n dimensions: Geometry, Divergence Theorem, Green PhL 3.22.11

This doc is really a supporting doc for **"Stakgold Chapter 7"** since below I derive the Divergence Theorem for 1+n dimensional evolution spacetime and then I apply that divergence theorem to the evolution cylinder for both heat and wave equations. But I want to store this doc in my "**geometry**" section since it contains basic data for n-dimensional geometry, such as the normal vector to a surface embedded in n dimensions. The divergence theorem in a sense is a geometrical theorem since it involves R^n space and volumes and surfaces within that space, these certainly being geometric concepts. I suppose such volumes and surfaces are the subject of **"calculus"** as well, especially when we get into the subject of derivatives and integrals (as appear in the divergence theorem). And of course all this stuff falls within the bailiwick of the Cartesian case of **"Curvilinear Coordinates"** as well. So really I have four candidate storage locations for this doc (bolded above), but I don't want to store multiple copies for editing reasons. So for now it goes into: geometry.

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The question is: how do n=3 geometric notions get extended to other values of n, in particular n > 3?
I will be using n = 4 as an example sometimes, and Cartesian in general. We are always talking Euclidean space R^n with a diag(1) metric tensor.

1. Vector and Gradient Vector

Certainly a vector is well defined

$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} + t\,\hat{\mathbf{t}}$ n=4

$\mathbf{r} = \Sigma_{i=1}^{n} x_i\,\hat{\mathbf{i}}$ n = n $x_i = \mathbf{r} \bullet \hat{\mathbf{i}}$

One vector of interest is the **gradient** vector and we have

$\nabla_4\varphi = \partial_x\varphi\,\hat{\mathbf{x}} + \partial_y\varphi\,\hat{\mathbf{y}} + \partial_z\varphi\,\hat{\mathbf{z}} + \partial t_x\varphi\,\hat{\mathbf{t}}$ n=4

$\nabla_n \varphi = \Sigma_{i=1}^{n} \partial_i \varphi \, \hat{\mathbf{i}}$ n=n

2. Dot Product

A dot product of two vectors is pretty clear

$\mathbf{r} \bullet \mathbf{s} = \Sigma_{i=1}^{n} r_i s_i$ n=n

3. Cross Product.

The cross product as I will define it here *requires* n-1 vectors on the RHS. This for n = 4

$\mathbf{a} = \mathbf{b} \times \mathbf{c} \times \mathbf{d}$ meaning $a_i = \varepsilon_{ijkm} b_j c_k d_m$ n=4

and this would by antisymmetric under the interchange of any two adjacent vectors on the RHS. You would find that this "cross product" is perp to all the component vectors, for example,

$\mathbf{c} \bullet \mathbf{b} \times \mathbf{c} \times \mathbf{d} = c_i \, \varepsilon_{ijkm} b_j c_k d_m = (c_i \, c_k \, \varepsilon_{ijkm}) \, b_j d_m = 0$ n=4

This notation is a little misleading because it suggests that maybe **b** x **c** means something when it in fact has no meaning for n = 4. Maybe I should say $\mathbf{a} = \boldsymbol{\varepsilon}(bcd)$.

4. Volume Element

The volume element is also pretty clear

$d^n r = dx_1 dx_2 ..... dx_n$ n=n

5. Area Element

The area element should be perpendicular to each of the $d\mathbf{x}_i$ and should have a magnitude that is the product of the dx_i . The ideal candidate is just the generalized cross product mentioned above. So we could say for n = 4,

$d\mathbf{A}_{ijk} = d\mathbf{x}_i \times d\mathbf{x}_j \times d\mathbf{x}_k$ => $dA_{ijk} = dx_i dx_j dx_k$ n=4

and the direction of $d\mathbf{A}_{ijk}$ is whatever it is. This definition is consistent with the n=3 situation where we normally say things like $d\mathbf{A}_{xy} = d\mathbf{x} \times d\mathbf{y}$.

