Fredholm's First Minor, Round 2 PhL 3.28.09

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This first section is a repeat of what appears at the start of Fredholm Determinant Round 2.

1. Preliminaries from Integral Equation Theory

Our Neumann form integral equation is this

$$\varphi = f + \lambda K \varphi$$

and the solution can be expressed in this manner:

$$\varphi = (1 - \lambda K)^{-1} f. = (1 + \lambda K + \lambda^2 K^2 + ...)\, f = f + \Sigma_{m=1} \lambda^m K^m f = (1 + \Gamma) f$$

All we did was use the high school algebra formula for 1/(1-x) as a series, and then defined this object

$$\Gamma = \Sigma_{m=1} \lambda^m K^m \qquad = \text{"the resolvent"}$$

The first few terms here are $\Gamma \approx \lambda K + \lambda^2 K^2$ so if $\lambda << 1$, we have $\Gamma \approx \lambda K$ so Γ is of order λ in general.

In all this, I like to think of K as a finite dimensional matrix and take the ∞ limit later on. So the object K^m is just the product of K matrices multiplied together. Looking at the above, we of course have

$$\varphi = f + \Gamma f$$

$$(1 - \lambda K)^{-1} = 1 + \Gamma$$

from which last we quickly conclude that

$$\Gamma = \lambda K + \lambda K \Gamma = \lambda K + \lambda \Gamma K$$

We also know the following fact from very elementary matrix theory,

$$(1 - \lambda K)^{-1} = [\ \text{cof}\ (1 - \lambda K)\ ]^T / \det(1 - \lambda K) = 1 + \Gamma$$

2. Definition of Fredholm's First Minor

In the previous section we found that the resolvent can be written as:

$$\Gamma = (1 - \lambda K)^{-1} - 1 = [\ \mathrm{cof}\ (1-\lambda K)\ ]^{T} / \det(\ 1 - \lambda K) - 1$$

which we regard as a matrix equation with these matrix elements

$$\Gamma_{ij} = (1 - \lambda K)^{-1}_{ij} - \delta_{i,j} = [\mathrm{cof}\ (1-\lambda K)]_{ji} / \det(\ 1 - \lambda K) - \delta_{i,j}$$

The RHS of this equation can be written as

$$\{\ [\mathrm{cof}\ (1-\lambda K)]_{ji} - \det(\ 1 - \lambda K)\ \delta_{i,j}\ \} / \det(1-\lambda K)$$

and it is traditional to define the quantity in {..} as N_{ij},

$$N_{ij} \equiv [\mathrm{cof}\ (1-\lambda K)]_{ji} - \det(\ 1 - \lambda K)\ \delta_{i,j}$$

Thus, we have

$$\Gamma_{ij} = N_{ij} / \det(1-\lambda K)$$

or

$$[\Gamma_{ij}/\lambda] = [N_{ij}/\lambda] / \det(1-\lambda K) \qquad (*)$$

We now assign these names:

$[N_{ij}/\lambda]$ ≡ **Fredholm's First Minor**,

$\det(1-\lambda K)$ ≡ **The Fredholm Determinant**.

There may be some ambiguity in the definition of the resolvent. Here are two choices

Γ_{ij} = "**the resolvent**" and is order λ^{1}

Γ_{ij}/λ = "**the resolvent**" and is order λ^{0}

In the continuum notation we rewrite (*) as

$$[\Gamma(x,y;\lambda) /\lambda] = [N(x,y;\lambda) /\lambda]/ \det(1-\lambda K)$$

where the fact that Γ and N depend on λ is shown explicitly. A standard notation people use is this:

$D_1(x,y; \lambda) \equiv [N(x,y;\lambda) /\lambda]$ Fredholm's First Minor (some people erroneously drop /λ)

$D(\lambda) \equiv \det(1-\lambda K)$ The Fredholm Determinant

and then the resolvent is given as

$$\Gamma(x,y;\lambda)/\lambda \;=\; D_1(x,y;\lambda)/D(\lambda) \qquad \text{// some people erroneously drop } /\lambda$$

So, if we know D_1 and D, then we know the resolvent and our integral equation is solved according to $\varphi = f + \Gamma f$ as shown in Section 1 above.

