Dot product of two vectors in sphericals PhL 3.26.05

Imagine the picture of the two vectors **a** and **b** plotted in sphericals. What is the dot product? We can compute it in two ways

$\mathbf{a} \bullet \mathbf{b} = ab\cos\gamma$ γ is the angle between the vectors

$\mathbf{a} \bullet \mathbf{b} = \Sigma a_i b_i$

Now

$a_x = a \sin\theta_a \cos\varphi_a$
$a_y = a \sin\theta_a \sin\varphi_a$
$a_z = a \cos\theta_a$

$b_x = b \sin\theta_b \cos\varphi_b$
$b_y = b \sin\theta_b \sin\varphi_b$
$b_z = b \cos\theta_b$

Then

$\mathbf{a} \bullet \mathbf{b} = ab \{ \sin\theta_a\cos\varphi_a \sin\theta_b\cos\varphi_b + \sin\theta_a\sin\varphi_a \sin\theta_b\sin\varphi_b + \cos\theta_a \cos\theta_b \}$

$= ab \{ \sin\theta_a\sin\theta_b\cos\varphi_a \cos\varphi_b + \sin\theta_a\sin\theta_b\sin\varphi_a \sin\varphi_b + \cos\theta_a \cos\theta_b \}$

$= ab \{ \sin\theta_a\sin\theta_b[\cos\varphi_a \cos\varphi_b + \sin\varphi_a \sin\varphi_b] + \cos\theta_a \cos\theta_b \}$

$= ab \{ \sin\theta_a\sin\theta_b\cos(\varphi_a - \varphi_b) + \cos\theta_a \cos\theta_b \}$

Therefore we can identify

$\cos\gamma = \cos\theta_a \cos\theta_b + \sin\theta_a\sin\theta_b\cos(\varphi_a - \varphi_b)$

Other forms:

Now if we replace polar angle θ by latitude angle ψ using $\theta = \pi/2 - \psi$ in one or both places, we get four forms including the first form

$\cos\gamma = \cos\theta_a \cos\theta_b + \sin\theta_a\sin\theta_b\cos(\varphi_a - \varphi_b)$
$\cos\gamma = \cos\theta_a \sin\psi_b + \sin\theta_a\cos\psi_b\cos(\varphi_a - \varphi_b)$
$\cos\gamma = \sin\psi_a \cos\theta_b + \cos\psi_a \sin\theta_b \cos(\varphi_a - \varphi_b)$ (*)
$\cos\gamma = \sin\psi_a \sin\psi_b + \cos\psi_a\cos\psi_b\cos(\varphi_a - \varphi_b)$

Still more forms can be obtained by introducing minus signs according to

$\sin(\pi/2 + \alpha) = \cos\alpha$
$\cos(\pi/2 + \alpha) = -\sin\alpha$

For example

$$\cos\gamma = \cos\theta_a \cos\theta_b + \sin\theta_a \sin\theta_b \cos(\varphi_a - \varphi_b)$$

$$\Rightarrow \cos\gamma = \sin(\pi/2 + \theta_a) \cos\theta_b - \cos(\pi/2 + \theta_a) \sin\theta_b \cos(\varphi_a - \varphi_b)$$

Another way to get a minus sign is to use

$$\cos(\varphi_a - \varphi_b) = -\cos(\pi + \varphi_a - \varphi_b) = -\cos(\varphi_a - \varphi_b - \pi)$$

Example: from sun position work I got these equations

$$x' = -R\,[\cos\theta_t \sin(\omega t)]$$
$$y' = R\,[\sin\theta_t \sin\theta_1 - \cos\theta_t \cos\theta_1 \cos(\omega t)]$$
$$z' = R\,[\sin\theta_t \cos\theta_1 + \cos\theta_t \sin\theta_1 \cos(\omega t)] \qquad \text{(E.9)}$$

Here θ_t is in fact a latitude type angle that gives the latitude of the sun on the celestial sphere. Meanwhile, θ_1 is the polar angle of an observation site. So adjust symbols to get

$$x' = -R\,[\cos\psi_t \sin(\omega t)]$$
$$y' = R\,[\sin\psi_t \sin\theta_1 - \cos\psi_t \cos\theta_1 \cos(\omega t)]$$
$$z' = R\,[\sin\psi_t \cos\theta_1 + \cos\psi_t \sin\theta_1 \cos(\omega t) \qquad \text{(E.9)}$$

The last one is case (*) above. The bracket there is just $\cos\gamma$ where γ is the angle between the sun's vector of length R, and the observer's up unit vector $\hat{\mathbf{z}}'$. If you draw a simple picture, you see that $z' = R\cos\gamma$.

Similarly you could find an interpretation of the y' [...]. It is just geometry.