6. Normal Vector

How do you find an n-dimensional vector **n** which is normal to a surface of dimension n-1 which is embedded in n dimensional space? (we allow symbol n to be overloaded here) I think I might know how to answer this question. See "surface geometry questions.doc" from which I quote and edit: [jump to the end to learn that $\hat{\mathbf{n}} = \nabla_{nd}(g) / |\nabla_{nd}(g)|$, and of course the n-gradient is shown above in item 1]

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0.1 Geometric meaning of the nD gradient $\nabla_{nd}(g)$.

Suppose you have a smooth function $g(x_1,x_2,....x_n)$. Then we know that for any point in R^n space, the following is true (to first order in d**r**),

$$g(\mathbf{r}+d\mathbf{r}) = g(\mathbf{r}) + \nabla_{nd}(g) \bullet d\mathbf{r}$$

Now suppose we consider the equation $g(x_1,x_2,....x_n) = 0$ which defines a surface in R^n of dimension n-1. Then for any point on this surface, we can say

$$g(\mathbf{r}+d\mathbf{r}) = \nabla_{nd}(g) \bullet d\mathbf{r}$$

Imagine our point on the surface surrounded by a little n-dimensional sphere of possible directions to move making d**r**. If we move in the direction $\nabla_{nd}(g)$, then we will get the largest possible change in the value of g for a given |d**r**|. Therefore, for a point on the surface $g(x_1,x_2,....x_n) = 0$, $\nabla_{nd}(g)$ points in the direction of the maximum change of g. [It points in the direction of maximum increase, and of course then $-\nabla_{nd}(g)$ points in the direction of maximum decrease.] On the other hand, suppose we move in a direction tangent to the surface, such that d**r** = ds $\hat{\mathbf{t}}$. In this case, since g = 0 at all points on the surface, we must have g(**r** + ds $\hat{\mathbf{t}}$**)** = 0. But from our first statement above, we know then that 0 = g(**r** + ds $\hat{\mathbf{t}}$**)** = $\nabla_{nd}(g) \bullet$ ds $\hat{\mathbf{t}}$. Since this dot product is 0, this tells that the vector $\nabla_{nd}(g)$ must be perpendicular to all possible $\hat{\mathbf{t}}$ **.** Thus, we may identify $\nabla_{nd}(g(r))$ as the normal to the surface g(**r**)=0 for **r** being a point on the surface g = 0. In other words, $\hat{\mathbf{n}} = \nabla_{nd}(g) / |\nabla_{nd}(g)|$. This, then, is how you can define the normal vector for a surface g = 0 in R_n.

Our conclusions are then these

For a point **r** located on the surface g(**r**) = 0:

(1) The vector $\nabla_{nd}(g)$ points in the direction of the normal to the surface, so that $\hat{\mathbf{n}} = \nabla_{nd}(g) / |\nabla_{nd}(g)|$.
(2) The function g has its maximal amount of change in this direction.

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7. Divergence Theorem

What can be said concerning the divergence theorem in n dimensions? [In this section, we just have n dimensions all on an equal footing and **A** is an n-dimensional vector field, etc]

$$\int dV\ \nabla \bullet \mathbf{A} = \int d\mathbf{S} \bullet \mathbf{A}$$

Well, dV means $dx_1.....dx_n$ as noted above, and $\nabla \bullet \mathbf{A} = \Sigma_{i=1}^{n} \partial_i A_i$. The integral on the right is over a boundary surface which is of just the type mentioned in Section 6 above. It is a surface of dimension n-1 embedded in an n dimensional space, so we could regard it as some $g(x_1,x_2,....x_n) = 0$. What the surface element means is this:

$$d\mathbf{S} = dS\, \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is specifically the unit vector found in Section 6 above. It is meant to be the choice of the two directions which is "outward", which might take some pondering to get exactly right in a general case. But assuming we have found that proper direction choice, we have

$$\int d\mathbf{S} \bullet \mathbf{A} = \int dS\, \hat{\mathbf{n}} \bullet \mathbf{A}$$