Note Added: We shall learn below that

$$N_{ij} = \lambda K_{ij} - \lambda^2 \Sigma_k \det_{ik;jk} + (\lambda^3/2!) \Sigma_{ab} \det_{iab;jab} - \ldots$$

$$\Rightarrow\; [N_{ij}/\lambda] = K_{ij} - \lambda \Sigma_k \det_{ik;jk} + (\lambda^2/2!) \Sigma_{ab} \det_{iab;jab} - \ldots$$

We already saw in Section 1 that $\Gamma_{ij} \approx \lambda K_{ij}$ for $\lambda << 1$ and of course $\det(1-\lambda K) \approx 1$.
Thus, our equation above $[\Gamma_{ij}/\lambda] \;=\; [N_{ij}/\lambda] / \det(1-\lambda K)$ becomes $K_{ij} \approx K_{ij}$. This is just to show that we have all our λ's in the right place. Then D_1 and D are order λ^0 and that is why D_1 is defined as it is.

We already have an expansion for $D(\lambda)$ as a power series in λ which is useful in problems when λ is small. That expansion is this (we are now back in matrix notation for a long time)

$$D(\lambda) = \det(1 - \lambda K) = 1 + \Sigma_{n=1}^{N} (-\lambda)^n/n! \;\; \Sigma_{i1,i2,i3\ldots\, in=1}^{N} \det_{i1\; i2\; i3\ldots in}$$

which is derived in our document on Fredholm Determinant, Version 2.

Our goal is to find a corresponding power series expansion for $D_1(x,y;\lambda)$ which, in matrix notation, means we need a power series expansion in λ for N_{ij} . At first glance, this does not seem an immensely hard problem. We have

$$N_{ij} = [\text{cof}\,(1-\lambda K)]_{ji} - \det(1 - \lambda K)\,\delta_{i,j}$$

3. Attempting the Direct Method for finding N_{ij}

Our problem is simply to write N_{ij} as a power series in λ, where

$$N_{ij} = [\text{cof}\,(1-\lambda K)]_{ji} - \det(1 - \lambda K)\,\delta_{i,j}$$

It turns out that finding this power series in λ is a much harder problem than one would think. In our document on subdeterminants of a matrix, we know that

$$[\text{cof}\,(1-\lambda K)]_{ji} = (-1)^{i+j} \text{minor}(1-\lambda K)_{ji} \;=\; (-1)^{i+j} \det_{\text{rows}\neq j;\,\text{cols}\neq i}$$

where the det notation is the N-1 x N-1 subdeterminant you get by crossing out row j and column i.

It is convenient at this point to set $-\lambda K = B$ and then when we are done we can replace B by $-\lambda K$ to get our result. Thus, we are interested in

$$N_{ij} = [\text{cof}\,(1+B)]_{ji} - \det(1+B)\,\delta_{i,j}$$

$$[\text{cof}\,(1+B)]_{ji} = (-1)^{i+j}\,\text{minor}(1+B)_{ji} = (-1)^{i+j}\,\det_{\text{rows}\neq j;\,\text{cols}\neq i}$$

where $\det_{\text{rows}\neq j;\,\text{cols}\neq i}$ is the N-1 x N-1 subdeterminant of the matrix (1+B) where we have crossed out row j and column i. We assume of course that B is an NxN matrix.

It would *seem* that all we have to do is compute things like:

" all terms in $\det_{\text{rows}\neq j;\,\text{cols}\neq i}$ which are of order B^n "

that is, all terms which have n factors of B_{ab} in them, since these would become λ^n terms in our power series expansion.

With a large amount of painful effort, I was able to show that, if we write

$$\text{minor}(1+B)_{ji} = \det_{\text{rows}\neq j;\,\text{cols}\neq i} = B^0 \text{ term} + B^1 \text{ terms} + (B^2 \text{ and higher terms}),$$

then we get

B^0 term: $\delta_{i,j}$

B^1 terms: $\{B_{j,i}\,(-1)^{i+j+1}\}(1 - \delta_{i,j}) + \{\text{tr}(B) - B_{ii}\}\delta_{i,j}$

$= \text{tr}(B)\,\delta_{i,j} - (-1)^{i+j}\,B_{j,i}$

Meanwhile, if we expand det(1+B) and keep only terms of B^0 and B^1 order we get

$$\det(1+B) \approx 1 + \text{tr}(B)$$

as we well know by now. Using all this information we find that

$$N_{ij} = [\text{cof}\,(1+B)]_{ji} - \det(1+B)\,\delta_{i,j}$$

$$= (-1)^{i+j}\,\text{minor}(1+B)_{ji} - \det(1+B)\,\delta_{i,j}$$

$$\approx (-1)^{i+j}\,\{\,\delta_{i,j} + \text{tr}(B)\,\delta_{i,j} - (-1)^{i+j}\,B_{j,i}\,\} - \{1 + \text{tr}(B)\}\,\delta_{i,j}$$

$$= -B_{ik}$$

So the reward we get from all this work is this:

$N_{ij} = -B_{ij}$ + order(B^2) and higher.

