sigma_{n=1}^{\infty} (-1)^n / n! \Sigma_{k1,k2...kn} \det_{k1\ k2...kn} \}$

The n=1 term is just $- \Sigma_{k1} \det_{k1} = - \Sigma_{k1} K_{k1,k1} = - \text{tr}(K)$. So

$\{\} = 1 - \text{tr}(K) + \Sigma_{n=2}^{\infty} (-1)^n / n! \Sigma_{k1,k2...kn} \det_{k1\ k2...kn}$

and this is a nice way to express det(1-K) I think. Thus, we have now verified this det(1-K) part of our answer "to all orders", which is a step forward.

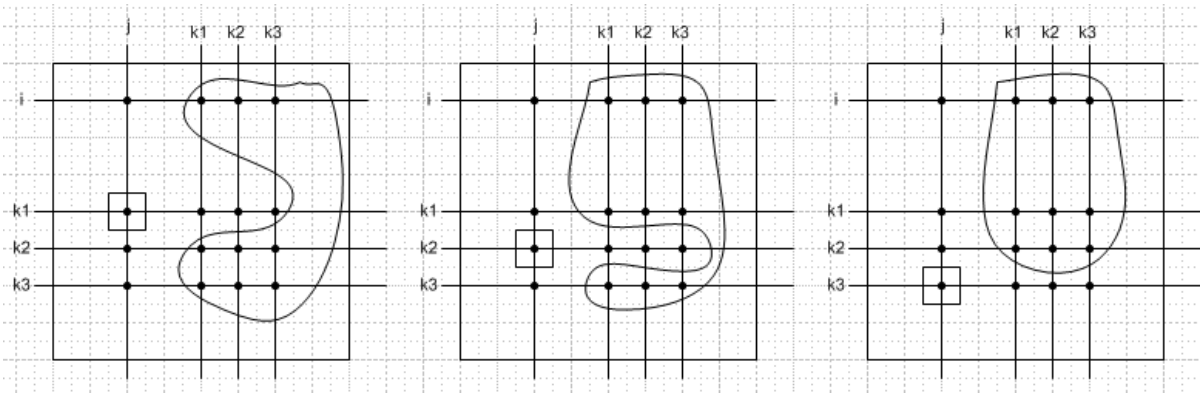
Now back to those "other terms" which are not as easy. The big question is: why are all the other terms equal to each other? To answer this, consider an n=3 set of terms in our sum for N_{ij} :

$$N_{ij} = K_{ij} + \Sigma_{n=1}^{\infty} (-1)^n / n! * [\ \Sigma_{k1,k2...kn}\ \det_{i\ k1\ k2...kn\ ;\ j\ k1\ k2...kn}\ ]$$

n=3 term: $(-1)^3 / 3!\ \Sigma_{k1,k2,k3}\ \det_{i\ k1\ k2\ k3\ ;\ j\ k1\ k2\ k3}$

This is a sum of a set of 4x4 subdeterminants. Imagine that we "work our way" down the first column doing our expansion thing, and we have already treated the top item in the column above, and here we are interested in the three remaining terms. Here is a picture (!) showing the situation:

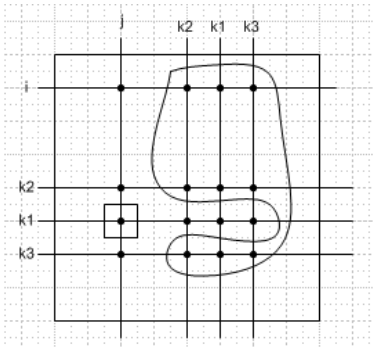
$-K_{k1,j}\ \det_{i\ k2\ k3;\ k1\ k2\ k3}$ + $K_{k2,j}\ \det_{i\ k1\ k3;\ k1\ k2\ k3}$ $-K_{k3,j}\ \det_{i\ k1\ k2;\ k1\ k2\ k3}$



The three pictures show the terms we get marching down the first column. But remember that we are summing over all possible values of k_1, k_2 and k_3 and we are just showing a particular possible position these three indices could have somewhere in the sum. The position i,j is fixed.

What we need now are "the magic words" and here they are.

Focus on the leftmost picture. Somewhere in the $\Sigma_{k1,k2,k3}$ triple sum, the "k bars" will take the following positions which are different from the positions shown top left. We have the k1 and k2 bars swapped:



We still have our "selection box" on the k1,j element, and we still are encircling the exact same 9 intersection points. That is to say, we still have rows i, k2 and k3 contributing, the same as in the leftmost

of our triple figure above. We have to draw our encirclement appropriately to capture these same 9 elements. [but I could not make these pictures do anything for me!]

