Diagonalization of Convolution Equations PhL 3.26.05

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1. Expansion, Projection, Completeness, Orthogonality, Addition Theorem

I need more precision on this subject, want to know why we need $\int dg = 1$ and perhaps a few other things. One source is my thesis page 45, but a proof of this requirement is missing. Vilenkin's book I know has this somewhere, but it is a big book. Hermann's book on Fourier Analysis on Groups I think has what I need on page 53 which I will write out in the notation of my thesis page 45:

$$f(g) = \Sigma_\sigma \Sigma_{k,k'} f^\sigma_{kk'} D^\sigma_{k'k}(g) = \Sigma_\sigma d^\sigma \text{ trace}[f^\sigma D^\sigma(g)] \qquad \text{//expansion, in matrix notation}$$

where d^σ is the dimensionality of the matrix representation σ. This would be (2j+1) for rotation group eg. This is the "expansion" formula which I unfortunately omitted from my thesis. The projection that goes with it is this, according to Hermann, [unless otherwise noted, $\int dg$ implies integration over the entire group parameter space]

$$f^\sigma = \int dg\, f(g)\, D^\sigma(g^{-1}) \qquad \text{// matrix form}$$

$$f^\sigma_{kk'} = \int dg\, f(g)\, D^\sigma_{kk'}(g^{-1}) \qquad \text{// show matrix elements}$$

The representations are *unitary* irreducible representations so

$$D^\sigma_{kk'}(g^{-1}) = D^\sigma_{kk'^{-1}}(g) = D^\sigma_{kk'^\dagger}(g) = D^\sigma_{k'k}(g)^*$$

So we can summarize what we have so far:

<u>matrix form:</u>

$f^\sigma = \int dg\, f(g)\, D^\sigma(g^{-1})$ // projection

$f(g) = \Sigma_\sigma\, d^\sigma\, tr[\, f^\sigma\, D^\sigma(g)]$ // expansion

showing matrix indices:

$f^\sigma_{kk'} = \int dg\, f(g)\, D^\sigma_{kk'}(g^{-1}) = \int dg\, f(g)\, D^\sigma_{k'k}(g)^*$ // projection

$f(g) = \Sigma_\sigma\ d^\sigma\, \Sigma_{k,k'}\, f^\sigma_{kk'}\, D^\sigma_{k'k}(g)$ // expansion

Now let's use these and see what they imply. First, stick the projection into the expansion:

$$f(g) = \Sigma_\sigma\, d^\sigma\, tr[\, f^\sigma\, D^\sigma(g)] = \Sigma_\sigma\, d^\sigma\, tr[\{\int dg'\, f(g')\, D^\sigma(g'^{-1})\}D^\sigma(g)] = \int dg'\, f(g')\, \Sigma_\sigma\, d^\sigma\, tr[D^\sigma(g'^{-1})D^\sigma(g)]$$

The requirement would then be, to get f(g) as a result, (this is the completeness relation)

$\Sigma_\sigma\, d^\sigma\, tr[D^\sigma(g'^{-1})D^\sigma(g)] = \delta(g-g')$ where of course $\int dg'\, \delta(g-g') = 1$

Now stick the expansion into the projection,

$$f^\sigma = \int dg\, f(g)\, D^\sigma(g^{-1}) = \int dg\, \{\, \Sigma_{\sigma'}\, d^{\sigma'}\, tr[\, f^{\sigma'}\, D^{\sigma'}(g)]\}\, D^\sigma(g^{-1})$$

I need to get the integral acting on the two D functions, so better write out indices

$$f^\sigma_{kk'} = \int dg\, \{\, \Sigma_{\sigma'}\, d^{\sigma'}\, tr[\, f^{\sigma'}\, D^{\sigma'}(g)]\}\, D^\sigma(g^{-1})_{kk'}$$

$$= \int dg\ \Sigma_{\sigma'}\, d^{\sigma'}\ f^{\sigma'}{}_{mn}\, D^{\sigma'}(g)_{nm}\, D^\sigma(g^{-1})_{kk'}$$

$$= \Sigma_{\sigma'}\, d^{\sigma'}\ f^{\sigma'}{}_{mn} \int dg\ D^{\sigma'}(g)_{nm}\, D^\sigma(g^{-1})_{kk'}$$

