Facts about ellipses PhL 10.17.09

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My Definition of Ellipse: I take as the definition of an ellipse the curve $x^2/a^2 + y^2/b^2 = 1$. Always assume centered at the origin, and assume $a \geq b$. Distance from center to either focus is c where $c^2 = a^2 - b^2$. Eccentricity is given by $\varepsilon = c/a$ and is 0 for a circle and is 1 for a super thin major-axis ellipse.

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(1) I have always known "the string theorem" for ellipses, $|z-c| + |z+c| = 2a$, but can't remember ever proving it. I give the brute force Cartesian coordinates proof where you square twice to eliminate all square roots and it just falls out from the algebra. This is the way my texts all do it as well. I don't know of a faster more intuitive proof. This is just a "fact" about ellipses. Some people define an ellipse by the string property, then derive its Cartesian equation.

(2) I did a similar brute-force Cartesian proof that $| z + \sqrt{z^2-1} | = K$ defines an ellipse in the z plane. Here a lot more "bruteness" is required and I did all the details in another document referenced below. This derivation gives no intuitive clue whatsoever about why this should be true.

(3) Anticipating some kind of plane geometry proof of $| z + \sqrt{z^2-1} | = K$, I derive here the famous laws of sines and cosines. Proofs are very trivial, but again, I don't remember ever actually doing these proofs.

(4) Here I attempted to relate the above ellipse facts to a detailed geometric drawing. The big problem is that the quantity $\sqrt{z^2-1}$ does not seem to fit in at all into the triangle geometry of the drawing. After drawing a good picture and writing down all the facts I could think of related to this picture, I did an abortive attempt to show that the equation $| z + \sqrt{z^2-1} | = K$ gives an ellipse. My conclusion is that this method is even worse than the Cartesian brute force method, so I abandoned it.

(5) Here I thought about elliptical coordinates, which are the 2D versions of oblate spheroidal coordinates mentioned by Jim and Richard. This led me to a good proof that $| z + \sqrt{z^2-1} | = K$ is an ellipse, which I then summarize better in section (6) below.

(6) Here I prove that $| z + \sqrt{z^2-1} | = K$ and $|z-1| + |z+1| = K+K^{-1}$ have the *same* solution locus. Since I know the locus of the second equation is an ellipse, I conclude that $| z + \sqrt{z^2-1} | = K$ describes the same ellipse. To show that both equations have the same locus, I transform from z to $s = z + \sqrt{z^2-1}$. In this new s-space, the equations become $|s| = K$ and $|s + s^{-1} + 2| + |s + s^{-1} - 2| = 2(K + K^{-1})$. I then show

with a *small* amount of algebra that the second equation describes a circle of radius K in s-space, which is of course the same locus that is defined by |s| = K. Since the loci are the same in s-space, they must be the same back in z-space.

(7) Here I show that $| z - \sqrt{z^2-1} | = K$ describes the same ellipse as $| z + \sqrt{z^2-1} | = K$, but this ellipse is on the second Riemann sheet if the ellipse $| z + \sqrt{z^2-1} | = K$ is on the first Riemann sheet. We can think of both functions $z \pm \sqrt{z^2-1}$ as having a branch cut on (-1,1) through which ellipses "drop" when K passes through 1, and move from one sheet to the other where they can start growing again.

1. Prove the string theorem for ellipses.

Claim: The locus of points in the z plane which satisfy $|z-c| + |z+c| = 2a$ form an ellipse centered at the origin with one semi-axis a, distance between from origin to either focus c, and other semi-axis $b = \sqrt{a^2- c^2}$.

Proof: $z = x+iy$ complex notation

$|x+iy+c| + |x+iy-c| = 2a$

$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$

$(x+c)^2 + y^2 + (x-c)^2 + y^2 + 2\sqrt{(x+c)^2 + y^2}\sqrt{(x-c)^2 + y^2} = 4a^2$ // square first time

$x^2 + c^2 + 2x + y^2 + x^2 + c^2 - 2x + y^2 + 2\sqrt{(x+c)^2 + y^2}\sqrt{(x-c)^2 + y^2} = 4a^2$

$2x^2 + 2c^2 + 2y^2 + 2\sqrt{(x+c)^2 + y^2}\sqrt{(x-c)^2 + y^2} = 4a^2$

$x^2 + c^2 + y^2 + \sqrt{(x+c)^2 + y^2}\sqrt{(x-c)^2 + y^2} = 2a^2$

$2a^2 - x^2 - c^2 - y^2 = \sqrt{(x+c)^2 + y^2}\sqrt{(x-c)^2 + y^2}$

$[2a^2 - x^2 - c^2 - y^2]^2 = [(x+c)^2 + y^2)][(x-c)^2 + y^2)]$ // square second time

$[2a^2 - x^2 - c^2 - y^2]^2 - [(x+c)^2 + y^2)][(x-c)^2 + y^2)] =?= 0$

Throw this into Maple and let it do the algebra without errors,

```
restart;
f1 := (2*a^2 - x^2 - c^2 - y^2)^2;
```

$$f1 := (2a^2 - x^2 - c^2 - y^2)^2$$

```
f2 := ((x+c)^2 + y^2)*((x-c)^2 + y^2);
```

$$f2 := ((x+c)^2 + y^2)((x-c)^2 + y^2)$$

```
-(f1-f2)/4;
```

$$-\frac{1}{4}(2a^2 - x^2 - c^2 - y^2)^2 + \frac{1}{4}((x+c)^2 + y^2)((x-c)^2 + y^2)$$

```
simplify(%);
```

$$-a^4 + x^2a^2 + a^2c^2 + a^2y^2 - x^2c^2$$

```
collect(%,x);
```

$$(a^2 - c^2)x^2 - a^4 + a^2c^2 + a^2y^2$$

'

and you see that all terms higher than quadratic (like x^4 or x^2y^2) cancel.
We want to set this = 0 and see what we get:

