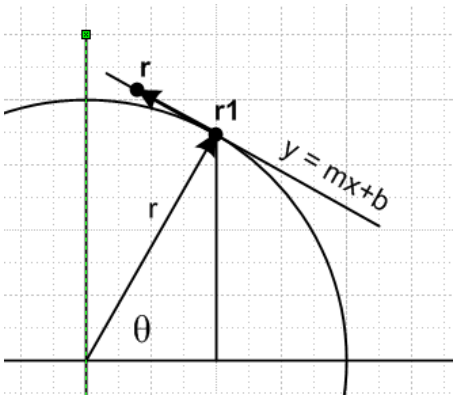
Problem: What is the equation of a line tangent to a circle? PhL 9.10.09

First, here is the picture:



Method 1. What are m and b in terms of r and θ ? Start with:

$(\mathbf{r}-\mathbf{r}_1)\bullet\mathbf{r}_1 = 0$ // line is tangent to the circle at point r_1

Write out the first equation to get

$[\ (x,y) - (x_1, y_1)\ ] \bullet (x_1, y_1) = 0$

$(x-x_1)x_1 + (y-y_1)y_1 = 0 \quad \Rightarrow \quad (x-x_1)x_1 = -(y-y_1)y_1$

$-(x-x_1)x_1/y_1 = (y-y_1) \quad \Rightarrow \quad y = y_1 - (x_1/y_1)x + x_1^2/y_1 = mx+b$

Therefore

$m = -(x_1/y_1) = -\cot\theta$

$b = y_1 + x_1^2/y_1 = (y_1^2 + x_1^2)/y_1 = r^2/y_1 = r/\sin\theta$

So the line tangent to a circle of radius r at location θ is given by

$y = [-\cot\theta\ ]\ x + r/\sin\theta$

Method 2.

$mx-y-b = 0 \Rightarrow \quad \mathbf{n} = (m,-1)$ // unnormalized normal vector to line

$\hat{\mathbf{n}} = (m,-1)/\sqrt{m^2+1} = \hat{\mathbf{r}}_1 = \mathbf{r}_1/r = (x_1, y_1)/r$

Thus we have two equations

$m/\sqrt{m^2+1} = x_1/r = \cos\theta$

$-1/\sqrt{m^2+1} = y_1/r = \sin\theta$

Divide these equations to get

$-m = \cot\theta$

Now insist that r_1 is on the line $y = mx+b$ so that

$y_1 = [-\cot\theta]\, x_1 + b$

$r\sin\theta = [-\cot\theta]r\cos\theta + b = -\, r\cos^2\theta/\sin\theta + b$

Then

$b = r\sin\theta + r\cos^2\theta/\sin\theta = r(\sin^2\theta + \cos^2\theta)/\sin\theta = r/\sin\theta$

and we get the same result as by Method 1.