A geometry problem PhL 8.18.09

Problem: what locus is described by

$$| z - a | = \alpha |z-b| \qquad \alpha > 0$$

That is to say, the ratio of the distances from two points is a constant.

Solution:

Square both sides and do the usual stuff:

$$(x-a_x)^2 + (y-a_y)^2 = \alpha^2 \{ (x-b_x)^2 + (y-b_y)^2 \}$$

$$(x-a_x)^2 - \alpha^2(x-b_x)^2 + (y-a_y)^2 - \alpha^2 (y-b_y)^2 = 0$$

$$(1-\alpha^2) x^2 +(-2a_x+2\alpha^2b_x)x + (1-\alpha^2) y^2 + (-2a_y+2\alpha^2b_y)y + (a_x^2- \alpha^2b_x^2 + a_y^2- \alpha^2b_y^2) = 0$$

$$x^2 +(-2a_x+2\alpha^2b_x)/ (1-\alpha^2) x + y^2 + (-2a_y+2\alpha^2b_y)/ (1-\alpha^2) y + (a_x^2- \alpha^2b_x^2 + a_y^2- \alpha^2b_y^2)/ (1-\alpha^2) = 0$$

$$x^2 +A_x'x + y^2 + A_y' y + B = 0$$

where

$$A_x'/2 = (-a_x+\alpha^2b_x)/ (1-\alpha^2)$$
$$A_y'/2 = (-a_y+\alpha^2b_y)/ (1-\alpha^2) \qquad B = (a_x^2- \alpha^2b_x^2 + a_y^2- \alpha^2b_y^2)/ (1-\alpha^2)$$

Now complete the square,

$$x^2 +A_x'x + (A_x'/2)^2+ y^2 + A_y' y + (A_y'/2)^2 + B - (A_x'/2)^2- (A_y'/2)^2 = 0$$

$$(x + A_x'/2)^2+ (y + A_y'/2)^2 + (B - (A_x'/2)^2- (A_y'/2)^2) = 0$$

This is a circle! The center of the circle is at

$$x_0 = - A_x'/2 = - (-a_x+\alpha^2b_x)/ (1-\alpha^2)$$
$$y_0 = - A_y'/2 = - (-a_y+\alpha^2b_y)/ (1-\alpha^2)$$

The squared radius of the circle is this:

$$R^2 = (A_x'/2)^2+ (A_y'/2)^2 - B$$

$$= - (a_x^2- \alpha^2b_x^2 + a_y^2- \alpha^2b_y^2)/ (1-\alpha^2) + (-a_x+\alpha^2b_x)^2/ (1-\alpha^2)^2+ (-a_y+\alpha^2b_y)^2/ (1-\alpha^2)^2$$

So this is the circle implied by $| z - a| = \alpha |z-b|$.

Now first let's put a and b on the x axis. Then

$$x_0 = -A_x'/2 = -(-a_x+\alpha^2 b_x)/(1-\alpha^2)$$
$$y_0 = -A_y'/2 = -(-a_y+\alpha^2 b_y)/(1-\alpha^2)$$
$$R^2 = -(a_x^2-\alpha^2 b_x^2 + a_y^2-\alpha^2 b_y^2)/(1-\alpha^2) + (-a_x+\alpha^2 b_x)^2/(1-\alpha^2)^2 + (-a_y+\alpha^2 b_y)^2/(1-\alpha^2)^2$$

$$x_0 = -A_x'/2 = -(-a_x+\alpha^2 b_x)/(1-\alpha^2)$$
$$y_0 = -A_y'/2 = -(-0+\alpha^2 0)/(1-\alpha^2) = 0$$
$$R^2 = -(a_x^2-\alpha^2 b_x^2 + 0^2-\alpha^2 0^2)/(1-\alpha^2) + (-a_x+\alpha^2 b_x)^2/(1-\alpha^2)^2 + (-0+\alpha^2 0)^2/(1-\alpha^2)^2$$
$$= -(a_x^2-\alpha^2 b_x^2)/(1-\alpha^2) + (-a_x+\alpha^2 b_x)^2/(1-\alpha^2)^2$$
$$1/(1-\alpha^2)^2 * \{ -(a_x^2-\alpha^2 b_x^2)(1-\alpha^2) + (-a_x+\alpha^2 b_x)^2 \}$$