$(a^2 - c^2)\, x^2 + a^2y^2 = a^4 - a^2c^2$

$b^2\, x^2 + a^2y^2 = a^2(a^2 - c^2) = a^2b^2$

$x^2/a^2 + y^2/b^2 = 1$ QED

2. Prove that $| z + \sqrt{z^2-1} | = K$ is an ellipse

The locus of points which satisfy the equation $| z + \sqrt{z^2-1} | = K$ form an ellipse which has this equation:

$x^2/a^2 + y^2/b^2 = 1$

a = semimajor axis distance = $(K^2 + 1)/2K$ $a+b = K$ $a-b = 1/K$
b = semiminor axis distance = $|K^2 - 1| /2K$

c = distance center to focus = 1 because:

$c^2 = a^2 - b^2 = (a+b)(a-b) = K(1/K) = 1$

eccentricity = $\varepsilon = c/a = 1/a = 2K/(K^2+1)$ // circle has $\varepsilon = 0$

as $K \to 1$, $\varepsilon \to 1$ and $b \to 0$ so we get the super-thin ellipse

Corollary before proof: Imagine the picture drawn for this problem with foci at ± 1. Suppose we scale up the entire picture by a factor c > 0. Then the foci are at ±c, and the resulting ellipse is scaled up by a linear factor of c as well. This means that the a,b,c given above all get multiplied by c, and it means that K is multiplied by c since all distances are scaled up by c. So make the following replacements:

$a \to ca$ $b \to cb$ $1 \to c$ (focus distance) $K \to cK$

The locus of points which satisfy the equation $| z + \sqrt{z^2-c^2} | = K$ form an ellipse which has this equation:

$$x^2/a^2 + y^2/b^2 = 1$$

a = semimajor axis distance = $c(K^2+ 1)/2K$ $a+b = cK$ $a-b = c/K$
b = semiminor axis distance = $c|K^2- 1| /2K$

c = distance center to focus = c because:

$$c^2 = a^2- b^2 = (a+b)(a-b) = cK * c/K= c^2$$

eccentricity = ε = $c/a = 2K/(K^2+1)$ // circle has $\varepsilon = 0$

as $K \to 1$, $\varepsilon \to 1$ and $b \to 0$ so we get the super-thin ellipse

Proof:

In document "find 2D wire equipotential curves in x y.doc" I do a brute force Cartesian-coordinates proof of the above claim with c = 1. I might say I had lots of problems doing the algebra right, and I had help from Maple in debugging my errors but finally got things right.

The first step is to find the real and imaginary parts of $\sqrt{z^2-1}$ which I named a,b and which have no relation to ellipse semi axes, it just happens (unfortunately) that I picked these two letters,

$$\sqrt{z^2-1} = \sqrt{(x+iy)^2-1} = a + ib$$

The results of this calculation of a and b are as follows:

$a^2 = [D + R] / 2$ $a^2 + b^2 = R$
$b^2 = [- D + R] / 2$ $a^2 - b^2 = D$
$D \equiv (x^2 - y^2 - 1)$ $R \equiv \sqrt{D^2 + 4x^2y^2}$ $xy = ab$

For z in the first quadrant, we show that a>0 and b>0. For any other quadrant, the signs may be obtained from the formula xy = ab shown above.

The second step is to knock out some algebra as follows, where I show selected steps only. The game here is to get rid of the square root R object.

$$| z + \sqrt{z^2-1} | = K$$

$$| x + iy + a + ib | = K$$

$(x + a)^2 + (y+b)^2 = K^2$

$K^2 - (a^2 + b^2 + x^2 + y^2) = 2(ax +by)$

$[K^2 - (a^2 + b^2 + x^2 + y^2)]^2 = 4 (ax+by)^2 = 4(a^2x^2 + b^2y^2 + 2abxy)$

Using the a,b equations shown above, this last becomes (again, just showing selected "checkpoints")

$[K^2 - (R + x^2 + y^2)]^2 = 4(a^2x^2 + b^2y^2 + 2x^2y^2)$

$[K^2 - (R + x^2 + y^2)]^2 = 2D(x^2–y^2) + 2R(x^2+ y^2) + 8x^2y^2$

$K^4 + (R + x^2 + y^2)^2 - 2K^2 (R + x^2 + y^2) = 2D(x^2–y^2) + 2R(x^2+ y^2) + 8x^2y^2$

$K^4 + R^2 + (x^2 + y^2)^2 -2K^2 (x^2 + y^2) + 2R (- K^2) = 2D(x^2–y^2) + 8x^2y^2$

$K^4 + R^2 + (x^2 + y^2)^2 -2K^2 (x^2 + y^2) - 2D(x^2–y^2) - 8x^2y^2 = 2RK^2$

At this point we show that the non-K terms on the LHS add up to 1

$R^2 + (x^2 + y^2)^2 - 2D(x^2–y^2) - 8x^2y^2 = 1$

leaving us with

$K^4 - 2K^2 (x^2 + y^2) + 1 = 2RK^2$

When we now square both sides, all square roots are finally gone and we have

$[K^4 - 2K^2 (x^2 + y^2) + 1]^2 = 4K^4 [(x^2 - y^2 - 1)^2 + 4x^2y^2]$

$4K^2(K^2- 1)^2x^2 + 4K^2(K^2+ 1)^2 y^2 = (K^4-1)^2$

$\Rightarrow \quad x^2/A^2 + y^2/B^2 = 1$

A = semimajor axis distance = $(K^2+ 1)/2K$ $\quad A+B = K \quad A-B = 1/K$
B = semiminor axis distance = $|K^2– 1| /2K$