One might wonder *why* these calculations are so hard (as claimed by your friendly writer). It seems to relate to the difficulty of dealing with $\det_{rows \neq j; cols \neq i}$ where the set "rows" is all integers 1 to N *except* the integer j, and "cols" is all integers 1 to N *except* the integer i. For each case of j and i, it almost seems to be a completely separate problem you have to go ponder.

For example, consider:

N = 6 j = 2 i = 4 want to know: $\det_{13456;\ 12356}$

We can use our general formula for subdeterminants which says

$$\det_{rows;cols} = \det_{r1\ r2..rn;\ c1\ c2..cn} = \Sigma_{i1,i2,..in \in \{c1\ c2\ ..cn\}}$$

$$(1+B)_{r1,i1}\,(1+B)_{r2,i2}\,(1+B)_{r3,i3}...\,(1+B)_{rn,in}\,\varepsilon^{i1\ i2\ ..in}$$

with $\varepsilon^{c1\ c2\ ..cn} = 1$ and rows,cols both assumed to be in natural order

This is not a "diagonal" subdeterminant as we encountered with the Fredholm Determinant. So we write this out as

$$\det_{13456;\ 12356} = \Sigma_{i1,i2,..i5 \in \{12356\}}$$

$$(1+B)_{1,i1}\,(1+B)_{3,i2}\,(1+B)_{4,i3}...\,(1+B)_{6,i5}\,\varepsilon^{i1\ i2\ ..i5} \qquad \text{with } \varepsilon^{12356} = 1$$

Then we have to partition all the terms into B^0 and B^1 and B^2 and B^3 and so on, which is a similar problem to what we faced in the Fredholm determinant situation, but worse since things are non-diagonal. Even if we do all this work, we have only obtained results for the N_{42}.

I hope the reader is at least convinced it is a messy approach. There is a better way. The problem is that the above method deals with the scalar object N_{ij} and we really need a method that deals with N as a matrix entity.

4. Using a recursion method to "discover" the nature of the N_{ij} series

We know from our earlier work above (see Sections 1 and 2) that

$$N = \det(1-\lambda K)\,\Gamma = \det(1-\lambda K)\,(\Gamma + 1) - \det(1-\lambda K)$$

$$= \det(1-\lambda K)\,(1-\lambda K)^{-1} - \det(1-\lambda K)$$

Therefore

$$[\,N + \det(1-\lambda K)\,]\,(1-\lambda K) = \det(1-\lambda K)$$

$$N - \lambda NK + \det(1-\lambda K)\,(1-\lambda K) = \det(1-\lambda K)$$

$$N - \lambda NK - \det(1-\lambda K)\lambda K = 0$$

This says that N must satisfy the above matrix equation. Let's write our series solution as

$$N = \Sigma_{n=0} M^{(n)} \lambda^n$$

where the $M^{(n)}$ are *matrices*. We then get

$$\Sigma_n M^{(n)} \lambda^n \; - \; \lambda K \;\det(1-\lambda K) \; - \Sigma_n M^{(n)} \lambda^{n+1} K \; = 0$$

or

$$\Sigma_{n=0} M^{(n)} \lambda^n \; - \; \lambda K \;\{ \Sigma_{n=0}^{N} \; (-\lambda)^n /n! \;\; \Sigma_{i1,i2,i3\ldots\; in=1}^{N} \; \det_{i1\;i2\;i3\ldots in} \} \; - \Sigma_{n=0} M^{(n)} \lambda^{n+1} K \; = 0$$

Rewrite the first sum as

$$\Sigma_{m=-1} \, M^{(m+1)} \lambda^{m+1} \; = \Sigma_{n=-1} \, M^{(n+1)} \lambda^{n+1} \; = M^{(0)} + \Sigma_{n=0} \, M^{(n+1)} \lambda^{n+1} \, .$$

Then we can combine the three sums on n to get

$$M^{(0)} + \Sigma_{n=0} \, \lambda^{n+1} \, \{ \, M^{(n+1)} - (-1)^n/n! \; \Sigma_{i1,i2,i3\ldots\; in=1}^{N} \; \det_{i1\;i2\;i3\ldots in} K - M^{(n)} K \} \; = 0$$

This tells us that $M^{(0)} = 0$ *and* that we have this recursion relation

$$M^{(n+1)} = \; M^{(n)} K + (-1)^n/n! \; * \; \Sigma_{i1,i2,i3\ldots\; in=1}^{N} \; \det_{i1\;i2\;i3\ldots in} K$$

