Notes on hand-written notes in the Fredholm Section of the Math I Binder PhL 3.22-29.09

(see "Fredholm World" section of math notes for details)

In the 1974 time frame I was moving into a thesis involving the "multiperipheral integral equation", and I guess this thing is Fredholm-like and that got my interested in this topic. Some of the notes in this section refer to this equation, and many refer to J values where Fred Det = 0, but $J \neq \lambda$.

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1. Solving a (Fredholm) Matrix Equation with Symmetric Kernel via Eigenvectors 1p

We are given some mysterious quantity probably related to Regge Theory, but just think of this as a math problem: (V is some kind of potential, J is probably angular momentum)

$$A = V + (J-\beta)^{-1}\ VA$$

This is an integral equation for A, and J-β is just some number. We can solve this like so:

$$(1- (J-\beta)^{-1}V)A = V \qquad A = (1- (J-\beta)^{-1}V)^{-1}V = [(J-\beta-V)/(J-\beta)]^{-1} * V$$

$$= V(J-\beta)/ (J-\beta-V)$$

Suppose we have $V\ |i\rangle = \lambda_i\ |i\rangle$ as eigenvalues and eigenstates for operator V. Let the capital letter states be "some other states", perhaps of some Hamiltonian, Then

$$A_{IK} = \langle I|A|K\rangle = \Sigma_j \langle I|\ V(J-\beta)/ (J-\beta-V)\ |j\rangle\langle j\ |\ K\rangle = \Sigma_j \langle I|\ \lambda_j(J-\beta)/ (J-\beta-\lambda_j)\ |j\rangle\langle j\ |\ K\rangle$$

$$= (J-\beta)\ \Sigma_j \langle I\ |j\rangle\ [\lambda_j/(J-\beta-\lambda_j)\ ]\ \langle j\ |\ K\rangle$$

So maybe this object A_{IK} has poles at $J_i = \beta + \lambda_i$ and who knows, maybe these are Regge poles. Again, this is just a nothing math problem really.

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2. Comments on Fredholm Theory (generic, A = K + APK) (1 page)

Just a small discussion which considers an integral equation written as $\varphi = f + \varphi K$ where the K is to the right of the φ, a little different from ordinary, Then another equation of form A = K + APK where P is a diagonal matrix. Solve this usual thinking of K' = PK. I mention that you need the Fred Det and the Fred Minor to write down a solution, and I guess I am assuming a small λ.

I then claim two "theorems" at the bottom of the page.

Class 2: With general kernel, you can solve for small λ in the continuum sense, and you might do a lattice approximation to get a matrix problem.

Class 1: If you have a *separable* kernel of degree N, you can solve by an N x N matrix technique. This subject is discussed in Stakgold Section 3.8 subsection 2, see meta notes.

I don't know where these "class" names came from.

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3. The Theory of Solving Integral Equations with Factorizable Kernels (6 pages)

This is a set of 6 numbered pages (6 1-sided pages), but seems to touch upon other topics. Subtopics are numbered, so I will go with those numbers here. I first deal with a separable kernel, then treat the factorizable one as a special case.

I must have been dealing a lot with integral equations of this form, but I don't have any real reference in my notes to the context.

1. Equation of interest is A = K + APK where P is a diagonal matrix. "true kernel" is PK.

2. Let $k(1,2) = \Sigma_{i=1}^{n} L_i(1) M_i R_i(2)$, so we throw in an extra weight thing M_i into each term, could include it with either side. This is our **separable kernel**.

3. Here I just derive a sequence of numbered equations, defining $\mathcal{L}_i(1)$ as shown.

4. Here I define matrix X_{ij} (in the space of the separability index), and I think of M and L as being diagonal matrices and I end up with $\mathcal{L}(1) = L(1) + \mathcal{L}(1)\, M\, X$ as a matrix equation in separability space. The "1" indicates a continuous coordinate like x.