C = distance center to focus = 1 because:

$C^2 = A^2- B^2 = (A+B)(A-B) = K(1/K) = 1$

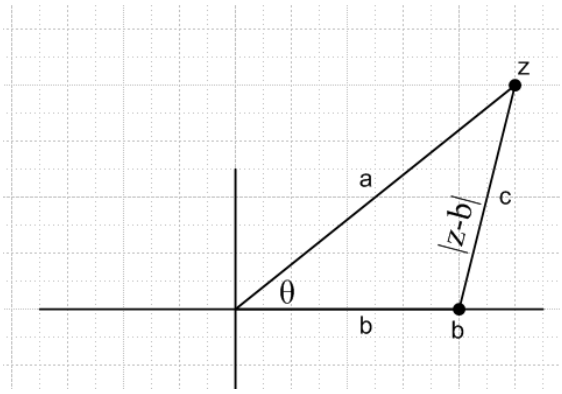
eccentricity = $\varepsilon = C/A = 1/A = 2K/(K^2+1)$ // circle has $\varepsilon = 0$

as $K \rightarrow 1$, $\varepsilon \rightarrow 1$ and $B \rightarrow 0$ so we get the super-thin ellipse surrounding our wire

3. Prove the Law of Cosines and Law of Sines

The law of cosines says: $c^2 = a^2 + b^2 - 2ab\cos(\theta_c)$

Proof: Here is a general triangle with an interesting alignment so one corner is at complex z.



Let $z = re^{i\theta}$. Then want to show that $|z-b|^2 = r^2 + b^2 - 2rb\cos(\theta)$

Write:

$$c^2 = |z-b|^2 = |re^{i\theta} - b|^2 = (re^{i\theta} - b)(re^{-i\theta} - b) = r^2 + b^2 - 2rb\cos(\theta)$$ QED

The law of sines says

$a/\sin(\theta_a) = b/\sin(\theta_b) = c/\sin(\theta_c)$

Proof : $\theta = \theta_c$

$\sin(\theta_c) = y/a$ $\sin(\pi-\theta_a) = y/c$ // two right triangles in the picture

$\sin(\pi-\theta_a) = -\sin(\theta_a - \pi) = \sin(\theta_a)$

Therefore: $y = a\sin(\theta_c) = c\sin(\theta_a)$ => $a/\sin(\theta_a) = c/\sin(\theta_c)$ QED

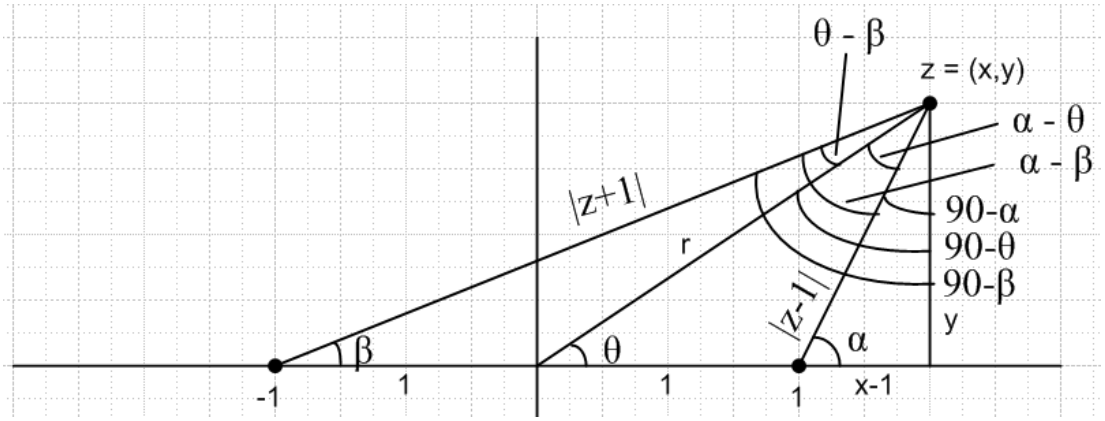
4. Is there a connection between the two ellipse forms shown above?

That is to say, can we make a simple connection between these two ways of representing an ellipse with focus at c = ± 1:

$$|z + \sqrt{z^2-1}| = K$$

$|z-1| + |z+1| = K + K^{-1}$ // using the above formula saying $2a = (K^2 + 1)/K$

We know for sure from all the work above that they really do represent the same ellipse. I played with this for a while, there are *many* facts you can write down, and geometry and trigonometry come into play. First of all, I think the basic picture can only help, where every distance and angle is labeled,



Here are some Law of Cosines: 2θ

left triangle:	$1 = \|z+1\|^2 + r^2 - 2r\ \|z+1\| \cos(\theta-\beta)$	1
	$r^2 = 1 + \|z+1\|^2 - 2\ \|z+1\| \cos(\beta)$	2
	$\|z+1\|^2 = 1 + r^2 - 2r \cos(\pi-\theta)$	3
right triangle:	$1 = \|z-1\|^2 + r^2 - 2r\ \|z-1\| \cos(\alpha-\theta)$	4
	$r^2 = 1 + \|z-1\|^2 - 2\ \|z-1\| \cos(\pi-\alpha)$	5
	$\|z-1\|^2 = 1 + r^2 - 2r \cos(\theta)$	6
combined triangle:	$2^2 = \|z+1\|^2 + \|z-1\|^2 - 2\ \|z+1\|\ \|z-1\| \cos(\alpha-\beta)$	7
	$\|z+1\|^2 = 2^2 + \|z-1\|^2 - 2\ 2\ \|z-1\| \cos(\pi-\alpha)$	8
	$\|z-1\|^2 = 2^2 + \|z+1\|^2 - 2\ 2\ \|z-1\| \cos(\beta)$	9