<u>For n = 0</u> this says:

$$M^{(1)} = \; M^{(0)} K + \quad 1 \; K \; = 0K + K = K$$

<u>For n =1</u> we then get

$$M^{(2)} = \; M^{(1)} K - \; \Sigma_{i1=1}^{N} \; \det_{i1} \; = M^{(1)} K - K \; \mathrm{tr}(K) \; = K^2 - K\mathrm{tr}(K)$$

which says

$$M^{(2)}{}_{ij} = (K^2)_{ij} - K_{ij} \; \mathrm{tr}(K) \; = \Sigma_k \, K_{ik} K_{kj} \; - K_{ij} \, \Sigma_k \, K_{kk}$$

$$= \Sigma_k \{ \, K_{ik} K_{kj} - K_{ij} \, K_{kk} \, \} = - \, \Sigma_k \, \det_{ik;jk}$$

Thus, we have already found the first three terms of our series

$$N_{ij} = \Sigma_{n=0} M^{(n)}{}_{ij} \; \lambda^n \; = \; \lambda^0 \;\; 0 + \lambda^1 K_{ij} - \lambda^2 \, \Sigma_k \, \det_{ik;jk}$$

For n = 2 we get

$$M^{(3)} = M^{(2)}K + (-1)^2/2!\ \Sigma_{i1,i2=1}^{N}\ det_{i1\ i2}\ K$$

$$M^{(3)}{}_{ij} = \Sigma_k M^{(2)}{}_{ik}K_{kj} + (1/2)\ \Sigma_{i1,i2=1}^{N}\ det_{i1\ i2}\ K_{ij}$$

$$= -\Sigma_k\ \Sigma_a\ det_{ia;ka}\ K_{kj} + (1/2)\Sigma_{i1,i2=1}^{N}\ det_{i1\ i2}\ K_{ij}$$

$$= -\Sigma_{ab}\ det_{ia;ba}\ K_{bj} + (1/2)\ \Sigma_{ab}\ det_{ab;ab}\ K_{ij}$$

Now anticipating the result, let's consider $det_{iab;jab}$. If we "expand this going down the first column" we get the following: (later we present a general formula for writing out this expansion)

$$det_{iab;jab} = K_{ij}\ det_{ab;ab} - K_{aj}det_{ib;ab} + K_{bj}\ det_{ia;ab}$$

If we now sum this on ab, the last two terms turn out to be equal and we get

$$\Sigma_{ab}\ det_{iab;jab} = K_{ij}\ \Sigma_{ab}\ det_{ab;ab} - 2!\ \Sigma_{ab}\ det_{ib;ab}K_{aj}$$

which we can solve to get

$$K_{ij}\ \Sigma_{ab}\ det_{ab;ab} = \Sigma_{ab}\ det_{iab;jab} + 2!\ \Sigma_{ab}\ det_{ib;ab}K_{aj}$$

Therefore from above we have

$$M^{(3)}{}_{ij} = -\Sigma_{ab}\ det_{ia;ba}\ K_{bj} + (1/2)\Sigma_{ab}\ det_{ab;ab}\ K_{ij}$$

$$= -\Sigma_{ab}\ det_{ia;ba}\ K_{bj} + (1/2)\Sigma_{ab}\ det_{iab;jab} + \Sigma_{ab}\ det_{ib;ab}K_{aj}$$

$$= -\Sigma_{ab}\ det_{ib;ab}\ K_{aj} + (1/2)\ \Sigma_{ab}\ det_{iab;jab} + \Sigma_{ab}\ det_{ib;ab}K_{aj}$$

$$= (1/2)\ \Sigma_{ab}\ det_{iab;jab}$$

Thus our series so far is

$$N_{ij} = \Sigma_{n=0}\ \lambda^n\ M^{(n)}{}_{ij} = \lambda K_{ij} - \lambda^2\ \Sigma_k\ det_{ik;jk} + (\lambda^3/2!)\ \Sigma_{ab}\ det_{iab;jab} - ...$$

This leads us to conjecture that the full series is this:

$$N_{ij}/\lambda = K_{ij} - \lambda\ \Sigma_k\ det_{ik;jk} + (\lambda^2/2!)\ \Sigma_{ab}\ det_{iab;jab} - (\lambda^3/3!)\ \Sigma_{abc}\ det_{iabc;jabc} + ...$$

$$= K_{ij} + \Sigma_{n=1}^{\infty}\ (-\lambda)^n/\ n!\ *\ [\ \Sigma_{k1,k2...kn}\ det_{i\ k1\ k2...kn\ ;\ j\ k1\ k2...kn}\ ]$$