So in computing this surface integral, at each point on the surface we have to compute $\hat{\mathbf{n}}$, and then the thing we integrate is $\hat{\mathbf{n}} \bullet \mathbf{A}$. You can write this as A_n , but n is not just a 1,2,3... index, it is $\hat{\mathbf{n}} \bullet \mathbf{A}$ where $\hat{\mathbf{n}}$ is varying continuously along your surface g=0. We might jazz up our divergence theorem with some extra labels

$$\int_{\mathbf{R_n}} d^n x\, \nabla \bullet \mathbf{A}(\mathbf{x}) = \int d\mathbf{S_n} \bullet \mathbf{A}(\mathbf{x}) = \int_{\sigma_n} dS_n\, \hat{\mathbf{n}}(\mathbf{x}) \bullet \mathbf{A}(\mathbf{x})$$

I have changed notation now so that dS_n is a patch on a surface embedded in R_n space. The patch itself then has dimension n-1. The surface is called σ_n and is also of dimension n-1.

We can always think of the surface patch as a vector in this sense (overloading symbol n)

$$d\mathbf{S_n} = dS_n\, \hat{\mathbf{n}} \qquad \hat{\mathbf{n}} = \nabla_{nd}(g) / |\nabla_{nd}(g)| \qquad g(x) = 0 \text{ is our surface } \sigma_n$$

If we think of the n-dot-product as $a \bullet b = a_i b_i$ with implied index summation i = 1..n, we can write the above divergence theorem this way

$$\int_{\mathbf{R_n}} d^n x\, \partial_i A_i(\mathbf{x}) = \int d\mathbf{S}_{n,i}\, A_i(\mathbf{x}) = \int_{\sigma_n} dS_n\ \hat{n}(x)_i\, A_i(x) \qquad \text{div theorem for n = n}$$

where all coordinates are on an equal footing. If we like we could also state this for n+1 dimensions

$$\int_{\mathbf{R_{n+1}}} d^{n+1} x\, \partial_i A_i(\mathbf{x}) = \int d\mathbf{S}_{n+1,i}\, A_i(\mathbf{x}) = \int_{\sigma_n} dS_{n+1}\ \hat{n}(x)_i\, A_i(x) \qquad \text{div theorem for n = n+1}$$

where now i = 1,2...n+1. In the section below, we will change to index μ = 0,1....n to replace this i index, just a renaming of the coordinates. Then we will let μ = 0 represent the time coordinate on a special footing and then the remaining μ = 1,2...n will be n-dimensional spatial coordinates. This is all just a matter of labeling, there is no change whatsoever in the content of the divergence theorem.

8. Divergence theorem for Time Evolution Stak p 195 Cylinder

How does this apply to Stakgold's picture on page 195 where we have n spatial dimensions and a time dimension, so we have n+1 dimensions? I am going to use some new notation here:

$\partial_\mu = (\partial_0, \nabla)$ $\quad x_\mu = (t,\mathbf{x})$ $\quad dV = dt\, dx$ $\quad dx = d^n x$ means $dx_1....dx_n$

$A_\mu = (A_0,\mathbf{A})$ $\quad n_\mu = (n_0, \mathbf{n})$ $\quad \mu = 0,1,2....n$

Luckily this is not hydrodynamics, so our spatial boundary stays the same as t varies in his picture which will simplify things somewhat. We need better notation. The first step is to restate the divergence theorem in n+1 dimensions, where x is an n+1 vector, a point in space, $x = (\mathbf{x},t)$ if you like.

$$\int_{\mathbf{R}_{n+1}} dt\, dx\, \partial_\mu A_\mu(x) = \int (dS_n)_\mu\, A_\mu(x) = \int_{\sigma_{n+1}} dS_{n+1}\ \hat{n}(x)_\mu\, A_\mu(x)$$

This form of the divergence theorem would apply to any boundary σ_{n+1} of region R_{n+1}, not just the cylindrical boundary that Stak uses. But now we want to apply the theorem to the cylinder. The volume inside is n+1 dimensions, the surface is n dimensions. The Big Question is now this:

How do we write dS_{n+1} and $\hat{n}(x)$ on the top, bottom, and sides of this cylinder?

I am trying to understand what that cylinder picture means, it is deceptively simple. There is a purely spatial region Stak calls R which never changes, nor does its boundary change. I will call this region R_n and I will call its boundary σ_n and a spatial patch on this boundary will be dS_n . I want to write this

$dS_{n+1}^{(side)} = dt\, dS_n$

but I am not yet convinced this is right.