Here are some Law of Sines:

left triangle:	$\sin(\beta)/r = \sin(\pi-\theta)/\|z+1\| = \sin(\theta-\beta)$	10
right triangle:	$\sin(\theta)/\|z-1\| = \sin(\pi-\alpha)/r = \sin(\alpha-\theta)$	11
combined triangle:	$\sin(\beta)/\|z-1\| = \sin(\pi-\alpha)/\ /\|z+1\| = \sin(\alpha-\beta)/2$	12

Then we know a lot about all the angles

$\sin(\alpha) = y/|z-1|$ $\cos(\alpha) = (x-1)/\ |z-1|$ $\tan(\alpha) = y/(x-1)$

$\sin(\beta) = y/|z+1|$ $\cos(\beta) = (x+1)/|z+1|$ $\tan(\beta) = y/(x+1)$

$\sin(\theta) = y/r$ $\cos(\theta) = x/r$ $\tan(\theta) = y/x$

And of course we know these distances

$|z+1|^2 = (x+1)^2 + y^2$ $|z-1|^2 = (x-1)^2 + y^2$

I am just writing down everything I can think of that might prove helpful.

Maybe our starting position then is this:

$$| z + \sqrt{z^2-1} | = K$$

$$| re^{i\theta} + e^{i(\alpha+\beta)/2} \sqrt{|z^2-1|} | = K$$

$$| r + e^{i[(\alpha+\beta)/2-\theta]} \sqrt{|z+1|} \sqrt{|z-1|} | = K$$

$$r^2 + |z+1| |z-1| + 2r \sqrt{|z+1|} \sqrt{|z-1|} \cos[(\alpha+\beta)/2-\theta] = K^2$$

Our motivation here is to expose the various "distances" like $|z+1|$ as soon as possible. Then

$$[K^2 - r^2 - |z+1| |z-1|]^2 = 4r^2 |z+1| |z-1| \cos^2[(\alpha+\beta)/2-\theta] = 4r^2 |z+1| |z-1| (1 + \cos [(\alpha+\beta)-2\theta])$$

At this point let's define the shorthand symbols $a = |z+1|$ and $b = |z-1|$ so the above says

$$[K^2 - r^2 - ab]^2 = 4r^2 ab(1 + \cos[(\alpha+\beta)-2\theta]) \qquad (*)$$

So at least we have something with distances, no square roots, and some angle action. But the question is: how are we ever going to isolate the terms $|z-1| + |z+1|$? Well, let's do a bit from that end :

$a+b = q$ $q = \text{constant} = K + K^{-1}$

$a^2 + b^2 + 2ab = q^2$ $\Rightarrow 2ab = (q^2 - a^2 - a^2)$

and maybe we can use this later below. But we have to deal with the angle issue in (*) at some point, so might as well attack it right now:

$$\cos(\alpha+\beta-2\theta) = \cos((\alpha-\theta) + (\beta-\theta)) = \cos(\alpha-\theta) \cos(\beta-\theta) - \sin(\alpha-\theta) \sin(\beta-\theta)$$

Then we can use our laws of sines and cosines to get rid of all angles. We have from above

$1 = b^2 + r^2 - 2 r b \cos(\alpha-\theta) \Rightarrow \cos(\alpha-\theta) = [b^2 + r^2 -1]/(2rb)$
$1 = a^2 + r^2 - 2 r a \cos(\beta-\theta) \Rightarrow \cos(\beta-\theta) = [a^2 + r^2 -1]/(2a)$

$\sin(\theta)/b = \sin(\alpha-\theta)$
$\sin(\theta)/a = \sin(\theta-\beta)$

Then

$$\cos(\alpha+\beta-2\theta) = [b^2 + r^2 -1]/(2rb) * [a^2 + r^2 -1]/(2ra) - \sin^2(\theta)/(ba)$$

$$= ([b^2 + r^2 -1]/(2r) * [|z+1|^2 + r^2 -1]/(2r) - \sin^2(\theta)) / (|z-1|\,|z+1|)$$

$$= ([b^2 + r^2 -1] * [a^2 + r^2 -1] - 4r^2\sin^2(\theta)) / (4r^2 b\, a)$$

$$\Rightarrow \quad 4r^2 b\, a \cos(\alpha+\beta-2\theta) = b^2 + r^2 -1] * [a^2 + r^2 -1] - 4r^2\sin^2(\theta)$$

$$\text{and} \quad 4r^2 ba\ 1 = 4r^2 ba\ 1$$

so add these last two lines to get

$$4r^2 ba\,[\ 1 + \cos(\alpha+\beta-2\theta)\ ] = [b^2 + r^2 -1]\ [a^2 + r^2 -1] + 4r^2\, ba - \sin^2\theta$$

Now recall result (*) from above

$$[K^2 - r^2 - ab\ ]^2 = 4r^2\, ab(\ 1 + \cos\ [(\alpha+\beta)/2-\theta]\ ) \qquad (*)$$

which we can now rewrite as

$$[K^2 - r^2 - ab\ ]^2 = [b^2 + r^2 -1]\ [a^2 + r^2 -1] + 4r^2\, ba - \sin^2\theta$$

so we are now down to only distances plus the one angle θ. We can use $2ab = (q^2 - a^2 - a^2)$ from "the other direction" to replace ab everywhere, but we are still stuck with r and θ which have to be removed!