If we want this to come out being $f^\sigma_{kk'}$, we require this to be true

$$\int dg\ D^{\sigma'}(g)_{nm}\, D^\sigma(g^{-1})_{kk'} = (1/d^\sigma)\, \delta_{\sigma\sigma'}\ \delta_{mk}\ \delta_{nk'}$$

or

$$\int dg\ D^{\sigma'}(g)_{nm}\, D^\sigma(g)_{k'k}{}^* = (1/d^\sigma)\, \delta_{\sigma\sigma'}\ \delta_{nk'}\ \delta_{mk}$$

So here then are our statements of completeness and orthogonality:

$\Sigma_\sigma\, d^\sigma\, tr[D^\sigma(g'^{-1})D^\sigma(g)] = \delta(g-g')$ // completeness

$$\int dg\ D^{\sigma'}(g)_{nm}\, D^{\sigma}(g^{-1})_{kk'} = (1/d^{\sigma})\, \delta_{\sigma\sigma'}\ \delta_{mk}\ \ \delta_{nk'} \qquad \text{// orthogonality}$$

Just for the record, an orthogonality result of this form appears in Vilenkin p 160 (4). My D functions are of course normalized in such a way above to get 1 on the RHS of completeness.

When we say that $D^{\sigma}(g)_{nm}$ is a representation of the group, it follows that

$$\Sigma_b\, D^{\sigma}(g_1)_{ab}\, D^{\sigma}(g_2)_{bc} = D^{\sigma}(g_1 g_2)_{ac} \qquad \text{or} \qquad D^{\sigma}(g_1)\, D^{\sigma}(g_2) = D^{\sigma}(g_1 g_2)$$

where on the right we show the matrix form. The leftmost form is the "addition theorem" for the particular representation σ of the group G, but it is really just "the group property" of a representation.

2. Application to the abelian group G = SO(2).

In this case, everything is 1D matrices, so can ignore "trace" and ignore lower indices altogether. Then perhaps we try this: (note that $d^{\sigma} = 1$)

$$D^{\sigma}(\theta) = e^{-i\sigma\theta} \qquad \sigma = \text{integers}$$

Aside: Suppose we put a constant out front like C

$$D^{\sigma}(\theta) = C\, e^{-i\sigma\theta} \qquad D^{\sigma}(\theta)^{-1} = [C\, e^{-i\sigma\theta}]^{-1} = (1/C)\, e^{+i\sigma\theta}$$
$$D^{\sigma}(\theta)^{\dagger} = D^{\sigma}(\theta)^{*} = C^{*}\, e^{+i\sigma\theta}$$

If you want $D^{\sigma}(\theta)$ to be *unitary*, then you must have $C = e^{-i\delta}$ so $1/C = C^{*}$. We of course choose C = 1. The point is that you cannot throw in factors of π or what have you in the definition of D. Unitarity determines that scale or magnitude of D.

Then how does the above completeness come out?

$$\Sigma_{\sigma}\, \mathrm{tr}[D^{\sigma}(g')D^{\sigma}(g)^{*}] = \delta(g-g')$$

$$\Sigma_{\sigma}\, e^{-i\sigma\theta'}\, e^{i\sigma\theta} = = \delta(g-g') \qquad \text{// see Spec. The. p 14 (13.2), on } -\pi,\pi$$
$$\text{// see Spec. The. p 13 (13.1)}$$

So if we want to use our notation above, we must accept that $\delta(g-g') = 2\pi\, \delta(\theta'-\theta)$. Then we are forced to take $dg = d\theta/2\pi$ in order to get

$$\int dg\, \delta(g-g') = \int d\theta/2\pi\ \ 2\pi\, \delta(\theta'-\theta) = 1$$

and we thus end up with

$$\int dg = \int d\theta/2\pi = 1$$

Orthogonality then says:

$$\int dg\ D^{\sigma'}(g)\ D^{\sigma}(g^{-1}) = \delta_{\sigma\sigma'}$$

$$\int d\theta/2\pi\ e^{i(\sigma-\sigma')\theta} = \delta_{\sigma\sigma'} \qquad \text{// integrate cos(mx) of sin(mx) over 1 cycle, get 0, m} \neq 0 \text{ integer}$$

So for this simple case, things seem to work OK.