5. Then I issue a clarification of how the vectors L and $\mathcal{L}$ were extended to matrices.

$\mathcal{L}(1)_{1i} = L(1)_{1i} + \Sigma_{jk}\ \mathcal{L}(1)_{1i}\, M_{jk}\, X_{ki}$ // details

$\mathcal{L}(1) = L(1) + \mathcal{L}(1)\, M\, X$ // matrix equation

6. Solve this matrix equation to get

$\mathcal{L}(1)\,[\,1 - MX\,] = L(1) \qquad \mathcal{L}(1) = [\,1 - MX\,]^{-1}\,L(1)$

and then I shuffle the result around a bit.

7. Now that we know $\mathcal{L}(1)$, jam it back into equation (4) to get our solution for A.

8. The **solution for A** can be written in an interesting symmetric form, and all the key results are then surrounded by **red pencil boxes**. This is "the general separable case".

9. If we now assume n = 0, our separable case becomes the **factorizable case**. Things become the red box on page 3, and I verified that the result for A is correct. The solution A is just a multiple of K.

10. Here I just redo the factorizable case as a double check.

11. Suppose the determinant $|M^{-1} - X|$ is a function of complex variable p and has a zero $(p-p_0)$. I then claim without proof that the cofactor matrix C_{ij} factorizes at such a pole if X is symmetric. I then claim that it follows that our solution A(1,3) has a **factorizable residue at such a pole**, and this is written out bottom of page 3. We are still in the n-separable model here. You see the two factors. This would be valid near the pole.

12. I restate the previous result that the pole residue factors.

13. In the factorizable limit n=1, we still have a factorizable pole residue, no surprise.

14. In any event, you have to compute X_{ij} which is $\int d2\; R_i(2)P(2)L_j(2)$, and somewhere we have been assuming actually that X is symmetric.

Maybe I will later encounter a proof of my claim that the pole residue factors.

<u>Page 5: Extension of Theory to Include a Double-Separable Kernel</u>

1. The kernel now has this form: $k(1,2) = \Sigma_{i,j=1}^{n}\, L_i(1)M_{ij}R_j(2)$

It turns out that the entire formalism presented in the earlier sections applies here. The difference is that before M was a diagonal matrix, now it is not. I again put the results in a red box. Nowadays I would have used the notation $A(1,3) = L^{T}(1)\,(M^{-1}-X)^{-1}\,R(3)$ instead of over-arrows.

<u>Page 6: Alternative Form of the General Solution</u>

1. I open with a comment that seems unimportant.

2. Here I just state the "alternative form" -- it is just a rehash of the previous result, but now it involves the determinant |1-MX| instead of $|M^{-1} - X|$.

3. Yet another form for A(1,3) is given, another rehash.

4. If MX combined is factorizable as a matrix, so $M_{ij} = A_i Bj$ (a new factorizable concept), but then I rewrite this as $M_{ij} = f_j k_j$ k . In this case, K factorizes as shown. But if it factorizes, then we get to use your trivial solution for A(1,3) as shown.

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4. Relation between Fredholm Theory and the Green's Function of ODE Theory (1 p)

I am so weak on Green's right now that I have to just omit this page for the moment. ********

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5. An application of Fredholm Theory (Jones and Teplitz tech note) 1 page

This is a technical comment about something appearing in a paper by Jones and Teplitz. We again have some kind of situation where det(K-1) = 0 and this is a pole of some solution φ and it is a pole in angular momentum. I draw a total blank on this, since I don't really have that reference handy.

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6. Fredholm Theory Revisited: the matrix viewpoint. (4 pages)

This is **four pages written on white lined paper** with my sharp Rapidograph pen or a sharp fountain pen. Again, no date. There are 13 "items" circled, I will number them here. Everything here makes complete sense to me now in 2009 after reading Stak Chap 3, and there is some new information not in Stak: I am going to add the factors λ where I know they go, though they are not present in my notes.

1. $\varphi = f + \lambda K\varphi$ form of the inhomo equation in Neumann form.