I am wearing down. I thought there would be some very fast way to do this. But I can see that whether you do this in Cartesians or "angles", it is still going to be painful. So I will let the subject rest. I have shown conclusively already that these two equations describe the same ellipse:

$$| z + \sqrt{z^2-1}\ | = K$$

$$|z-1| + |z+1| = K + K^{-1} \qquad \text{// using the above formula saying } 2a = (K^2 + 1)/K$$

which ellipse is this

$$x^2/a^2 + y^2/b^2 = 1$$

a = semimajor axis distance = $(K^2+ 1)/2K$ $\qquad 2a = K + K^{-1}$
b = semiminor axis distance = $|K^2 - 1|\ /2K$
c = distance center to focus = 1 because:

Notice that both forms tell us that |z| = K/2 for large K and z.

Maybe there is some better "coordinate system" that makes all this stuff obvious, like this one:

5. Elliptical coordinates approach to showing equivalence

These coordinates are given by (specializing to focus distance = ±1)

$x = ch\mu \cos\nu$
$y = sh\mu \sin\nu$ $\qquad z = x+iy = ch(\mu+i\nu) = ch(w)$

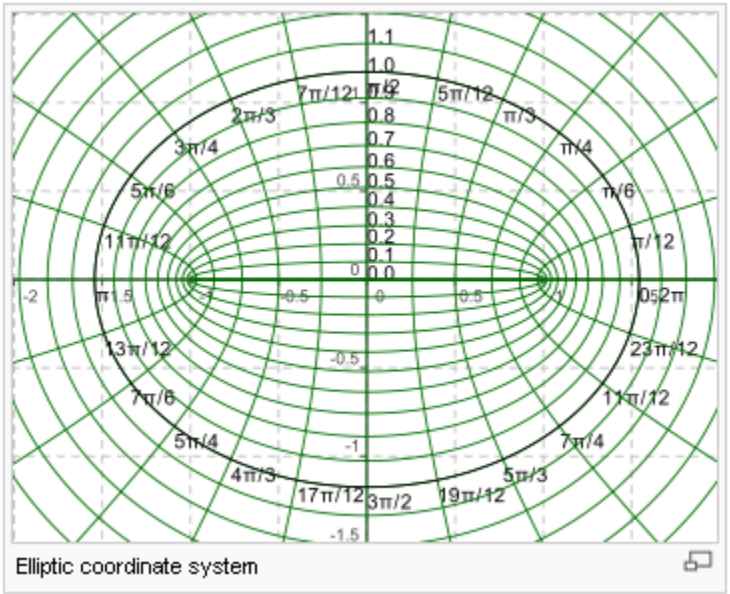
Notice that we then have

$$x^2/ch^2\mu + y^2/sh^2\mu = 1 \quad \Rightarrow \quad \text{ellipse with } a = ch\mu \text{ and } b = sh\mu \text{ and } c = 1$$

So μ = constant means an **ellipse** in the (x,y) space, so label each ellipse by μ.
Similarly

$$x^2/\cos^2\nu - y^2/\sin^2\nu = 1$$

So ν = constant means a **hyperbola** in the (x,y) space, each hyperbola labeled by ν. Here is the wiki picture:



Then

$$z^2-1 = ch^2(w) - 1 = sh^2(w) \quad \Rightarrow \quad \sqrt{z^2-1} = sh(\omega)$$

Then we have

$$z + \sqrt{z^2-1} \;= \mathrm{ch}(w) + \mathrm{sh}(w) \;= e^w$$

$$z - 1 = \mathrm{ch}w - 1$$
$$z + 1 = \mathrm{ch}w + 1$$

Notice how we have now replaced the messy object $z + \sqrt{z^2-1}$ with the simple object e^w ! And then the other two objects are not too bad. This detangling of $\sqrt{z^2-1}$ is the big benefit here.

Then what we would have to show that these two things are the same:

$| z + \sqrt{z^2-1} | = K$ is really $|e^w| = K$

$|z-1| + |z+1| = K + K^{-1}$ is really $| \mathrm{ch}w - 1| + |\mathrm{ch}w + 1| = K + K^{-1}$

Then to show these were the same, we would have to show that

$$|e^w| + |e^{-w}| = | \mathrm{ch}w - 1| + |\mathrm{ch}w + 1| = (1/2) \{ | e^w + e^{-w} - 2| + | e^w + e^{-w} + 2| \}$$

Suppose we now define a *third* complex variable, call it $s = e^w$. Then we have to show that

$$|s| + 1/|s| = (1/2) \{ | s + s^{-1} - 2| + | s + s^{-1} + 2| \}$$

But now let $s = re^{i\theta}$ (we are now thinking about vectors in the s-plane) so this says

$$2(r + r^{-1}) = | re^{i\theta} + r^{-1}e^{-i\theta} - 2| + | re^{i\theta} + r^{-1}e^{-i\theta} + 2|$$

Now THIS looks like something we could easily verify or disprove. We have

$$| re^{i\theta} + r^{-1}e^{-i\theta} \pm 2| = | r(C_\theta + iS_\theta) + r^{-1}(C_\theta - iS_\theta) \pm 2|$$

$$= | C_\theta(r + r^{-1}) \pm 2 + i S_\theta(r - r^{-1}) |$$

Now shorthand says $a = r + r^{-1}$ and $b = r - r^{-1}$ so we have

$$| re^{i\theta} + r^{-1}e^{-i\theta} \pm 2| = | (a C_\theta \pm 2) + i bS_\theta | = \sqrt{(a C_\theta \pm 2)^2 + (bS_\theta)^2}$$