Here is another way to arrive at $\int dg = 1$ more generally. First, our orthogonality from above is this,

$$\int dg\ D^{\sigma'}(g)_{nm}\ D^{\sigma}(g^{-1})_{kk'} = (1/d^{\sigma})\ \delta_{\sigma\sigma'}\ \delta_{mk}\ \delta_{nk'}$$

For a 1D irreducible representation of the group, this says:

$$\int dg\ D^{\sigma'}(g)\ D^{\sigma}(g^{-1}) = \delta_{\sigma\sigma'}$$

But we know there is always the 1D representation called the trivial representation where $D^{\sigma}(g) = 1$ for any g. If we set $\sigma = \sigma'$ in the above for this trivial representation, we get

$$\int dg\ 1 * 1 = 1 \qquad \Rightarrow \qquad \int dg = 1$$

While we're here, we might as well state the projection and expansion formulas for G = SO(2). First, here are the general formulas,

$$f^{\sigma}_{kk'} = \int dg\ f(g)\ D^{\sigma}_{kk'}(g^{-1}) = \int dg\ f(g)\ D^{\sigma}_{k'k}(g)^* \qquad \text{// projection}$$

$$f(g) = \Sigma_{\sigma}\ d^{\sigma}\ \Sigma_{k,k'}\ f^{\sigma}_{kk'}\ D^{\sigma}_{k'k}(g) \qquad \text{// expansion}$$

and then for our 1D case we remove all indices and set $d^{\sigma} = 1$ and $D^{\sigma}(g) = e^{-i\sigma\theta}$ and $dg = d\theta/2\pi$ and σ=n

$$f^{n} = (1/2\pi) \int d\theta\ f(\theta)\ e^{+in\theta} \qquad \text{// projection} \qquad \text{//Complex Fourier Series Transform}$$

$$f(\theta) = \Sigma_{n}\ f^{n}\ e^{-in\theta} \qquad \text{// expansion}$$

$$(1/2\pi) \int d\theta\ e^{in\theta}\ e^{-in'\theta} = \delta_{nn'} \qquad \text{// orthogonality}$$

$$(1/2\pi) \Sigma_{n}\ e^{-in\theta'}\ e^{in\theta} = \delta(\theta-\theta') \qquad \text{// completeness}$$

This is the usual Fourier Series written in exponential notation. If we write out the exponential we get

$$f^n = (1/2\pi) \int d\theta\, f(\theta) (\cos(n\theta) + i \sin(n\theta))$$
$$f(\theta) = \Sigma_n f^n (\cos(n\theta) - i \sin(n\theta))$$

If we define these real coefficients (we assume f(θ) is real) ,

$$a_n = (1/\pi) \int d\theta\, f(\theta) \cos(n\theta) \qquad b_n = (1/\pi) \int d\theta\, f(\theta) \sin(n\theta)$$

where we note in passing that $a_{-n} = a_n$ and that $b_{-n} = - b_n$, we then have

$$2f^n = a_n + ib_n$$
$$f(\theta) = \Sigma_n (a_n + ib_n) (\cos(n\theta) - i \sin(n\theta))/2 = (1/2) \Sigma_n a_n \cos(n\theta) + (1/2) \Sigma_n b_n \sin(n\theta)$$

where the cross terms must vanish because f(θ) = real, so we don't write them out. At this point, sums Σ_n are still over all integers. We can break out n = 0 and reflect the rest. The n=0 sine term gives nothing and we then write

$$\Sigma_{n<0} a_n \cos(n\theta) + \Sigma_{n>0} a_n \cos(n\theta) = \Sigma_{n>0} a_{-n} \cos(-n\theta) + \Sigma_{n>0} a_n \cos(n\theta) = 2 \Sigma_{n>0} a_n \cos(n\theta)$$

$$\Sigma_{n<0} b_n \sin(n\theta) + \Sigma_{n>0} b_n \sin(n\theta) = \Sigma_{n>0} b_{-n} \sin(-n\theta) + \Sigma_{n>0} b_n \sin(n\theta) = 2 \Sigma_{n>0} b_n \sin(n\theta)$$