2. think of the above as a matrix equation

3. Write it as $(1 - \lambda K)\varphi = f$ and then write formal solution as

$$\varphi = (1 - \lambda K)^{-1} f. = (1 + \lambda K + \lambda^2 K^2 + ...) f = f + \Sigma_{m=1} \lambda^m K^m f$$

If we are doing matrices, we say K^m but if coordinate, we would say k_m for the iterated kernel.
Recall that $A^{-1} = \hat{A} / |A|$ where $\hat{A}$ = classical adjoint = $\text{cof}(A)^T$ so think of $(1 - \lambda K)^{-1}$ that way.

4. Mikhlin's resolvent is $\Gamma(x,s) = \Sigma_{m=1} \lambda^m k_m(x,s) = \Sigma_{m=1} \lambda^m K^m$. Thus, we can write our solution above in this form:

$$\varphi = f + \Gamma f = (1+\Gamma) f .$$

Also we can say

$$1 + \Gamma = (1 - \lambda K)^{-1}$$

$\Rightarrow \quad (1-\lambda K)(1+\Gamma) = 1 \quad \Rightarrow \quad -\lambda K + \Gamma - \lambda K\Gamma = 0 \quad \Rightarrow \quad \Gamma = \lambda K + \lambda K\Gamma$

But we can also do it from the other side, so we then would get $\Gamma = \lambda K + \lambda\Gamma K$.

5. Convert the above items into matrix form:

$\varphi = f + \Gamma f$ // already shown in 4 above
$\Gamma = \Sigma_{m=1} \lambda^m K^m$ = sum of powers of matrix K // shown 4 above, matrix form here
$k_m = K^m = KKK...K$

6. Here I quote my two results above: $\Gamma = \lambda K + \lambda K\Gamma = \lambda K + \lambda\Gamma K$.

7. Nothing new

8. Writing out items of 6 in integral notation

9. Homogeneous equation is the EV problem. Fine.

10. Fredholm's Fourth Theorem: Here I just give a rehash of the Alternative Theorem, nothing new here.

11. To get the eigenvalues of $K\psi = \mu\psi$, write as $(K - \mu)\psi = A\psi = 0$. If $\det A \neq 0$, then invert to find only the trivial solution $\psi = 0$. So to have actual EF's, need $\det A = 0$ or $\det(K-\mu 1) = 0$.

Recast this with $\lambda = 1/\mu$. Then we have $\lambda K\psi = \psi$, which we write as $(1-\lambda K)\psi = 0$. Then our condition for true EF's would be: $\det(1-\lambda K) = 0$ which we would solve for the EV's λ_i as the roots. It is just the secular equation, no big deal. But in the limit $n \rightarrow \infty$, **this is the Fredholm Determinant**!

12. **Theorem**: the eigenfunctions of symmetric K span the range of Kf=g, and this is something we know from Stakgold. In fact, you can make the same claim for eigenfunctions of non-zero eigenvalues, since $\mu=0$ maps into the nullspace. If we include the $\mu=0$ EF's, we get a complete set for L_2, a fact we also know from Stak but not quoted here. So assume that the φ_i are a complete set for L_2, then :

Since the φ_i are then a complete set, we can say this:

$1 = \Sigma_i \ |\varphi_i\rangle\langle\varphi_i|$ // operator statement

$K = \Sigma_i \ \mu_i \ |\varphi_i\rangle\langle\varphi_i|$ // operator statement

$\langle x|K|y\rangle = k(x,y) = \Sigma_i \ \mu_i \langle x \ |\varphi_i\rangle\langle\varphi_i| \ y\rangle = \Sigma_i \ \mu_i\varphi_i(x)\bar{\varphi}_i(y)$

where we for the first time think of our integral operator K as an operator in the QM HS. So we get this nice way to expand the kernel:

$k(x,y) = \Sigma_i \ \mu_i\varphi_i(x)\bar{\varphi}_i(y)$

where the φ_i are the eigenfunctions of K and μ_i are the eigenvalues. We could write all this in matrix notation as follows:

$1 = \Sigma_i \, v_i v_i^\dagger$ $\quad$ v_i = a basis vector = eigenfunction of K = complete set

$Kv_i = \mu_i v_i$ $\quad$ EF statement

$K = \Sigma_i \, v_i v_i^\dagger = \Sigma_i \, K \, v_i v_i^\dagger = \Sigma_i \, \mu_i \, (v_i v_i^\dagger)$

$K_{mn} = \Sigma_i \, \mu_i \, (v_i v_i^\dagger)_{mn}$

13. Says do not confuse kernels with Green's Functions, but I know that in fact Green's Functions make the kernels of integral equations equivalent to an ODE with BCs.