Notice that $a^2 - b^2 = 4$, a key fact needed below $s = e^w = \exp(\mathrm{ch}^{-1}z)$ $z = \mathrm{ch}(\ln(s))$

Then **here is what we have to show is true:**

$$2a = \sqrt{(a C_\theta + 2)^2 + (bS_\theta)^2} + \sqrt{(a C_\theta - 2)^2 + (bS_\theta)^2} \qquad a^2 - b^2 = 4$$

It does not seem very likely does it. But go ahead and try. Square both sides:

$$4a^2 = (a\,C_\theta + 2)^2 + (bS_\theta)^2 + (a\,C_\theta - 2)^2 + (bS_\theta)^2 + 2\sqrt{(a\,C_\theta + 2)^2 + (bS_\theta)^2}\,\sqrt{(a\,C_\theta - 2)^2 + (bS_\theta)^2}$$

$$4a^2 = 2a^2\,C_\theta^2 + 8 + 2b^2\,S_\theta^2 + 2\sqrt{(a\,C_\theta + 2)^2 + (bS_\theta)^2}\,\sqrt{(a\,C_\theta - 2)^2 + (bS_\theta)^2}$$

$$2a^2 = a^2\,C_\theta^2 + 4 + b^2\,S_\theta^2 + \sqrt{(a\,C_\theta + 2)^2 + (bS_\theta)^2}\,\sqrt{(a\,C_\theta - 2)^2 + (bS_\theta)^2}$$

$$2a^2 = a^2\,C_\theta^2 + 4 + b^2\,(1 - C_\theta^2) + \sqrt{(a\,C_\theta + 2)^2 + (bS_\theta)^2}\,\sqrt{(a\,C_\theta - 2)^2 + (bS_\theta)^2}$$

$$2a^2 = (a^2 - b^2)\,C_\theta^2 + 4 + b^2 + \sqrt{(a\,C_\theta + 2)^2 + (bS_\theta)^2}\,\sqrt{(a\,C_\theta - 2)^2 + (bS_\theta)^2}$$

$$2a^2 = 4C_\theta^2 + 4 + b^2 + \sqrt{(a\,C_\theta + 2)^2 + (bS_\theta)^2}\,\sqrt{(a\,C_\theta - 2)^2 + (bS_\theta)^2}$$

Now isolate and square both sides

$$[\,2a^2 - 4C_\theta^2 - 4 - b^2\,]^2 = [\,(a\,C_\theta + 2)^2 + (bS_\theta)^2][(a\,C_\theta - 2)^2 + (bS_\theta)^2\,]$$

It still seems very unlikely this is true, but carry on:

$$\text{RHS} = [\,(a\,C_\theta + 2)^2 + (bS_\theta)^2][(a\,C_\theta - 2)^2 + (bS_\theta)^2\,]$$

$$= (a\,C_\theta + 2)^2(a\,C_\theta - 2)^2 + (bS_\theta)^2\,\{\,(a\,C_\theta + 2)^2 + (a\,C_\theta - 2)^2\,\} + (bS_\theta)^4$$

$$= (a^2\,C_\theta^2 - 4)^2 + (bS_\theta)^2\,\{\,2\,a^2C_\theta^2 + 8\,\} + (bS_\theta)^4$$

$$= (a^2\,C_\theta^2 - 4)^2 + b^2\,(1 - C_\theta^2)\,\{\,2\,a^2C_\theta^2 + 8\,\} + b^4\,(1 - C_\theta^2)^2$$

$$= C_\theta^4\,\{\,a^4 - 2a^2b^2 + b^4\} + C_\theta^2\,\{\,-8a^2 - 8b^2 + 2a^2b^2 - 2b^4\} + \{\,16 + 8b^2 + b^4\}$$

$$= C_\theta^4\,\{\,(a^2 - b^2)^2\} + 2C_\theta^2\,\{\,-4a^2 - 4b^2 + a^2b^2 - b^4\} + \{\,16 + 8b^2 + b^4\}$$

$$= 16\,C_\theta^4 + 2C_\theta^2\,\{\,-4a^2 - 4b^2 + a^2b^2 - b^4\} + \{\,16 + 8b^2 + b^4\}$$

$$\text{LHS} = [\,2a^2 - 4C_\theta^2 - 4 - b^2\,]^2$$

$$= 4a^4 + 16C_\theta^4 + 16 + b^4 - 16\,a^2\,C_\theta^2 - 16a^2 - 4a^2b^2 + 32\,C_\theta^2 + 8b^2\,C_\theta^2 + 8b^2$$

$$= 16C_\theta^4 + C_\theta^2\,\{-\,16\,a^2 + 32 + 8b^2\} + \{\,4a^4 + 16 + b^4 - 16a^2 - 4a^2b^2 + 8b^2\,\}$$

At least the C_θ^4 terms match. Now let's see if the C_θ^2 terms match

$$\{\,-4a^2 - 4b^2 + a^2b^2 - b^4\} \stackrel{?}{=} \{-\,16\,a^2 + 32 + 8b^2\}/2$$

$-4a^2 - 4b^2 + a^2b^2 - b^4 + 8a^2 - 16 - 4b^2 = 0$?

$- 8b^2 + a^2b^2 - b^4 + 4a^2 - 16$

$b^2(-8 + a^2 - b^2) + 4a^2 - 16$

$b^2(-4) + 4a^2 - 16 \quad = \quad 4(a^2 - b^2) - 16 \quad = 16 - 16 = 0$!!!