********** try again **********

Reminder of our rule from above:

First column expansion theorem rules:
(1) alternate signs as usual
(2) the K_{ab} factors as follows:
first index sequences through row group, second index is always first of col group
(3) on the new dets, the col group is always the same: last of the original col group.
(4) on the new dets, the row group is obtained by deleting, one at a time, one digit from the original row group. I have written in "delete k" for the second term and others to illustrate

Now let's go down the column omitting the first term which we have already dealt with. We assume some generic case here with some largish value of n. So here we are treating all those "other terms" all together.

$\det_{\mathbf{i\ k1\ k2...kn\ ;\ j\ k1\ k2...kn}}$

$- K_{\mathbf{k1,j}}$ det **i ** k2 k3 k4 k5...kn ; k1 k2 k3...kn** **1**

$+ K_{\mathbf{k2,j}}$ det **i k1 ** k3 k4 k5...kn ; k1 k2 k3...kn** **2**

$- K_{\mathbf{k3,j}}$ det **i k1 k2 ** k4 k5...kn ; k1 k2 k3...kn** **3**

$+ K_{\mathbf{k4,j}}$ det **i k1 k2 k3 ** k5...kn ; k1 k2 k3...kn** **4**

$- K_{\mathbf{k5,j}}$ det **i k1 k2 k3 k4 **...kn ; k1 k2 k3...kn** **5**

Here we have shown 5 terms just to have a good sample so we can see what is happening. Each term has a complete summation $\Sigma_{\mathbf{k1\ k2\ k3....kn}}$ but we are suppressing this summation symbol. We mark the "deleted" row group index with **.

Now let's swap two k summation indices on each of lines 2,3,4,5 such that the leading K factor is always $K_{\mathbf{k1},j}$. Then just rewrite:

$\det_{\mathbf{i\ k1\ k2...kn\ ;\ j\ k1\ k2...kn}}$

			What we did
$- K_{\mathbf{k1,j}}$	det **i ** k2 k3 k4 k5...kn ; k1 k2 k3 k4 k5...kn**	**1**	
$+ K_{\mathbf{k1,j}}$	det **i k2 ** k3 k4 k5...kn ; k2 k1 k3 k4 k5...kn**	**2**	2 ↔ 1
$- K_{\mathbf{k1,j}}$	det **i k3 k2 ** k4 k5...kn ; k3 k2 k1 k4 k5...kn**	**3**	3 ↔ 1
$+ K_{\mathbf{k1,j}}$	det **i k4 k2 k3 ** k5...kn ; k4 k2 k3 k1 k5...kn**	**4**	4 ↔ 1
$- K_{\mathbf{k1,j}}$	det **i k5 k2 k3 k4 **...kn ; k5 k2 k3 k4 k1...kn**	**5**	5 ↔ 1

Notice that these swaps affect both the row group and the column group. In the column group we actually do a swap. In the row group, one of the two swapees is missing, so we just do 1→2, 1→3 etc. so it is only the first k index of the row group that is affected.

At this point we are still free to relabel the summation indices k2,k3...kn any way we want. So let's do that now:

$\det_{\mathbf{i\ k1\ k2...kn\ ;\ j\ k1\ k2...kn}}$

	line #	What we did
$- K_{\mathbf{k1,j}}$ det **i ** k2 k3 k4 k5...kn ; k1 k2 k3 k4 k5...kn**	**1**	
$+ K_{\mathbf{k1,j}}$ det **i k2 ** k3 k4 k5...kn ; k2 k1 k3 k4 k5...kn**	**2**	**nothing**
$- K_{\mathbf{k1,j}}$ det **i k2 k3 ** k4 k5...kn ; k2 k3 k1 k4 k5...kn**	**3**	k2,k3→k3.k2
$+ K_{\mathbf{k1,j}}$ det **i k2 k3 k4 ** k5...kn ; k2 k3 k4 k1 k5...kn**	**4**	k4,k2,k3→ k2,k3,k4
$- K_{\mathbf{k1,j}}$ det **i k2 k3 k4 k5 **...kn ; k2 k3 k4 k5 k1...kn**	**5**	5,2,3,4 → 2,3,4,5

At this point, the row groups are all the same! To make the column groups the same, we have to shift k1 to the left a number of places equal to line number - 1. The even numbered lines like 2 and 4 pick up a minus sign because we do an odd number of adjacent swaps to achieve the required shift. The odd numbered lines maintain their sign. Doing this, we get