So here is a summary of interesting new things from these notes:

$(1-\lambda K)(1+\Gamma) = 1$

$\mu\Gamma = K + \Gamma K = K + K\Gamma$ $\quad$ $\mu = 1/\lambda$

$\det(1-\lambda K) = 0$ $\quad$ gives eigenvalues $\lambda_i = 1/\mu_i$ of $K\varphi_i = \mu_i \varphi_i$

When you think of this in the limit $n \to \infty$, you have the full Fredholm Determinant.

Derivative of Fredholm's First Minor $\quad$ **// two sides of a page, white paper**

Think of a finite n x n situation and you want to know

$\text{cof}^T(1-K)$ $\quad$ K = n x n matrix

Look up stuff in matrix notes:

To compute a cofactor of a matrix "a" I know from matrix notes:

cof(a_{pq}) $= (-1)^{p+q}$ minor(a_{pq}). and

minor(a_{pq}) = det of the N-1 x N-1 matrix you get by crossing out row p and column q.

$\det(a) = \Sigma_q \, a_{pq} \, \text{cof}(a_{pq})$ [across row p] $= \Sigma_p \, a_{pq} \, \text{cof}(a_{pq})$ [down column q] (10-2)

$\det(a) = \varepsilon_{ijk\ell} \, a_{1i} \, a_{2j} \, a_{3k} \, a_{4\ell}$

$\text{minor}(a_{32}) = - \varepsilon_{ij2\ell} \, a_{1i} \, a_{2j} \, a_{4\ell}$ $\quad$ // a_{3k} is missing, and we set k = 2

So how would we apply this to a = 1-K?

[This turned out to be a real bear, see Fredholm's First Minor Round 2 for derivation!]

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7. Comment on Fredholm (Koplik, multiperipheral IE) (5 yellow pages)

First page. The first page is in reference to some 1974 thesis work by a guy named **Joel Koplik**. He is there now on the web at CUNY in New York Physics Dept. He seems to have departed the particle realm:



Research Interests:

- Molecular fluid mechanics - dynamics of surfactants in solution - interfacial boundary conditions - nanoscale flows on patterned substrates - non-Newtonian extrusion flows
- Fluid and particulate flow in geological fractures and random porous media
- Vortex dynamics in superfluid turbulence

Somewhere I supposedly have notes on his **thesis**. The idea is that you start off with an integral equation which is in fact the "**multiperipheral" integral equation** where I guess you tack on an extra link in the chain, it is all fuzzy now. My notes say that when you **diagonalize this into the J plane**, the integration goes away and you are left with a **simple matrix equation** which is in "**channel space**", whatever that means, there are a finite number of "channels". So we have F = G + GSF as your matrix equation for F. All my little pictures are specific to this application and I don't know what they mean.

Second page. **Genuine Fredholm Determinant**. Just some reasonable comments. The idea is that the angular momentum **J is just a parameter** that appears in the various familiar Fredholm objects like the resolvent, and it is not really the eigenvalue of an inhomo Fredholm equation. In fact such a thing λ or μ is often just set to 1 in the integral equation. Eventually we are interested in values of J for which the integral equations Fredholm determinant might have a zero. I comment on how the real Fred Det is a continuum thing that is mysterious unless you look at the famous series for it which I regard as a small K expansion. I note that **"trace expansion"** of the Fred Det which seems to be a closed form for the whole thing by there are then an infinite number of trace T_m terms in the exponent. I note that you only have the famous **1 + tr(K) if the kernel factorizes**. I comment that λ in terms of λK might be regarded as some kind of particle physics "coupling strength". Finally, I comment on a paper that uses "the trace approximation" and this time I call it **Chew, Rogers and Snider**, elsewhere I call this just **CRS**. So I could look there to see how angular momentum J gets into this subject of Fredholm integral equations.