So our result is then

$$f(\theta) = a_0/2 + \Sigma_{n=1} a_n \cos(n\theta) + \Sigma_{n=1} b_n \sin(n\theta)$$

$$a_n = (1/\pi) \int d\theta\, f(\theta) \cos(n\theta) \qquad b_n = (1/\pi) \int d\theta\, f(\theta) \sin(n\theta)$$

which agrees with Schaum p 131 with L = π. This is the traditional Fourier Series.

Comments. Hermann says that even in 1969 when he wrote his Fourier Group stuff, the entire field was poorly developed. This suggests that in 2010 there must be some better standard book on this subject. Maybe I should take a look for this supposed book. // I downloaded lots of books from the usual place.

3. Diagonalization of Convolution Equations

Here we show such a convolution equation, along with the addition theorem noted above and projection:

$$A(g) = \int dg_1 B(g_1) C(g_2) \quad \text{where } g_2 = g_1^{-1} g. \qquad \text{// convolution equation } g = g_1 g_2 \quad (*)$$

$$\Sigma_b D^\sigma(g_1)_{ab} D^\sigma(g_2)_{bc} = D^\sigma(g_1 g_2)_{ac} \qquad \text{// "addition theorem" for group rep}$$

$$f^{\sigma}_{kk'} = \int dg\, f(g)\, D^{\sigma}_{kk'}(g^{-1}) \qquad \text{// projection (Hermann)}$$

If we apply the projection operator $\int dg\, D^{\sigma}_{kk'}(g^{-1})$ to both sides of the integral equation we get $A^{\sigma}_{kk'}$ on LHS of (*). Meanwhile, we know that $g^{-1} = g_2^{-1}g_1^{-1}$ since $g = g_1g_2$. Moreover, consider :

$$I \equiv \int dg\, h(g) = \int dg'\, h(g) = \int d(g_1g)\, h(g)$$

As g moves over G, so does g' where $g' = g_1g$ with g_1 some *fixed* group element. This fact is known as the rearrangement theorem. Symbolically, then, since this is true for any h(g), we write as an operator,

$$\int dg = \int d(g_1g) = \int dg' \qquad \text{where } g' = g_1g$$

Thus the RHS of (*) becomes [here we use the above line as $\int dg_2 = \int d(g_1g_2) = \int dg$]

$$A^{\sigma}_{kk'} = \int dg\, D^{\sigma}_{kk'}(g^{-1}) \int dg_1 B(g_1)C(g_1^{-1}g)$$

$$= \int dg_2\, D^{\sigma}_{kk'}(g_2^{-1}g_1^{-1}) \int dg_1 B(g_1)C(g_2)$$

$$= \int dg_2\, \Sigma_{k"}\, D^{\sigma}_{kk"}(g_2^{-1})\, D^{\sigma}_{k"k'}(g_1^{-1}) \int dg_1 B(g_1)C(g_2)$$

$$= \Sigma_{k"} \{ \int dg_2\, C(g_2)\, D^{\sigma}_{kk"}(g_2^{-1}) \} \{ \int dg_1 B(g_1)\, D^{\sigma}_{k"k'}(g_1^{-1}) \}$$

$$= \Sigma_{k"} B^{\sigma}_{k"k'} C^{\sigma}_{kk"}$$

$$= \Sigma_{k"} C^{\sigma}_{kk"} B^{\sigma}_{k"k'}$$

or in matrix form

$$A^{\sigma} = C^{\sigma}\, B^{\sigma}$$

If we apply † to this equation, we get

$$(A^{\sigma})^{\dagger} = (B^{\sigma})^{\dagger}\, (C^{\sigma})^{\dagger}$$

where the dagger projection would be the second line below,

$$f^{\sigma} = \int dg\, f(g)\, D^{\sigma}(g^{-1})$$

$$(f^{\sigma})^{\dagger} = \int dg\, f(g)\, [D^{\sigma}(g^{-1})]^{\dagger} = \int dg\, f(g)\, D^{\sigma}(g)$$

This is the projection I use in my Thesis I page 46 (6.3) and I do that only because it has the benefit of keeping things in A,B,C order in both the integral and diagonalized equations.