Now let's check to see if the constant terms match:

$\{ 16 + 8b^2 + b^4 \} =? \{ 4a^4 + 16 + b^4 - 16a^2 - 4a^2b^2 + 8b^2 \}$

$16 + 8b^2 + b^4 - 4a^4 - 16 - b^4 + 16a^2 + 4a^2b^2 - 8b^2 = 0$?

$-4a^4 + 16a^2 + 4a^2b^2$

$-a^4 + 4a^2 + a^2b^2$

$-a^2 + 4 + b^2 \quad = 4 - (a^2 - b^2) = 0$!!!

As shown below, none of this algebra is really needed

6. Showing equivalence by doing a change of variable

Here we give a more compact presentation of the previous section.

We do a change of variables from z to s :

$s = z + \sqrt{z^2-1} \quad$ with inverse $\quad z = ch[\ln(s)] = (1/2)(s + s^{-1})$

This mapping was discussed on page 94-5 of Ahlfors where we saw that it maps circles in s space into ellipses in z space. Below we are going to see this exact thing happening.

In this new s variable we have

$2(z \pm 1) = (s + s^{-1}) \pm 1 = [s + s^{-1} \pm 2]$

In s-space, our two z-space equations (the two we want to show have the same locus)

$| z + \sqrt{z^2-1} | = K \quad \rightarrow \quad |s| = K$

$|z-1| + |z+1| = K + K^{-1} \quad \rightarrow \quad |s + s^{-1} + 2| + |s + s^{-1} - 2| = 2(K + K^{-1})$

The first equation obviously describes a circle in s-space with radius r = K. We shall now show that the second equation describes *this same circle*. Consider the LHS of the second equation with $s = re^{i\theta}$,

$$\text{LHS} = \quad |s + s^{-1} + 2| + |s + s^{-1} - 2|$$

$$= | re^{i\theta} + r^{-1}e^{-i\theta} + 2| + | re^{i\theta} + r^{-1}e^{-i\theta} - 2|$$

$$= \sqrt{(a\,C_\theta + 2)^2 + (bS_\theta)^2} + \sqrt{(a\,C_\theta - 2)^2 + (bS_\theta)^2}$$

where

$$a = r + r^{-1} \qquad b = r - r^{-1} \qquad \Rightarrow \qquad a^2 - b^2 = 4$$

A look inside these radicals shows that each argument is a perfect square:

$$(a\,C_\theta \pm 2)^2 + b^2S_\theta^2 = (a\,C_\theta \pm 2)^2 + (a^2-4)\,S_\theta^2$$

$$= a^2C_\theta^2 + 4 \pm 4aC_\theta + a^2\,S_\theta^2 - 4\,S_\theta^2 = a^2 + 4 \pm 4aC_\theta - 4\,S_\theta^2 = a^2 + 4 \pm 4aC_\theta - 4\,(1 - C_\theta^2)$$

$$= a^2 \pm 4aC_\theta + 4\,C_\theta^2 = (a \pm 2C_\theta)^2$$

The function a(r) = r + 1/r has a minimum value of 2, so $(a \pm 2C_\theta)$ is always positive so we take these to be the square roots which are our "distances".,

$$|s + s^{-1} + 2| + |s + s^{-1} - 2|$$

$$= \sqrt{(a\,C_\theta + 2)^2 + (bS_\theta)^2} + \sqrt{(a\,C_\theta - 2)^2 + (bS_\theta)^2}$$

$$= (a + 2C_\theta) + (a - 2C_\theta) = 2a$$

so we have now shown that

$$|s + s^{-1} + 2| + |s + s^{-1} - 2| = 2\,(r + r^{-1})$$

So, as we vary θ, $s = re^{i\theta}$ describes a circle of radius r and the above equation is true for any θ. Thus, the locus of this equation is a circle of radius r which we identify with K. But this is the same as the locus of our first equation above, which was |s| = r = K.

Conclusion: In s-space, these two equations describe the same circle of radius K:

$$|s| = K$$

$$|s + s^{-1} + 2| + |s + s^{-1} - 2| = 2(K + K^{-1})$$

If we map these two equations back into z-space, they must describe the same locus in z-space. The z-space equations are these

$$| z + \sqrt{z^2-1} | = K$$

$$|z-1| + |z+1| = K + K^{-1}$$

We know from Section 1 above that the second equation describes an ellipse with focal distance c = 1 and with semimajor axis $(K + K^{-1})/2$. Therefore, the first equation must describe this same ellipse.

7. One extra detail.

If you go back, you see that everything works the same way for $| z - \sqrt{z^2-1} | = K$, something mentioned in Ahlfors. So this thing is also an ellipse in z-space. It is easy to show that the limit of $z - \sqrt{z^2-1}$ as $|z| \to \infty$ is 0 (being of order 1/z). So the very large ellipses for large |z| described by the equation $| z - \sqrt{z^2-1} | = K$ correspond to very small values of K. When K increases to .99, you have a super thin ellipse. as K goes to 1.01, you can think of this ellipse as falling through the branch cut between -1 and 1 and appearing on the second Riemann sheet. As K increases to ∞, the ellipses grow larger on this second sheet. *Meanwhile*, back on the first sheet, the equation $| z + \sqrt{z^2-1} | = K$ describes a set of ellipses which shrink down from large to small as K moves down from ∞ to 1, then THESE ellipses move onto the second sheet as K = .99 and below. So really, the two signs $z \pm \sqrt{z^2-1}$ do the same thing, but with the meaning of first and second sheet reversed.