Suppose $h(x_1,....x_n) = 0$ is the equation of the boundary surface σ_n. It is not a function of t according to my non-hydrodynamic comment above. But what then is the equation of the cylinder surface σ_{n+1} ?

What is the equation of a cylinder surface in regular 3D space, a regular cylinder? Well, the equation of the bottom surface is $t = 0$ and $x^2+y^2 \le R^2$, the top surface is $t = T$ and $x^2+y^2 \le R^2$, and the side surface is then $x^2+y^2 = R^2$ and $0 \le t \le T$. So we might try this for the Stak cylinder:

bottom surface: $t = 0$ and x in R_n (that is, x lies inside this region in n dim spatial space)
top surface: $t = T$ and x in R_n (that is, x lies inside this region in n dim spatial space)
side surface: x on S_{n-1} and $0 \le t \le T$

How would we write dS_n for these three cases?

bottom surface: $dS_{n+1} = d^n x$

top surface: $dS_{n+1} = d^n x$

side surface: $dS_{n+1} = dt\ dS_n$ // using my conjecture from above

How would we write $\hat{\mathbf{n}}$ for these three cases? For the bottom surface, the equation we write generally as g = 0 is given by t=0. That is to say, g(**x**,t) = t. Then the normal is $n_\mu = \nabla_{nd}(g) / |\nabla_{nd}(g)| = (1,\mathbf{0})$ and the correct sign for the bottom surface going out is then $n_\mu = (-1,\mathbf{0})$. For the top surface g(**x**,t) = t-T and n_μ = (+1,**0**) We just keep applying our lessons from early sections of this little doc. For the side surface, g = h as noted above and we have

$$n_\mu = \nabla_{nd}(h) / |\nabla_{nd}(h)| = (0,\mathbf{n})$$

So we can enhance our last conclusions this way:

bottom surface:	$dS_{n+1} = d^n x$	$n_\mu = (-1,\mathbf{0})$	$(d\mathbf{S}_{n+1})_\mu = -\ d^n x\ \ (1,\mathbf{0})$
top surface:	$dS_{n+1} = d^n x$	$n_\mu = (1,\mathbf{0})$	$(d\mathbf{S}_{n+1})_\mu = d^n x\ \ (1,\mathbf{0})$
side surface:	$dS_{n+1} = dt\ dS_n$	$n_\mu = (0,\mathbf{n})$	$(d\mathbf{S}_{n+1})_\mu = dt\ dS_n\ (0,\mathbf{n})$

I think I could also say at this point that

$$d\mathbf{S}_n = dS_n\ \mathbf{n}$$

as a comment about the surface σ_n which has nothing to do with time.

Now we can attempt to evaluate the RHS of our divergence theorem on the boundary of the Stak cylinder:

$$\int_{\sigma_{n+1}} dS_{n+1}\ \hat{n}(x)_\mu\ A_\mu(x) = \text{top piece} + \text{bottom piece} + \text{side piece}$$

We have

$$\text{top piece} = \int_{top} d^n x\ A_0(T,\mathbf{x})$$

$$\text{bot piece} = -\int_{bot} d^n x\ A_0(0,\mathbf{x})$$

$$\text{side piece} = \int_{\sigma_n\ side} dt\ dS_n\ \hat{\mathbf{n}}(\mathbf{x}) \bullet \mathbf{A}(t,\mathbf{x}) = \int_0^T dt \int dS_n\ \hat{\mathbf{n}}(\mathbf{x}) \bullet \mathbf{A}(t,\mathbf{x})$$

Notice that $\hat{\mathbf{n}}(\mathbf{x})$ does not depend on t because the spatial surface boundary never changes.