This is of course the correct known result. But if we did not know the correct result, this recursion method would have led us to the correct result. The general formula for $M^{(n)}{}_{ij}$ must be

$$M^{(3)}{}_{ij} = (1/2)\,\Sigma_{ab}\,\det_{iab;jab} = (1/2!)\,\Sigma_{k1,k2}\,\det_{i\ k1\ k2;j\ k1\ k2}$$

$$M^{(n+1)}{}_{ij} = (-1)^n/n! * [\,\Sigma_{k1,k2\ldots kn}\ \det_{i\ k1\ k2\ldots kn}\,;\ {}_{j\ k1\ k2\ldots kn}\,]$$

or

$$M^{(n)}{}_{ij} = (-1)^{n-1}/(n-1)! * [\,\Sigma_{k1,k2\ldots kn-1}\ \det_{i\ k1\ k2\ldots kn-1}\,;\ {}_{j\ k1\ k2\ldots kn-1}\,]$$

5. An Induction Proof of the N_{ij} series

We can now show "by induction" that our conjectured series is indeed correct.. Assume that the above is true for n;

$$M^{(n)}{}_{ij} = (-1)^{n-1}/(n-1)! * [\,\Sigma_{k1,k2\ldots kn-1}\ \det_{i\ k1\ k2\ldots kn-1}\,;\ {}_{j\ k1\ k2\ldots kn-1}\,]$$

We have our recursion relation as follows:

$$M^{(n+1)} = M^{(n)}K + (-1)^n/n! * \Sigma_{i1,i2,i3\ldots\ in=1}{}^{N}\ \det_{i1\ i2\ i3\ldots in}K$$

which we first rewrite showing matrix elements, and then we install $M^{(n)}{}_{ij}$ from above with j=k:

$$M^{(n+1)}{}_{ij} = \Sigma_k\,\{M^{(n)}{}_{ik}\}K_{kj} + (-1)^n/n! * \Sigma_{i1,i2,i3\ldots\ in\ in=1}{}^{N}\ \det_{i1\ i2\ i3\ldots in\ in}K_{ij}$$

$$= \Sigma_k\,\{\ (-1)^{n-1}/(n-1)! * [\,\Sigma_{k1,k2\ldots kn-1}\ \det_{i\ k1\ k2\ldots kn-1}\,;\ {}_{k\ k1\ k2\ldots kn-1}\,]\}\,K_{kj}$$

$$+ (-1)^n/n! * \Sigma_{i1,i2,i3\ldots\ in=1}{}^{N}\ \det_{i1\ i2\ i3\ldots in}K_{ij}$$

$$= \{\,(-1)^n/n!* \Sigma_k\,[-n\,\Sigma_{k1,k2\ldots kn-1}\ \det_{i\ k1\ k2\ldots kn-1}\,;\ {}_{k\ k1\ k2\ldots kn-1}\,]\}\,K_{kj}$$

$$+ (-1)^n/n! * \Sigma_{i1,i2,i3\ldots\ in=1}{}^{N}\ \det_{i1\ i2\ i3\ldots in}K_{ij}$$

The desired induction result is this (copying from end of previous section)

$$M^{(n+1)}{}_{ij} = (-1)^n/n! * [\,\Sigma_{k1,k2\ldots kn}\ \det_{i\ k1\ k2\ldots kn}\,;\ {}_{j\ k1\ k2\ldots kn}\,]$$

We conclude our induction proof of the general form of the series if we can show that

$$[\,\Sigma_{k1,k2\ldots kn}\ \det_{i\ k1\ k2\ldots kn}\,;\ {}_{j\ k1\ k2\ldots kn}\,]$$

$$= \Sigma_k\,[-n\,\Sigma_{k1,k2\ldots kn-1}\ \det_{i\ k1\ k2\ldots kn-1}\,;\ {}_{k\ k1\ k2\ldots kn-1}\,]\,K_{kj}$$

$$+ \Sigma_{i1,i2\ldots\ in}\ \det_{i1\ i2\ldots in}\ K_{ij} \qquad (*)$$

As we did in one of our lower terms above, we need to do a "first column expansion" of this determinant:

$$\det_{\mathbf{i\ k1\ k2\ldots kn}} ;\ _{\mathbf{j\ k1\ k2\ldots kn}}$$

We now write down some general rules for doing one of these expansions:

First column expansion theorem rules:
(1) alternate signs as usual
(2) the $K_{\mathbf{ab}}$ factors are as follows:
first index sequences through row group, second index is always first of col group
(3) on the new dets, the col group is always the same: last of the original col group.
(4) on the new dets, the row group is obtained by deleting, one at a time, one digit from the original row group.