$\det_{\mathbf{i\ k1\ k2...kn\ ;\ j\ k1\ k2...kn}}$

	line #
$- K_{\mathbf{k1,j}}$ det **i ** k2 k3 k4 k5...kn ; k1 k2 k3 k4 k5...kn**	**1**
$- K_{\mathbf{k1,j}}$ det **i k2 ** k3 k4 k5...kn ; k1 k2 k3 k4 k5...kn**	**2**
$- K_{\mathbf{k1,j}}$ det **i k2 k3 ** k4 k5...kn ; k1 k2 k3 k4 k5...kn**	**3**
$- K_{\mathbf{k1,j}}$ det **i k2 k3 k4 ** k5...kn ; k1 k2 k3 k4 k5...kn**	**4**
$- K_{\mathbf{k1,j}}$ det **i k2 k3 k4 k5 **...kn ; k1 k2 k3 k4 k5...kn**	**5**

Now all lines are exactly the same. It seems pretty clear that this proof method extends to however many "lines" there are when we do our column expansion. In our example here our starting subdeterminant dimension was n+1 so there will be n "lines" which are all equal. Thus we can just represent all these lines as n times the first line. So we would say

$\det_{\mathbf{i\ k1\ k2...kn\ ;\ j\ k1\ k2...kn}}$

$= - n * K_{\mathbf{k1,j}}\ \det_{\mathbf{i\ k2\ k3...kn;\ k1\ k2\ k3...kn}}$

Now go back to our original form

$$N_{ij} = K_{ij} + \Sigma_{n=1}^{\infty} (-1)^n / n! * [\, \Sigma_{k1,k2\ldots kn}\ \det_{i\ k1\ k2\ldots kn\ ;\ j\ k1\ k2\ldots kn}\,]$$

The "other terms" contribution to the RHS is going to be this:

$$\Sigma_{n=1}^{\infty} (-1)^n / n!\ \Sigma_{k1,k2\ldots kn} \{\, \det_{i\ k1\ k2\ldots kn\ ;\ j\ k1\ k2\ldots kn} \}$$

$$= \Sigma_{n=1}^{\infty} (-1)^n / n!\ \Sigma_{k1,k2\ldots kn} \{ -n * K_{k1,j}\ \det_{i\ k2\ k3\ldots kn;\ k1\ k2\ k3\ldots kn} \}$$

$$= \Sigma_{n=1}^{\infty} (-1)^{n-1} / (n-1)!\ \Sigma_{k1} [\Sigma_{k2\ldots kn}\ \det_{i\ k2\ k3\ldots kn;\ k1\ k2\ k3\ldots kn}]\ K_{k1,j}$$

$$= \Sigma_{n=1}^{\infty} (-1)^{n-1} / (n-1)!\ \Sigma_{k} [\Sigma_{k2\ldots kn}\ \det_{i\ k2\ k3\ldots kn;\ k\ k2\ k3\ldots kn}]\ K_{k,j}$$

$$= \Sigma_{k} \{ \Sigma_{n=1}^{\infty} (-1)^{n-1} / (n-1)! [\Sigma_{k2\ldots kn}\ \det_{i\ k2\ k3\ldots kn;\ k\ k2\ k3\ldots kn}] \}\ K_{k,j}$$

$$= \Sigma_{k} \{ \Sigma_{n=1}^{\infty} (-1)^{n-1} / (n-1)! [\Sigma_{k1\ldots kn-1}\ \det_{i\ k1\ k2\ldots kn-1;\ k\ k1\ k2\ldots kn-1}] \}\ K_{k,j}$$

Consider now the object in {...} :

$$\Sigma_{n=1}^{\infty} (-1)^{n-1} / (n-1)! [\Sigma_{k1\ldots kn-1}\ \det_{i\ k1\ k2\ldots kn-1;\ k\ k1\ k2\ldots kn-1}]$$

Let n-1 = m to write this as

$$\Sigma_{m=0}^{\infty} (-1)^m / (m)! [\Sigma_{k1\ldots km}\ \det_{i\ k1\ k2\ldots km;\ k\ k1\ k2\ldots km}]$$

$$= \det_{ik} + \Sigma_{m=1}^{\infty} (-1)^m / (m)! [\Sigma_{k1\ldots km}\ \det_{i\ k1\ k2\ldots km;\ k\ k1\ k2\ldots km}]$$

$$= K_{ik} + \Sigma_{m=1}^{\infty} (-1)^m / (m)! [\Sigma_{k1\ldots km}\ \det_{i\ k1\ k2\ldots km;\ k\ k1\ k2\ldots km}]$$

$$= N_{ik} \qquad !!!$$

Thus we have shown that

$$\text{"other terms"} = \Sigma_k N_{ik} K_{kj}$$

$$N_{ij} = \det(1-K)\ K_{ij} + \Sigma_k N_{ik} K_{kj}$$

which is our desired matrix equation

$$N = \det(1-K)\ K + NK.$$