Now go back to our integral equation and its diagonalization:

$$A(g) = \int dg_1 B(g_1) C(g_2) \qquad \text{where } g_2 = g_1^{-1} g.$$

$$A^{\sigma} = C^{\sigma}\, B^{\sigma}$$

Suppose $B(g_1)$ is the unknown function in our "integral equation". Then we can first compute

$$B^{\sigma} = (C^{\sigma})^{-1} A^{\sigma}$$

which means we compute the inverse of some finite dimensional matrix, do a matrix multiplication and then we have the finite matrix B^{σ} . Then

$$B(g) = \Sigma_{\sigma}\, d^{\sigma}\, \mathrm{tr}[\, B^{\sigma}\, D^{\sigma}(g)]$$

So to get the full answer, we have to compute B^{σ} for ALL representations, and as d^{σ} gets larger, this requires more and more work. The hope is that B(g) is a "low frequency" function so you only need to do this with the lowest several representations. This is the way low energy partial wave scattering analysis works.

4. Diagonalization of an SO(2) Integral Equation

Our integral equation of interest is this

$$A(g) = \int dg_1 B(g_1) C(g_2) \qquad \text{where } g_2 = g_1^{-1} g. \qquad \text{// convolution equation } \; g = g_1 g_2$$

or

$$A(\theta) = \int d\theta_1/2\pi\, B(\theta_1)\, C(\theta - \theta_1) \qquad (*)$$

We read off the diagonalized equation (set $\sigma = n$ as traditional)

$$A^{n} = B^{n}\, C^{n} \qquad (**)$$

where we change the BC ordering since just numbers. The various projections and expansions are:

$$A^n = (1/2\pi) \int d\theta\, A(\theta)\, e^{+in\theta} \qquad \text{// projection}$$

$$A(\theta) = \Sigma_n\, A^n\, e^{-in\theta} \qquad \text{// expansion}$$

and similarly for B and C. If our unknown function is B(θ), then we have the following solution to our integral equation:

$$B(\theta) = \Sigma_n\, (A^n/C^n)\, e^{-in\theta}$$

Note added 1.8.11. Is this diagonalization valid for any functions A,B,C? I think each of these functions has to be "defined on the group SO(2)". A function F(θ) is defined on the group means (1) it is a functions which is defined on (0,2π), and (2) F(θ) = F(θ+2π). I think a general condition is that F(θ) has to be at least piecewise continuous everywhere on the group space, and that does mean F(θ) = F(θ+2π). So consider the function F(θ) = $1/(a^2+\theta^2)$ for θ in (-∞,∞). This function does NOT have F(θ) = F(θ+2π) so our diagonalization would not apply to it. It would apply if for example you considered instead the function F(θ) = $1/(a^2+\theta^2)$ for (0,2π) and you replicated this in each 2π interval to make F(θ) be periodic.

5. Another application: the Translation Group T(1) and the Fourier Transform

This obvious group (translations in 1 dimension) does not seem to have a standard name such as SO(2). This is an abelian non-compact group (parameter range is unbounded) and we can write everything as in the SO(2) case except now θ covers the range (-∞,∞), AND at the same time the representation label σ, instead of being integers, becomes a continuous real variable. We can then do these comparisons with SO(2):

$$f^\sigma = (1/2\pi) \int_{-\pi}^{\pi} d\theta\, f(\theta)\, e^{+i\sigma\theta} \qquad \text{// projection} \qquad \text{// Fourier Series}$$

$$f(\theta) = \Sigma_{n=-\infty}^{\infty}\, f^\sigma\, e^{-i\sigma\theta} \qquad \text{// expansion}$$

$$f^\sigma = (1/2\pi) \int_{-\infty}^{\infty} d\theta\, f(\theta)\, e^{+i\sigma\theta} \qquad \text{// projection} \qquad \text{// Fourier Transform}$$

$$f(\theta) = \int_{-\infty}^{\infty} d\sigma\, f^\sigma\, e^{-i\sigma\theta} \qquad \text{// expansion}$$