So here we go,

$$\det_{\mathbf{i\ k1\ k2\ldots kn}} ;\ _{\mathbf{j\ k1\ k2\ldots kn}} =$$

$$K_{\mathbf{ij}} \det{}_{\mathbf{k1\ k2\ldots kn\ ;\ k1\ k2\ldots kn}} + \text{other terms}$$

Let's ignore the "other terms" for the moment. If we apply $\Sigma_{\mathbf{k1,k2\ldots kn}}$ to both sides of this equation we immediately generate our desired term (*) above, which I now rewrite as

$$+ \Sigma_{\mathbf{i1,i2\ldots\ in}} \det_{\mathbf{i1\ i2\ldots in;\ i1\ i2\ldots in}} K_{\mathbf{ij}} \qquad (*)$$

where I expand the notation for the "diagonal" subdeterminants. Therefore, now we have only to show that:

$$\Sigma_{\mathbf{k1,k2\ldots kn}} (\text{other terms}) =$$

$$-n\ \Sigma_{\mathbf{k}} \left[\Sigma_{\mathbf{k1,k2\ldots kn-1}} \det_{\mathbf{i\ k1\ k2\ldots kn-1}} ;\ _{\mathbf{k\ k1\ k2\ldots kn-1}} \right] K_{\mathbf{kj}} \qquad (**)$$

If we can show this, our induction proof is complete. Using our "first column expansion rules" ,

First column expansion theorem rules:
(1) alternate signs as usual
(2) the $K_{\mathbf{ab}}$ factors are as follows:
first index sequences through row group, second index is always first of col group
(3) on the new dets, the col group is always the same: last of the original col group.
(4) on the new dets, the row group is obtained by deleting, one at a time, one digit from the original row group,

we now write out these "other terms". We assume that n is some large integer, and we will write out the first 5 of the "other terms" with the understanding that there are really n "other terms". This will clearly show the pattern that develops. So:

$\det_{\mathbf{i\ k1\ k2...kn}}$; **j k1 k2...kn**

- $K_{\mathbf{k1,j}}$	det **i ** k2 k3 k4 k5...kn ; k1 k2 k3...kn**	**1**	
+ $K_{\mathbf{k2,j}}$	det **i k1 ** k3 k4 k5...kn ; k1 k2 k3...kn**	**2**	
- $K_{\mathbf{k3,j}}$	det **i k1 k2 ** k4 k5...kn ; k1 k2 k3...kn**	**3**	
+ $K_{\mathbf{k4,j}}$	det **i k1 k2 k3 ** k5...kn ; k1 k2 k3...kn**	**4**	
- $K_{\mathbf{k5,j}}$	det **i k1 k2 k3 k4 **...kn ; k1 k2 k3...kn**	**5**	

Each term has a complete summation $\Sigma_{\mathbf{k1\ k2\ k3....kn}}$ but we are suppressing this summation symbol. We mark the "deleted" row group index with **. To verify the above list, the reader will have to read the column expansion rules above. Remember that we are not writing the very first term in the expansion because we have already dealt with it.

Now let's swap two k summation indices on each of lines 2,3,4,5 such that the leading K factor is always $K_{\mathbf{k1}}$,j . Then just rewrite:

$\det_{\mathbf{i\ k1\ k2...kn}}$; **j k1 k2...kn**

			What we did
- $K_{\mathbf{k1,j}}$	det **i ** k2 k3 k4 k5...kn ; k1 k2 k3 k4 k5...kn**	**1**	
+ $K_{\mathbf{k1,j}}$	det **i k2 ** k3 k4 k5...kn ; k2 k1 k3 k4 k5...kn**	**2**	2 ↔ 1
- $K_{\mathbf{k1,j}}$	det **i k3 k2 ** k4 k5...kn ; k3 k2 k1 k4 k5...kn**	**3**	3 ↔ 1
+ $K_{\mathbf{k1,j}}$	det **i k4 k2 k3 ** k5...kn ; k4 k2 k3 k1 k5...kn**	**4**	4 ↔ 1
- $K_{\mathbf{k1,j}}$	det **i k5 k2 k3 k4 **...kn ; k5 k2 k3 k4 k1...kn**	**5**	5 ↔ 1

Notice that these swaps affect both the row group and the column group. In the column group we actually do a swap. In the row group, one of the two swapees is missing, so we just do 1→2, 1→3 etc. so it is only the first k index of the row group that is affected.