Third page: (really a 3 page set on yellow grid paper, red and blue writing). These notes say they are conclusions of the Mikhlin book. Aha! I write the **"Chew-Frazer" multiperipheral integral equation**. I write it in the form $\varphi = \varphi_o + \varphi G$ where there is a 1D continuum integral, and there is also a finite matrix sense to the equation, I guess this is the channel space thing. I show passive J labels on φ and φ_0 and G, the kernel. I then quote various results of Mikhlin which are now very familiar to me such as

$$\Gamma(x,s;\lambda) = D(x,s;\lambda)/D(\lambda)$$

On the second page of this group I point out again that you should not confuse this parameter J with the λ thing of the integral equation world. On the second half of this page, I show you put an integral equation on lattice and then you get a Fred Det that is N x N.

On the third page we are back to separable kernels again. Recall again that this also leads to an n x n problem where n is the number of terms in the separable kernel. So there are two somewhat different finite matrix ideas: (1) put IE on the lattice; (2) use separable kernel. At the time, I was wondering which of these ideas Chew was using in some paper.

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8. Mikhlin notes on integral equations (18 pages)

This Mikhlin book is Stakgold's first reference at the end of his Chapter 3. I don't have the book, but I see that back in the 1970's I got hold of this book from the LBL libraqry and took **18 pages of hand-written notes** (in blue and red). I am now going to review those notes and comment on them here.

I did not DATE my notes at this time of my life, so cannot tell how these notes fit timewise with my other integral equation notes. Very regrettable. Must have been in the range 1973-1976.

Facts about the book.

I wrote down some facts about the book I got, it was 2nd Edition 1964. First edition was I think 1957.

"Integral equations and their applications to certain problems: In mechanics, mathematical physics and technology" ISBN-10: B0006AV6SY

International series of monographs on pure and applied mathematics Volume 4.

Unknown Binding: 583 pages

Publisher: Pergamon Press; [distributed by Macmillan, New York] (1964)

Language: English

ASIN: B0007IVG5S

- **Hardcover:** 338 pages
- **Publisher:** Pergamon Press; 1st edition 1957
- **Language:** English
- **ASIN:** B0006AV6SY

Integral Equations & Their Applications to Certain Problems in Mechanics, Mathematical Physics and Technology

Buy Used (Marketplace): from $25.50

Category: Book
Author: S. G. Mikhlin
Publisher: Macmillan
Studio: Macmillan
Manufacturer: Macmillan
Label: Macmillan
Media: Hardcover
Edition: Second edition
ASIN: B001GAYHKM
Publication Date: 1964

This book can be bought used on the web for $45 or so. Marriott has the 1957 version. I think it was published several times in different ways.

Page by page summary of my notes:

I only comment on things I don't know (ie, that were not in Stakgold). Basic inhomo 2 equation is this:

$$\varphi(x) = f(x) + \lambda \int k(x,s)\varphi(s)ds \qquad\qquad \varphi = f + \lambda K\varphi$$

This is the Neumann-friendly scattering form. Stak usually uses $Ku = \mu u + f$, so we need to keep in mind the fact that $\lambda = 1/\mu$. For example, Stak shows that accumulation points are $\mu=0$, but for M that will be only at $\lambda = \infty$.

Numbers in what follows are page numbers of my 18 pages of notes.

1. Says first 133 pages is theory, then last 200 pages is the application, translated from Russian. Suggests a 333 page book.

definition of **Fredholm**: k(x,y) continuous on the square and has bounded double integral. Considers possibility of a weak or full singularity in the square. Weak means a pole with less power than 1, strong means a full pole.

2. Will stick to **one variable**, but multivariable is a minor generalization. However, 3D problems often have problems that 1D and 2D do not have.

Section 1.2 The Successive Approximation Method.