Often one thinks of θ = t, a time variable, and σ = ω, its conjugate frequency variable, or θ =x a spatial coordinate and then k as the wave number. There is always a question of how to distribute the 2π constant. In my Spectral Theory book, I include 1/2π in the expansion instead of the projection, as shown Chap 1, first two equations. Going with the above we have the following obvious orthogonality and completeness statements:

$$\int_{-\infty}^{\infty} d\theta/2\pi\; e^{i(\sigma-\sigma')\theta} = \delta(\sigma-\sigma') \qquad \text{// orthogonality}$$

$$\int_{-\infty}^{\infty} d\sigma/2\pi\; e^{-i\sigma(\theta'-\theta)} = \delta(\theta'-\theta) \qquad \text{// completeness}$$

The diagonalization of a convolution equation then works like this:

$$A(\theta) = \int d\theta_1/2\pi\, B(\theta_1)\, C(\theta-\theta_1) \qquad (*)$$

$$A^\sigma = B^\sigma\, C^\sigma \qquad (**)$$

where the various projections and expansions are:

$$A^\sigma = (1/2\pi) \int_{-\infty}^{\infty} d\theta\, A(\theta)\, e^{+i\sigma\theta} \qquad \text{// projection}$$

$$A(\theta) = \int_{-\infty}^{\infty} d\sigma\, A^\sigma\, e^{-i\sigma\theta} \qquad \text{// expansion}$$

and similarly for B and C. If our unknown function is B(θ), then we have the following solution to our integral equation:

$$B(\theta) = \int d\sigma\, (A^\sigma/C^\sigma)\, e^{-i\sigma\theta}$$

In my spectral theory paper, things look slightly different due to several issues:

(1) In that paper, I put the $1/2\pi$ in the expansion instead of the projection as in my thesis, as shown Chap 1 page 1. For this reason, the diagonalization which exactly matches what I just did above is given by (3.3) and (3.4) and not (3.1) and (3.2). It is just a question of where you put the 2π ! That is to say, equation (3.3) matches (*) above, while (3.4) matches (**), in terms of 2π factors.

(2) I also write the integral equation with the difference argument first, and I call that function B, so that means a B↔C swap compared to the above. So just accounting for this fact, the above reads

$$A(\theta) = \int d\theta_1/2\pi\, B(\theta-\theta_1)\, C(\theta_1)$$

$$A^\sigma = B^\sigma\, C^\sigma$$

I think this is more the standard way a "convolution equation" is written. If I had things to start over again, I would probably do it this way which is consistent with the integral operator notion A = B[C] and in particular its matrix interpretation $A_\theta = \Sigma_{\theta 1}\, B_{\theta,\theta 1} C_{\theta 1}$.

So here is a summary of the T(1) results just stated, converted to more familiar coordinates. Also, I have moved the

$$\int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = 2\pi\delta(k-k') \qquad \text{// orthogonality}$$

$$\int_{-\infty}^{\infty} dk\, e^{-ik(x'-x)} = 2\pi\delta(x-x') \qquad \text{// completeness}$$

$A(k) = (1/2\pi) \int_{-\infty}^{\infty} dx\, a(x)\, e^{+ikx}$ // projection

$a(x) = \int_{-\infty}^{\infty} dk\, A(k)\, e^{-ikx}$ // expansion (same for b and c below)

$a(x) = (1/2\pi) \int_{-\infty}^{\infty} dx'\, b(x-x')\, c(x')$ // convolution-form integral equation