At this point we are still free to relabel the summation indices k2,k3...kn any way we want. So let's do that now:

$\det_{\mathbf{i\ k1\ k2...kn}}$; **j k1 k2...kn**

		line #	What we did
- $K_{\mathbf{k1,j}}$	det **i ** k2 k3 k4 k5...kn ; k1 k2 k3 k4 k5...kn**	**1**	
+ $K_{\mathbf{k1,j}}$	det **i k2 ** k3 k4 k5...kn ; k2 k1 k3 k4 k5...kn**	**2**	**nothing**
- $K_{\mathbf{k1,j}}$	det **i k2 k3 ** k4 k5...kn ; k2 k3 k1 k4 k5...kn**	**3**	k2,k3→k3.k2

$+ K_{k1,j}\ \det_{i\ k2\ k3\ k4\ **\ k5...kn\ ;\ k2\ k3\ k4\ k1\ k5...kn}$ 4 k4,k2,k3→ k2,k3,k4

$- K_{k1,j}\ \det_{i\ k2\ k3\ k4\ k5\ **...kn\ ;\ k2\ k3\ k4\ k5\ k1...kn}$ 5 5,2,3,4 → 2,3,4,5

At this point, the row groups are all the same! To make the column groups the same, we have to shift k1 to the left a number of places equal to line number - 1. The even numbered lines like 2 and 4 pick up a minus sign because we do an odd number of adjacent swaps to achieve the required shift. The odd numbered lines maintain their sign. Doing this, we get

$\det_{i\ k1\ k2...kn\ ;\ j\ k1\ k2...kn}$

line #

$- K_{k1,j}\ \det_{i\ **\ k2\ k3\ k4\ k5...kn\ ;\ k1\ k2\ k3\ k4\ k5...kn}$ 1

$- K_{k1,j}\ \det_{i\ k2\ **\ k3\ k4\ k5...kn\ ;\ k1\ k2\ k3\ k4\ k5...kn}$ 2

$- K_{k1,j}\ \det_{i\ k2\ k3\ **\ k4\ k5...kn\ ;\ k1\ k2\ k3\ k4\ k5...kn}$ 3

$- K_{k1,j}\ \det_{i\ k2\ k3\ k4\ **\ k5...kn\ ;\ k1\ k2\ k3\ k4\ k5...kn}$ 4

$- K_{k1,j}\ \det_{i\ k2\ k3\ k4\ k5\ **...kn\ ;\ k1\ k2\ k3\ k4\ k5...kn}$ 5

Now all lines are exactly the same!!! It seems pretty clear that this proof method extends to however many "lines" there are when we do our column expansion. In our example here our starting subdeterminant dimension was n+1 so there will be n "lines" which are all equal. Thus we can just represent all these lines as n times the first line. So,

$\det_{i\ k1\ k2...kn\ ;\ j\ k1\ k2...kn}$ // "other terms" part

$= - n * K_{k1,j}\ \det_{i\ k2\ k3...kn;\ k1\ k2\ k3...kn}$

Remember that this has the implied summation $\Sigma_{k1\ k2\ k3....kn}$. Let's now rename these dummy summation indices as follows:

$k1\ k2\ k3....kn \rightarrow k\ k1\ k2....kn-1$

so the above becomes:

$= - n * \Sigma_{k\ k1\ k2....kn-1}\ K_{k,j}\ \det_{i\ k1\ k2...kn-1;\ k\ k1\ k2...kn-1}$

But this is seen to exactly match (**) above, so that concludes this somewhat gory induction proof! We have thus proven that the correct series for N_{ij} is this:

$[N_{ij}/\lambda] = K_{ij} - \lambda \Sigma_k \det_{ik;jk} + (\lambda^2/2!) \Sigma_{ab} \det_{iab;jab} - (\lambda^3/3!) \Sigma_{abc} \det_{iabc;jabc} + ...$

$= K_{ij} + \Sigma_{n=1}^{\infty} (-\lambda)^n/ n! * [\ \Sigma_{k1,k2...kn}\ \det_{i\ k1\ k2...kn\ ;\ j\ k1\ k2...kn}\ ]$

6. A restatement of the correct series for N_{ij}

We have already proven the formula above for the correct series N_{ij} in terms of K. Here I will first write this solution series in terms of B = -λK, then return later to K.