Continuing

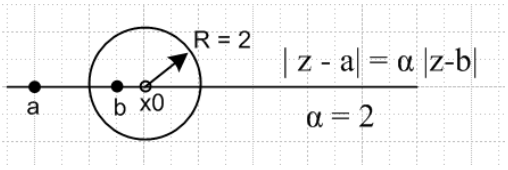
$$\{\} = -a_x^2 + \alpha^2 b_x^2 + a_x^2 + \alpha^4 b_x^2 - 2\alpha^2 a_x b_x + \alpha^2(a_x^2-\alpha^2 b_x^2)$$
$$= \alpha^2 b_x^2 + \alpha^4 b_x^2 - 2\alpha^2 a_x b_x + \alpha^2(a_x^2-\alpha^2 b_x^2)$$
$$= \alpha^2[b_x^2 + \alpha^2 b_x^2 - 2a_x b_x + (a_x^2-\alpha^2 b_x^2)]$$
$$= \alpha^2[b_x^2 + \alpha^2 b_x^2 - 2a_x b_x + a_x^2-\alpha^2 b_x^2]$$
$$= \alpha^2[b_x^2 - 2a_x b_x + a_x^2]$$
$$= \alpha^2(b_x-a_x)^2$$

To summarize, in the case where a and b are on the real axis, $|z-a| = \alpha|z-b|$ is a circle where:

$$x_0 = -(-a_x+\alpha^2 b_x)/(1-\alpha^2)$$
$$y_0 = 0$$

$$R^2 = \alpha^2/(1-\alpha^2)^2 * (b_x - a_x)^2 \qquad R = \{\alpha / |1-\alpha^2|\} \; |b_x - a_x|$$

I think this situation must look like this. Imagine $\alpha = 2$ so that $R = (2/3)|b_x - a_x| = 2$.



where I have done a precision scaled drawing. Notice that both left and right extremes of the circle give the right answer: 2/1 and 6/3. So the circle notion must be right.

Summary:

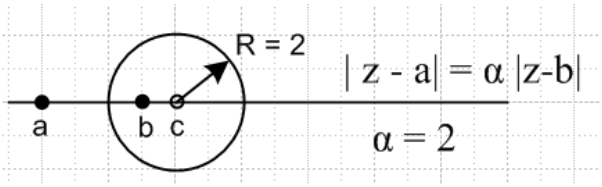
Consider the locus of points in the complex plane that solve this equation:

$$|z - a| = \alpha|z-b| \qquad \Rightarrow \qquad \alpha > 0$$

A. The solution locus is a circle of some radius R and center location c.

B. The circle center c is on the line connecting the points a and b, and lies outside the segment ab.

C. If $\alpha > 1$, then the ordering of these points is a, b, c as shown in this picture, which is correctly scaled,



D. If $\alpha < 1$, then the ordering is c, a, b and the circle encloses point a and not point b. This situation is obtained from C by doing the swap a↔b.

E. The specific center and radius of the circle are given by:

$$c_x = -(-a_x+\alpha^2 b_x)/(1-\alpha^2)$$
$$c_y = -(-a_y+\alpha^2 b_y)/(1-\alpha^2)$$
$$R^2 = c_x^2 + c_y^2 - (a_x^2 - \alpha^2 b_x^2 + a_y^2 - \alpha^2 b_y^2)/(1-\alpha^2)$$

F. If we put a and b on the x-axis, these results simplify to

$$c_x = -(-a_x+\alpha^2 b_x)/(1-\alpha^2)$$
$$c_y = 0$$
$$R = \{\alpha / |1-\alpha^2|\} \; |b_x - a_x|$$

G. If in addition we put point a at the origin, we get

$$c_x = -(\alpha^2 b_x)/(1-\alpha^2)$$
$$c_y = 0$$
$$R = \{\alpha / |1-\alpha^2|\} \; |b_x|$$

$$|c_x| = (\alpha^2 |b_x|)/|1-\alpha^2| = \alpha R$$

H. If $\alpha = 1^+$ then $R = \infty$ and $c_x = +\infty$. Think if this as the limit of the above picture as α reduces from 2 to 1. The center c moves out to the right and the radius increases. The circle becomes a vertical line bisecting the segment ab, and of course this is true for arbitrary a and b.