$A(k) = B(k)\, C(k)$ // its diagonalization by the Fourier Integral transform

Let's restate the above one more time with a different convention. Define

$a'(x) \equiv (1/2\pi)\, a(x)$

and we will then have

$A(k) = \int_{-\infty}^{\infty} dx\, a'(x)\, e^{+ikx}$ // projection

$a'(x) = (1/2\pi) \int_{-\infty}^{\infty} dk\, A(k)\, e^{-ikx}$ // expansion (same for b' and c' below)

$2\pi a'(x) = (1/2\pi) \int_{-\infty}^{\infty} dx'\, 2\pi b'(x-x')\, 2\pi c'(x')$

or

$a'(x) = \int_{-\infty}^{\infty} dx'\, b'(x-x')\, c'(x')$ // convolution-form integral equation

$A(k) = B(k)\, C(k)$ // its diagonalization by the Fourier Integral transform

This form is nicer because there is no factor of 2π either in the integral equation or in the diagonalized equation. So now let's re-summarize all the results in this convention, removing now the primes, and also changing the sign in the exponents to get a match to Bateman ET1 tables.

$\int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = 2\pi\delta(k-k')$ // orthogonality

$\int_{-\infty}^{\infty} dk\, e^{-ik(x-x')} = 2\pi\delta(x-x')$ // completeness

$A(k) = \int_{-\infty}^{\infty} dx\, a(x)\, e^{-ikx}$ // projection

$a(x) = (1/2\pi) \int_{-\infty}^{\infty} dk\, A(k)\, e^{+ikx}$ // expansion (same for b and c below)

$a(x) = \int_{-\infty}^{\infty} dx'\, b(x-x')\, c(x')$ // convolution-form integral equation

$A(k) = B(k)\, C(k)$ // its diagonalization by the Fourier Integral transform

There are two applications that come to mind of this diagonalization. The first is that one can use it to solve for one of the functions in the integrand, if the other two functions are known, and in this sense one is "solving an integral equation". The second application is just an idea. Maybe you know b and c, but you cannot evaluate the dx' integral to find a. If you can compute B and C, then you know A, and then maybe from A you can compute a. In any event, all this stuff is predicated on functions whose integrals converge.

6. Relation between group Convolution equations and Fred 1 Integral equations?

It is not easy to reconcile these two "worlds". The word "diagonalization" has a meaning in each case, but it is not the same meaning. I tried hard to force these worlds together in the following subsections, but they don't really fit.,

(a) Fredholm Integral Equation and solution using a complete basis of eigenfunctions

In the Fredholm integral equation world, we perhaps have an equation of this form: (here I am trying to express things in a form that can be compared to a later group theory interpretation)

$$A(g) = \int dg_1\, K(g, g_1)\, B(g_1)$$

where K is an integral operator that acts on functions defined on the group G so then A = KB and we have overloaded the symbol K. Now one of the Fredholm techniques is to take a function like A(g) and project it onto a complete set of basis functions φ_i. We might say (where i is a symbolic composite index)

$$A(g) = \Sigma_i\, A_i\, \varphi_i(g) \qquad\qquad A_i = \int dg\, \varphi_i(g)\, A(g)$$

Now suppose we can find the eigenvalues and eigenfunctions of the integral operator K whose kernel is $K(g, g_1)$. Then we have $K\varphi_i = \lambda_i\varphi_i$ and we can use the φ_i as our complete basis (providing K is Hermitian). Then we have $A(g) = \Sigma_i\, A_i\, \varphi_i(g)$ and we can insert this into our integral equation this way

$$A = K\,B \quad \Rightarrow \quad \Sigma_i\, A_i\, \varphi_i(g) = K\,\{\,\Sigma_i\, B_i\, \varphi_i(g)\,\} = \Sigma_i\, B_i\{\,K\,\varphi_i(g)\,\} = \Sigma_i\, B_i\lambda_i\, \varphi_i(g)$$

$$\Rightarrow \quad A_i = B_i\lambda_i \quad \Rightarrow \quad B_i = A_i/\lambda_i \quad \text{and } B(g) = \Sigma_i\, B_i\, \varphi_i(g)$$

