Definition of Focal Distance and Eccentricity for an Ellipse PhL 11.13.16

This document is a definition of the focal distance of an ellipse.

1. Start with this ellipse centered at the origin

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which can be written as

$$\frac{b^2x^2+a^2y^2}{a^2b^2} = 1 \qquad \Rightarrow \qquad b^2x^2 + a^2y^2 - a^2b^2 = 0$$

You could solve this for y^2 as follows:

$$(b/a)^2x^2 + y^2 - b^2 = 0 \qquad \Rightarrow \qquad y^2 = b^2 - (b/a)^2x^2 \qquad (1)$$

2. Consider the two points (±c,0) where c is at the moment just a number. Run a string from the left point (-c,0) to some (x,y) on the ellipse, and thence from (x,y) to the right point (c,0). Let the length of this string be 2L where in general L = L(x,y). The total length 2L of the string is this

$$2L(x,y) = \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} \equiv \sqrt{M} + \sqrt{P}$$

so

$$4L^2 = M + P + 2\sqrt{M}\sqrt{P}$$

$$4L^2 - M - P = 2\sqrt{M}\sqrt{P}$$

$$[4L^2 - M - P]^2 = 4MP$$

$$f = [4L^2 - M - P]^2 - 4MP = 0$$

Maple simplifies this result so that f/16 is the following

$$L^4 + (-x^2 - c^2 - y^2)L^2 + x^2c^2$$

Therefore,

$$L^4 - (x^2+y^2+c^2)L^2 + x^2c^2 = 0 \qquad (2)$$

$$L^4 - (x^2+[b^2 - (b/a)^2x^2]+c^2)L^2 + x^2c^2 = 0$$

For some *general* value of c, this is a quadratic equation in L^2 which one could solve and one would obtain some messy solution L(x; a,b,c). The length 2L is different for different points (x,y) on the ellipse.

But now go back to (2) above and just process things again :

$$L^4 - x^2L^2 - y^2L^2 - c^2L^2 + x^2c^2 = 0$$

$$L^4 - x^2L^2 - [b^2 - (b/a)^2x^2]L^2 - c^2L^2 + x^2c^2 = 0 \qquad \text{// using (1) here}$$

$$L^4 - x^2L^2 - b^2L^2 + (b/a)^2L^2x^2 - c^2L^2 + x^2c^2 = 0$$

$$a^2L^4 - x^2a^2L^2 - a^2b^2L^2 + b^2L^2x^2 - a^2c^2L^2 + x^2a^2c^2 = 0$$

$$x^2[-a^2L^2 + b^2L^2 + a^2c^2] + [a^2L^4 - a^2b^2L^2 - a^2c^2L^2] = 0$$

or

$$x^2[-a^2L^2 + b^2L^2 + a^2c^2] + a^2L^2[L^2 - b^2 - c^2] = 0$$

Question: Is there a value for c such that this equation is true for all x on the ellipse?

In order for this equation to be true for any x, we must have these two equations be valid

$$-a^2L^2 + b^2L^2 + a^2c^2 = 0$$
$$L^2 - b^2 - c^2 = 0 . \qquad \text{// } L^2 = b^2+c^2$$

The second says $L^2 = b^2+c^2$ and then the first says

$$-a^2(b^2+c^2) + b^2(b^2+c^2) + a^2c^2 = 0$$

$$-a^2b^2 - a^2c^2 + b^4 + b^2c^2 + a^2c^2 = 0$$

$$-a^2b^2 + b^4 + b^2c^2 = 0$$

$$-a^2 + b^2 + c^2 = 0$$

$$c^2 = a^2 - b^2$$

The conclusion is that if we set $c = \sqrt{a^2-b^2}$, then the distance 2L is the same for all x, and the distance is in fact 2L = 2a. This is the *only* value of c for which the string length will be constant.

We have just proven this theorem:

Theorem: Start with an ellipse with semi-axes a and b centered at the origin, a > b. Run a string from the left point (-c,0) to some (x,y) on the ellipse, and thence from (x,y) to the right point (c,0). Let the length of this string be 2L where in general L = L(x,y). Then:

(1) if we set $c = \sqrt{a^2-b^2}$, then the length of the string is 2L(x,y) = 2a which is independent of (x,y).

(2) If c is any other value, then the length of the string length is NOT constant.

Definition: The points ±c which make the string length constant are called "**the focal points**" of the ellipse.

Definition: The **eccentricity** of the ellipse is defined as $\varepsilon = c/a$. If $c \approx 0$, the two focal points are close to the center point and the ellipse is close to a circle. If $\varepsilon \to 1$, the ellipse is very wide and thin.