8. A theorem based on the above efforts. [added 11.29.09]

We showed above that these two loci describe the same ellipse

$$| z + \sqrt{z^2-1} | = K$$

$$|z-1| + |z+1| = K + K^{-1}$$

It is interesting and perhaps useful two write out both these loci in x,y coordinates. The second one is the easiest and we find that

$$\sqrt{(x-1)^2+ y^2} + \sqrt{(x+1)^2+ y^2} = K + K^{-1}$$

$$\left(\sqrt{(x-1)^2+ y^2} + \sqrt{(x+1)^2+ y^2} \right)^2 = K^2 + K^{-2} + 2$$

What is the corresponding equation that goes with the first line? Let's just extract some lines from our Cartesian proof above:

$$D \equiv (x^2 - y^2 - 1) \qquad R \equiv \sqrt{D^2 + 4x^2y^2}$$

$$K^4 - 2K^2(x^2 + y^2) + 1 = 2RK^2$$

Rewrite this last line as

$$K^4 + 1 = 2K^2 [\ x^2 + y^2 + \sqrt{(x^2 - y^2 - 1)^2 + 4x^2y^2}$$

$$(K^4 + 1)/K^2 = 2[\ x^2 + y^2 + \sqrt{(x^2 - y^2 - 1)^2 + 4x^2y^2}\ ]$$

$$K^2 + K^{-2} = 2[\ x^2 + y^2 + \sqrt{(x^2 - y^2 - 1)^2 + 4x^2y^2}\ ]$$

$$K^2 + K^{-2} + 2 = 2[\ (x^2 + y^2 + 1) + \sqrt{(x^2 - y^2 - 1)^2 + 4x^2y^2}\ ]$$

Comparing to the above, we get this interesting identity:

$$(\ \sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2}\ )^2 = 2[\ (x^2 + y^2 + 1) + \sqrt{(x^2 - y^2 - 1)^2 + 4x^2y^2}\ ]$$

Now suppose we multiply through by a constant a^2 with $a>0$. The above becomes,

$$(\ \sqrt{(ax-a)^2 + a^2y^2} + \sqrt{(ax+a)^2 + a^2y^2}\ )^2 = 2[\ (a^2x^2 + a^2y^2 + a^2) + \sqrt{(a^2x^2 - a^2y^2 - a^2)^2 + 4a^4x^2y^2}\ ]$$

Now define $x' = ax$, $y' = ay$ and rewrite the above, then after doing that, replace x',y' with x,y'

$$(1/2)\ (\ \sqrt{(x-a)^2 + y^2} + \sqrt{(x+a)^2 + y^2}\ )^2 = (x^2 + y^2 + a^2) + \sqrt{(x^2 - y^2 - a^2)^2 + 4x^2y^2}$$

which I will now call **Ellipse String Theorem 3**. The quantity being squared on the left is the ellipse string length. This is a non-trivial "fact" that is showing up right now in my comparison of my oblate spheroidal disk potential and the Jackson/Weber formula for the same, see elsewhere. But here is where I did all the work to get this theorem, so we might as well export it and then use it elsewhere. There are perhaps some square root sheet issues to keep track of.

Theorem Warning: The theorem was derived with x,y,a all real numbers. Even so, the theorem has some ambiguity depending on the relative values of the variables. For example, suppose we consider y = 0 to get where our theorem says

$$(1/2)\ (\ \sqrt{(x-a)^2} + \sqrt{(x+a)^2}\ )^2 = (x^2 + a^2) + \sqrt{(x^2 - a^2)^2}$$

If x > a, we are pretty content to say this

$$(1/2)\ (\ (x-a) + (x+a)\ )^2 = (x^2 + a^2) + (x^2 - a^2)$$

or

$$(1/2)\ (2x)^2 = 2x^2$$ which checks out.

But suppose -a < x < a. Then we wonder what the square roots mean. I claim in this case this is how you need to do to evaluate:

$(1/2)\,(\ (a-x)\ +\ (x+a)\ )^2\ =\ (x^2\ +a^2)+(a^2\ -\ x^2)$

$(1/2)\ (2a)^2\ =\ 2a^2$ which checks out

And finally, in the case $x < -a$, our evaluation will be this:

$(1/2)\,(\ (a-x)\ +[\ -(x+a)]\ )^2\ =(x^2\ +a^2)+(x^2-\ a^2)$

$(1/2)\ (\ -2x)^2\ =\ 2x^2$ which again checks out.

I don't have any way to write the theorem to remind one of these issues.

$$(1/2)\ (\ \sqrt{(x-a)^2+\ y^2}\ \ +\ \sqrt{(x+a)^2+\ y^2}\ )^2 =\ \ (x^2\ +y^2+a^2)+\sqrt{(x^2\ -\ y^2-\ a^2)^2+4x^2y^2}$$

We are on the positive branch of all three roots, so if you "take something out" , you need to make sure what you take out is positive, and that may require you to reverse the sign inside the parens.

Summary:

Ellipse Theorem 1: $|\ z \pm \sqrt{z^2-1}\ | = K$ is an ellipse (either sign OK, different sheets)

Ellipse Theorem 2: It is the same ellipse as $|z-1| + |z+1| = K + K^{-1}$

Ellipse Theorem 3:

$$(1/2)\ (\ \sqrt{(x-a)^2+\ y^2}\ \ +\ \sqrt{(x+a)^2+\ y^2}\ )^2 =\ \ (x^2\ +y^2+a^2)+\sqrt{(x^2\ -\ y^2-\ a^2)^2+4x^2y^2}$$