Assume that single integral of k is finite. Writes the Neumann series solution of the full inhomo with the usual iterated kernels

3. The CSI is called the **Bunyakovsky-Schwarz Inequality**! Series converges for $|\lambda| B^2 \leq 1$ where B^2 is the double integral of k^2. He then goes on to show that $|\lambda| B \leq 1$. Same idea as Stak.

4. The **resolvent** is this:

$$\Gamma(x,y;\lambda) = \Sigma_{m=1} \lambda^m k_m(x,y)$$

I have to add this: Neumann says

$$u = (1 - \lambda K)^{-1} f = (1 + \lambda K + \lambda^2 K^2 + ...) f = (1 + \Gamma) f \quad => \quad (1 - \lambda K)^{-1} = 1 + \Gamma$$

If series solution exists, then equation "has a resolvent". Comments that if series exists, you probably could have solved it more easily some other way, such as the ODE.

Section 1.3 Volterra

Series here works for any λ, as Stak showed, even if k has a weak singularity.

Section 1.4 Degenerate Kernels

This just means a **separable kernel** with a finite sum using a_i and b_i function, solution called φ (Stak uses p_i and g_i and u for these things.) Let $A_{ik} = \int a_k b_i$ and let $f_i = \int f b_i$ and let $c_i = \int b_i \varphi$. Get matrix equation $(1-\lambda A)\mathbf{c} = \mathbf{f}$, solve for the c, solution is then $\varphi = f + \lambda \mathbf{c} \bullet \mathbf{a}$. Pretty simple.

5. If some λ causes $\det (1-\lambda A) = 0$, such λ likely to be singularities in solution somehow.

Section 1.5 General Case of Fredholm Equation

M claims that you approximate that kernel with a separable one. Increase n for more accuracy. If symmetric, use $b_i = a_i$ = cos functions, for example. Does a Dirichlet problem example with ellipse.

Section 1.6 Systems of Integral Equations

Now you have a combined integral and matrix equation

$$\varphi_i(x) - \lambda \Sigma_k \int ds\, k_{ik}(x,s) \varphi_k(s) ds = f_i(x)$$

Stak never commented on this. M talks about this a little bit, I have not real notes. I comment that I like the M book a lot.

6.

Section 1.7 Applications of Approximate Integration Formulas

"mechanical quadrature" means fitting are with strips. Problem becomes n x n matrix equation. Then interpolate solution, no integration ever needed. He does a Dirichlet integral equation example, uses n = 4 and gets within 1% of the right answer!

Section 1.8 Fredholm's Theorems

Definition: regular means the resolvent exists for some λ

Def: characteristic values = eigenvalues. For such values, the resolvent does not exist, blows up.

7.

Fredholm Theorem #1: Any finite region of the λ plane can contain only a finite number of eigenvalues. This I understand from Stak since $\lambda = \infty$ is the only possible accumulation point ($\mu = 0$)

Fredholm Theorem #2: Every eigenvalue has a finite multiplicity ≥ 1. That is, there is at least one EF, and at most a finite number. This applies away from $\lambda = \infty$.

Def: **conjugate** kernel means the Hermitian conjugate of k(x,y) which is $\overline{k}(y,x)$

You can always write a conjugate integral equation too.

Fredholm Theorem #3: Compares an integral equation and its conjugate. If one has λ as EV, the other has $\overline{\lambda}$ as EV, and multiplicities are the same.

At this late point in the book, M defines a **scalar product** and states the CSI and the Δ inequality.

8. At this point, I tried to relate M's book to a certain **Chew** integral equation from a reference I call CRS which I have no idea what means, perhaps some Chew paper. Yes, some Chew Regge paper...

9. Comments about the adjoint operator = conjugate operator wrt to the scalar product.

Fredholm Theorem #4: The inhomo equation with some λ_0 has a solution if $f.\psi_i = 0$ where ψ_i are the eigenfunctions of K* for that value of λ_0. This is an idea of required "consistency conditions" on f.