Meanwhile, our LHS of the divergence theorem is this

$$\int_{\mathbf{R_{n+1}}} dt\, dx\, \partial_\mu A_\mu(x) = \int_0^T dt \int_{\mathbf{R_n}} d^n x\ \partial_\mu A_\mu(t,\mathbf{x})$$

and our divergence theorem for this cylinder boundary thing is now: (final result:)

$$\int_0^T dt \int_{\mathbf{R_n}} d^n x\ \partial_\mu A_\mu(t,\mathbf{x}) = \int d^n x\, [A_0(T,\mathbf{x}) - A_0(0,\mathbf{x})] + \int_0^T dt \int dS_n\ \hat{\mathbf{n}}(\mathbf{x}) \bullet \mathbf{A}(t,\mathbf{x})$$

So this is the divergence theorem in n+1 dimensions where x = (t,**x**) and where we have the Stakgold cylindrical boundary for σ_{n+1} and there is a fixed spatial region R_n with its spatial boundary σ_n.

9. Application to the heat-conduction equation

Green's Theorem 5.72 or 5.73, when written in our notation above, says this

$$vLu - uL^*v = \partial_\mu J_\mu$$

So we set $A_\mu = J_\mu$ in our results above and we get for our cylindrical boundary this divergence theorem:

$$\int_0^T dt \int_{\mathbf{R_n}} d^n x\ [vLu - uL^*v] = \int d^n x\, [J_0(T,\mathbf{x}) - J_0(0,\mathbf{x})] + \int_0^T dt \int dS_n\ \hat{\mathbf{n}}(\mathbf{x}) \bullet \mathbf{J}(t,\mathbf{x})$$

This is valid for any operator L and its (non-unique) current J_μ . On page 41 we showed that for the heat conduction operator, we have

$$L = (\partial_t - \nabla^2) \qquad J_\mu = (uv,\ u\nabla v - v\nabla u)$$

Thus we rewrite to get

$$\int_0^T dt \int_{\mathbf{R_n}} d^n x\ [\, v(t,\mathbf{x})\, (\partial_t - \nabla^2) u(t,\mathbf{x}) - u(t,\mathbf{x})\, (-\partial_t - \nabla^2) v(t,\mathbf{x})\,]$$

$$= \int d^n x\, [u(T,\mathbf{x})v(T,\mathbf{x}) - u(0,\mathbf{x})v(0,\mathbf{x})] + \int_0^T dt \int_{\sigma_n} dS_n\ \hat{\mathbf{n}}(\mathbf{x}) \bullet [\, u(t,\mathbf{x})\nabla v(t,\mathbf{x}) - v(t,\mathbf{x})\nabla u(t,\mathbf{x})\,]$$

where I show the arguments for all functions. We can of course rewrite it this way

$$\int_0^T dt \int_{\mathbf{R_n}} d^n x\ [\, v(t,\mathbf{x})\, (\partial_t - \nabla^2) u(t,\mathbf{x}) - u(t,\mathbf{x})\, (-\partial_t - \nabla^2) v(t,\mathbf{x})\,]$$

$$= \int d^n x\, [u(T,\mathbf{x})v(T,\mathbf{x}) - u(0,\mathbf{x})v(0,\mathbf{x})] + \int_0^T dt \int_{\sigma_n} dS_n\, [\, u(t,\mathbf{x})\partial_n v(t,\mathbf{x}) - v(t,\mathbf{x})\partial_n u(t,\mathbf{x})\,] \qquad (7.6)$$

where we replace $\hat{\mathbf{n}}(\mathbf{x}) \bullet \nabla$ by ∂_n, but of course what we really mean is shown in the previous form. This exactly matches Stak (7.6) where he uses dS for dS_n and σ for σ_n.

10. Application to the wave equation

Now while we are at it, we can give the result for the wave operator instead. Start with our general cylinder result for A = J,

$$\int_0^T dt \int_{R_n} d^n x \; \partial_\mu J_\mu(t,\mathbf{x}) = \int d^n x \, [J_0(T,\mathbf{x}) - J_0(0,\mathbf{x})] + \int_0^T dt \int dS_n \; \hat{\mathbf{n}}(\mathbf{x}) \bullet \mathbf{J}(t,\mathbf{x})$$

and this time the current is given by the square bracket in (5.81) on page 41 which says

$$J_\mu = (v\partial_0 u - u\partial_0 v, \; u\nabla v - v\nabla u)$$

This says that the final result will be the same as the heat operator result but we make this replacement in the first term on the RHS: $uv \rightarrow v\partial_0 u - u\partial_0 v$. So result is then