Here is the solution to our problem, where we assume that B is NxN :

$$-N_{ij} = -[\mathrm{cof}\,(1+B)]_{ji} + \det(1+B)\,\delta_{i,j}$$

$$= B_{ij} + \Sigma_k \det_{ik;jk} + (1/2!)\,\Sigma_{km} \det_{ikm;jkm} + (1/3!)\,\Sigma_{kmn} \det_{ikmn;jkmn} + \text{etc}$$

$$= B_{ij} + \Sigma_{n=1}^{N-1}\, 1/n!\, *\, [\, \Sigma_{k1,k2\ldots kn}\ \det_{i\ k1\ k2\ldots kn\ ;\ j\ k1\ k2\ldots kn}\,]$$

where the det objects appearing here are standard subdeterminants of the B matrix as we discussed in our separate document on subdeterminants. You can see that at least I got the first term right in my painful manual analysis of the B^0 and B^1 terms. This is certainly faint praise, since most of the interest lies in the remaining terms.

We can now set B = -λK as discussed above to get the actual power series result of interest. Note that $\det_{i\ k1\ k2\ldots kn\ ;\ j\ k1\ k2\ldots kn}$ is order B^{n+1} so we are going to get

$$-N_{ij} = -\lambda K_{ij} + \Sigma_{n=1}^{\infty}(-\lambda)^{n+1}/n!\, *\, [\, \Sigma_{k1,k2\ldots kn}\ \det_{i\ k1\ k2\ldots kn\ ;\ j\ k1\ k2\ldots kn}\,]$$

where now the dets are of K, not B. We can rewrite this as (and this is what we proved above)

$$[N_{ij}/\lambda] = K_{ij} + \Sigma_{n=1}^{\infty}(-\lambda)^n/n!\, *\, [\, \Sigma_{k1,k2\ldots kn}\ \det_{i\ k1\ k2\ldots kn\ ;\ j\ k1\ k2\ldots kn}\,]$$

and the thing on the right is then the **Fredholm's First minor** expansion $D^1{}_{ij}(\lambda)$. We can write this out in graphical notation as

$$[N_{ij}/\lambda] = K_{ij} - \lambda\Sigma_k \begin{vmatrix} Kij & Kik \\ Kkj & Kkk \end{vmatrix} + (\lambda^2/2!)\,\Sigma_{km} \begin{vmatrix} Kij & Kik & Kim \\ Kkj & Kkk & Kkm \\ Kmj & Kmk & Kmm \end{vmatrix} - (\lambda^3/3!)\,\Sigma_{kmn} \begin{vmatrix} Kij & Kik & Kim & Kin \\ Kkj & Kkk & Kkm & Kkn \\ Kmj & Kmk & Kmm & Kmn \\ Knj & Knk & Knm & Knn \end{vmatrix} + .$$

$$\det_{ik;jk} \qquad \det_{ikm;jkm} \qquad \det_{ikmn;jkmn}$$

You can see the "economy" of the det notation compared to writing everything out, where you can easily make an index error.

In the continuum sense we would write this as

$$D_1(x,y;\lambda) = k(x,y) + \Sigma_{n=1}^{\infty}(-\lambda)^n/n!\, *\, \int ds_1 ds_2 .. d_{sn}\ \det_{x\ s1\ s2\ldots sn;\ y\ s1\ s2\ \ldots sn}$$

$$= k(x,y) - \lambda \int ds_1 \begin{vmatrix} k(x,y) & k(x,s1) \\ k(s1,y) & k(s1,s1) \end{vmatrix} + (\lambda^2/2!) \int ds_1 ds_2 \begin{vmatrix} k(x,y) & k(x,s1) & k(x,s2) \\ k(s1,y) & k(s1,s1) & k(s1,s2) \\ k(s2,y) & k(s2,s1) & k(s1,s2) \end{vmatrix} - \ldots$$

$\det_{\texttt{x s1; y s1}}$ $\det_{\texttt{x s1 s2; y s1 s2}}$

Here is how this result appears in one book:

$$K\begin{pmatrix} x_1, \ldots, x_n \\ y_1, \ldots, y_n \end{pmatrix} = \begin{vmatrix} K(x_1, y_1) & \ldots & K(x_1, y_n) \\ & \ldots & \\ K(x_n, y_1) & \ldots & K(x_n, y_n) \end{vmatrix} = \det_{\texttt{x1 x2...xn; y1 y2 ...yn}}$$

$$D(x, y; \lambda) = K(x, y) + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_D \ldots \int_D K\begin{pmatrix} x, s_1, \ldots, s_n \\ y, s_1, \ldots, s_n \end{pmatrix} ds_1 \ldots ds_n.$$

This author uses a capital K for the kernel function whereas I use lower case k.