How would I relate this to a Stak thing? Recall Case 2 Section 3.4. We assume $\mu = \mu_m$ = some eigenvalue. We found in Stak that consistency conditions were $f.\varphi_m = 0$ for all eigenfunctions of K. But in Stak, we were dealing with symmetric K, so if you are more general, it is really K* that appears. I can rephrase this all as follows:

homo	$Ku = \mu u$	or	$(K-\mu I)u = 0$	or	$Au = 0$
inhomo	$Ku = \mu u + f$	or	$(K-\mu I)u = f$	or	$Au = f$

We know that the alternative theorem says $\overline{R_A} = (N_{A^*})^{\perp}$ and $(R_A)^{\perp} = N_{A^*}$ from Stak. So, in terms of operators A, suppose we have $A^*v_m = 0$ for some μ_m. Then v_m is in the nullspace of A*. The range of A

therefore has to be the perp space of N_{A*} and therefore we must have $f.v_m = 0$. **Thus, Fredholm's Theorem #4 is really a statement of the Alternative Theorem.** I think people actually call this Fredholm's Alternative Theorem.

I note that the left right sense of the inner product of M is the same as Stak.

10. M does a little review of matrix concepts including **nullspace and rank** (being number of solutions of the equation a**x** = **y**). Full rank means n solutions and no nullspace. M then gives short proofs of the four theorems, I took some notes on Fred #4 only. Then we have the formal statement:

Fredholm's Alternative: either Ax = y has solutions for all y, or the nullspace has non-trivial elements. Should really be talking A and A* as above, fine. As stated here, applies to symmetric A.

11. We now suddenly get some interesting claims made for the resolvent:

$$\Gamma(x,s;\lambda) = k(x,s) + \lambda \int k(x,t)\, \Gamma(t,s;\lambda)\, dt$$

$$\varphi(x) = f(x) + \lambda \int k(x,s)\varphi(s)ds$$

Note added: Mikhlin is defining Γ in this way:

$$\Gamma = [\, \Gamma_{me}/\lambda]$$

See Fredholm det R2 notes for implications. M then states two theorems which I prove in R2:

$$\int \Gamma(x,t{:}\lambda)\, \Gamma(t,s;\lambda)\, dt = \partial_{\lambda}\, \Gamma(x,s;\lambda)$$

$$\int \Gamma(x,x{:}\lambda)dx = \Sigma_{m=1}^{\infty} A_m \lambda^{m-1} \qquad \text{where } A_m = \int k_m(x,x)dx = \text{"trace" of } k_m = m^{th} \text{ trace}$$

12-13. This page of notes relates to making a "polynomial" estimate for k(x,s) and involves resolvent. This development then leads to the Fredholm Determinant $D_R(\lambda) = \det(1-\lambda A)$.

14. Then various claims about analyticity of the resolvent in λ. Then he writes expressions for something he calls $D_R(\lambda)$ as if this thing were maybe related to the "Resolvent". Maybe this is a finite n x n version of the full thing to come.

15. He then makes some new definitions:

$$\delta(\lambda) \equiv \int \Gamma(x,x{:}\lambda)dx = \text{function of } \lambda$$

$D(\lambda) \equiv \exp(- \oint \delta(\lambda')d\lambda')$

So now we have D_R and D flying around.

16. We end up with this interesting fact:

$\Gamma(x,s:\lambda) = D(x,s;\lambda)/D(\lambda)$ = Fredholm's First Minor / Fredholm's Determinant

$D(\lambda) = \exp [- \Sigma_n A_n \lambda^n/n]$ with A_n as shown above

which says this: eigenvalues = poles of resolvent = zeroes of $D(\lambda)$.

If λ is very small, we get this

$D(\lambda) \approx 1 - \lambda\, tr(k)$ " trace approximation for small λ"

We then get the famous λ expansions for $D(x,s;\lambda)$ and for $D(\lambda)$.

Claim: if kernel factors, then $D(\lambda) = 1 - \lambda\, tr(k)$ exactly.

17. Two general methods for finding the poles of the resolvent.

18. Here I have one page of notes on Stakgold!