$$\int_0^T dt \int_{R_n} d^n x \; [\, v(t,\mathbf{x}) \, (\partial_t^2 - \nabla^2) u(t,\mathbf{x}) - u(t,\mathbf{x}) \, (\partial_t^2 - \nabla^2) v(t,\mathbf{x}) \,]$$

$$= \int d^n x \, [\; \{ \, v(T,\mathbf{x}) \partial_t u(T,\mathbf{x}) - u(T,\mathbf{x}) \partial_t v(T,\mathbf{x}) \, \} \; - \{ \, v(0,\mathbf{x}) \partial_t u(0,\mathbf{x}) - u(0,\mathbf{x}) \partial_t v(0,\mathbf{x}) \, \} \;]$$

$$+ \int_0^T dt \int_{\sigma_n} dS_n \, [\, u(t,\mathbf{x}) \partial_n v(t,\mathbf{x}) - v(t,\mathbf{x}) \partial_n u(t,\mathbf{x}) \,] \qquad (7.7)$$

and this agrees with (7.7).

11. More on conjecture $dS_{n+1}{}^{(side)} = dt \, dS_n$

In 3D, here is how we would analyze the cylinder in the case R_n = disk of radius a = R_2. Then the cylinder is a real cylinder sitting in 3D space, and we can use vector notation for surface elements. We assume coordinates ρ,θ,t (instead of usual ρ,θ,z). [these notes were from an earlier document which is not gone, when I started pondering this question and got in trouble with cross products for n > 3]

Question #1: We have a 3D spacetime volume in the shape of a cylinder. How do you write expressions for surface patches on the boundary including sides and top and bottom?

Answer: If it really were a cylinder, we would say (dz = dt)

$$d\mathbf{S}_3{}^{(side)} = d\mathbf{t} \times d\mathbf{s} \qquad d\mathbf{s} = (a d\theta) \, \hat{\boldsymbol{\rho}}$$

This surface patch is perpendicular to both $\hat{\mathbf{t}}$ and $\hat{\boldsymbol{\rho}}$. As for the top, we would have

$$d\mathbf{S}_3{}^{(top)} = (\rho \, d\rho \, d\theta) \, \hat{\mathbf{t}} \qquad\qquad d\mathbf{S}_3{}^{(side)} = - \, d\mathbf{S}_3{}^{(top)}$$

If we now think just of the disk itself as an n = 2 dimensional object, we could say

$d\mathbf{S_2} = (R d\theta)\,\hat{\boldsymbol{\rho}}$ "area" element on disk boundary

$d^2x = dV_2 = (\rho\, d\rho\, d\theta)$ " volume" of piece of the disk

Then we can write our results above this way

$d\mathbf{S_3}^{(side)} = d\mathbf{t} \times d\mathbf{S_2}$ $\qquad d\mathbf{S_2}^{(side)} \bullet \hat{\mathbf{t}} = 0$

$d\mathbf{S_2}^{(top)} = d^2x\,\hat{\mathbf{t}}$ $\qquad d\mathbf{S_2}^{(top)} \bullet \hat{\mathbf{t}} = d^2x$

and the first line above is an example of our conjecture (which has no vector sense to it)

$dS_{n+1}^{(side)} = dt\, dS_n$

In this example, we started thinking of the cross section being a disk. We said d^2x is a piece of area on a disk, and $d\mathbf{S_1}$ is a piece of boundary on a disk. But if we replace "disk" with any other 2D closed shape, our equations above are not altered. If it were a square, $d\mathbf{S_2}$ would of course depend on which side of the square you were talking about, but we can still call it $d\mathbf{S_2}$. And area is still d^2x.

So basically $dS_{n+1}^{(side)} = dt\, dS_n$ says that we take a surface element of our spatial surface and just multiply it by dt to get a surface element in the n+1 dimension space (t,**x**). The t axis is perpendicular to all the spatial axes, and our "cylinder" sides are parallel to the t axis. Therefore, our surface element $dt\, dS_n$ in fact is a surface patch *on the side of the cylinde*r. We saw this explicitly for R = disk. In any number of dimensions, if you could actually draw the hyper cylinder, you would see that the sides of the cylinder are parallel to the t axis.