Ellipse in Polar Coordinates PhL 11.13.16

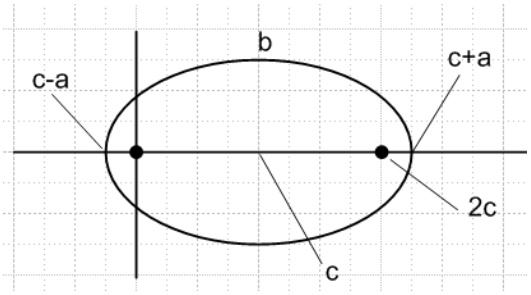
Start with an ellipse centered at the origin,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We want to shift this horizontally *to the right* so that the center of the ellipse is at $x_c = +c$.

$$\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

which says the $x_c = c$. Here is a picture of this shifted ellipse



This puts the left focal point at the origin!

Write as (1) as :

$$f \equiv b^2(x-c)^2 + y^2a^2 - a^2b^2 = 0 \ .$$

Now make these replacements:

$x = r\cos\theta$
$y = r\sin\theta$
$b = \sqrt{a^2-c^2}$
$c = \varepsilon a$

and do it all in Maple:

```
restart;
f := b^2*(x-c)^2 + a^2*y^2  - a^2*b^2;
```

$$f := b^2 (x-c)^2 + a^2 y^2 - a^2 b^2$$

```
x := r*cos(theta):
y := r*sin(theta):
c := e*a:
b := sqrt(a^2-c^2):
s := solve(f,r):
simplify(s[1]);
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$$\frac{(-1+e^2)a}{\cos(\theta)e+1}$$

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simplify(s[2]);
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$$\frac{(-1+e^2)a}{\cos(\theta)e-1}$$

There are two solutions which are

$$r_{\pm} = \frac{a(1-\varepsilon^2)}{1 \pm \varepsilon \cos(\theta)}$$

Why are there two solutions? Well, it was a quadratic equation in r. I know from my setup above if we set $\theta = \pi$ we should get r = a-c (see picture) . Let's test the two results

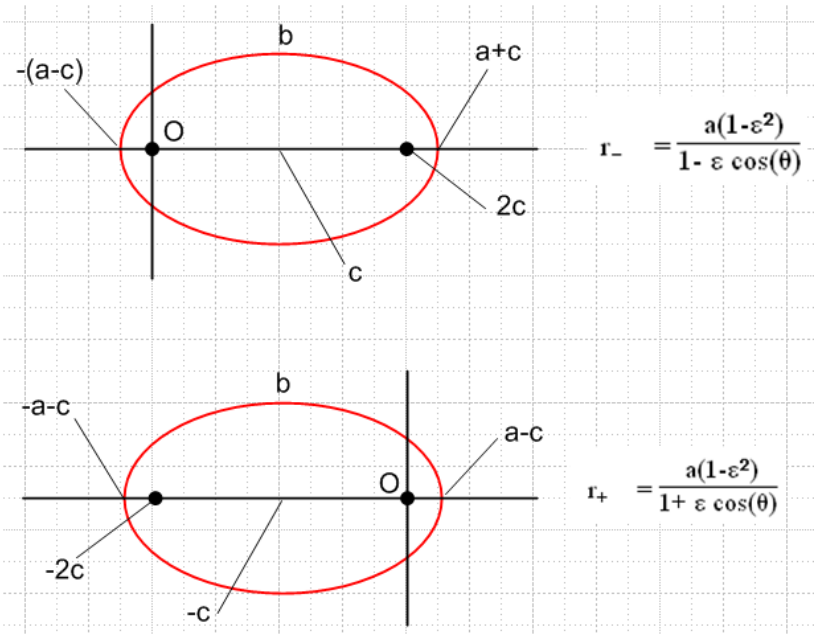
$$r_+ = \frac{a(1-\varepsilon^2)}{1+ \varepsilon \cos(\theta)} = \frac{a(1-\varepsilon^2)}{1- \varepsilon} = a(1+\varepsilon) = a + c$$

$$r_- = \frac{a(1-\varepsilon^2)}{1- \varepsilon \cos(\theta)} = \frac{a(1-\varepsilon^2)}{1+ \varepsilon} = a(1-\varepsilon) = a - c$$

Only r_- is correct, while r_+ is spurious for my positioning of the ellipse.

Suppose I started shifting the other way. That is like taking $c \rightarrow -c$ and that is like $\varepsilon \rightarrow -\varepsilon$. So in that case the correct solution is r_+.

Here are the conclusions:



$$r_- = \frac{a(1-\varepsilon^2)}{1-\varepsilon\cos(\theta)}$$

$$r_+ = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos(\theta)}$$

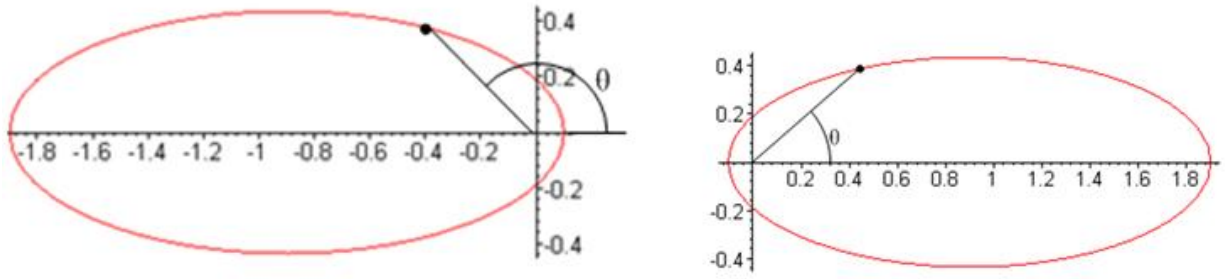
Fact: In either polar coordinate representation of the ellipse, the origin is at a focal point of the ellipse.

Final check:

$$r_+(\theta=0) = \frac{a(1-\varepsilon^2)}{1+\varepsilon\ 1} = a(1-\varepsilon) = a - c \qquad \text{correct}$$

$$r_-(\theta=0) = \frac{a(1-\varepsilon^2)}{1-\varepsilon\ 1} = a(1+\varepsilon) = a + c \qquad \text{correct}$$

Just as a final look, here are the angles θ for the two ellipse cases



In each case θ takes the full range 0 to 2π since the origin lies inside the ellipse.