What have we done here? Write things in QM notation. First, let $\varphi_i \rightarrow |i\rangle$,

$$K|i\rangle = \lambda_i|i\rangle \quad \Rightarrow \quad \langle j|K|i\rangle = \lambda_i\langle j|i\rangle = \lambda_i\delta_{ij} \quad \Rightarrow \quad K_{ij} = \lambda_i\,\delta_{ij}$$

So in this φ_i basis, we find that *our matrix K is diagonal*! We go on then to solve our A = KB :

$K|B\rangle = |A\rangle \quad \Rightarrow \quad \Sigma_j \langle i|Kj\rangle\langle j|B\rangle = \langle i|A\rangle \Rightarrow \quad K_{ii}\langle i|B\rangle = \langle i|A\rangle$

$\Rightarrow \lambda_i B_i = A_i \quad \Rightarrow \quad B_i = \lambda_i^{-1} A_i \quad \Rightarrow \quad |B\rangle = \Sigma_i |i\rangle\langle i|B\rangle = \Sigma_i \lambda_i^{-1} A_i |i\rangle$

Basically, our integral equation A = K B has been "diagonalized" into the form $A_i = K_{ii} B_i$ where in fact $K_{ii} = \lambda_i$. Recall Matrix binder Theorem 19C: A Hermitian matrix can be diagonalized by a unitary similarity. So our basis φ_i is the basis in which K is diagonal, and our diagonalized problem is then trivial to solve. Here we are just "diagonalizing" a matrix K, and a priori, this has no connection whatever to the group theory concept of diagonalizing a convolution equation. Our Fredholm diagonalization has been achieved by finding a complete set of basis functions φ_i which diagonalize the matrix K. We can regard these basis functions as defining a generalized "Fourier Transform" which would be this:

$A(g) = \Sigma_i A_i \varphi_i(g)$	expansion	$\langle g|A\rangle = \Sigma_i \langle g|i\rangle\langle i|A\rangle$
$A_i = \int dg\, \varphi_i(g) A(g)$	projection	$\langle i|A\rangle = \int dg \langle i|g\rangle\langle g|A\rangle$

(I am assume a real Hilbert space here) .

So the upshot is that we can use a Fourier Transform to "diagonalize" our Fredholm integral equation and make it trivial to solve. We must of course find the right set of basis functions, and those will be the functions which are the eigenfunctions of the operator K in our integral equation. (this gives us strong motivation perhaps to be able to find the eigenfunctions of an integral operator K!).

(b) The Convolution Integral Equation written as a Fredholm equation

Consider again the convolution equation $A(g) = \int dg_1 B(g_1) C(g_2)$ where $g_2 = g_1^{-1} g$. In order to make this *look like* a Fredholm equation, let's rewrite is this way,

$$A(g) = \int dg_1 B(g_1) C(g_1^{-1} g) = \int dg_1 K(g, g_1) B(g_1) \qquad K(g, g_1) \equiv C(g_1^{-1} g)$$

In the matrix sense we can think of A and B as vectors, and K as a matrix, so we have

$A_g = \Sigma_{g1} K_{g,g1} B_{g1}$ or $A = K B$

In this matrix equation A = KB, the vector A has an infinite number of components labeled by g, where g is an element of our group G. The matrix K is an ∞ x ∞ matrix that is "fully populated" with matrix elements.

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

where it just helps me to "see" the drawing of some kind of matrix. A label here like "1" means perhaps something like $\varphi_1,\theta_1,\psi_1$ for SO(3) Euler angles. We are trying to solve this matrix equation which requires us to invert the matrix K which is a painful thing to do. The matrix K is highly "non-diagonal".

So my point so far is that yes, we can *regard* our convolution equation as a Fredholm equation, and we might go so far as to assume that the kernel K is Hermitian. We might then try to find the eigenfunctions of K, and carry out the program outlined in (a) above.

But this is not quite what the group theoretic analysis does. In the group theory approach, we replace our Fourier Transform with a Fourier Transform on a Group, which is a different animal! We pause now to compare these two Fourier concepts, and then we attempt to make the Group version look as much like the non-group version as possible. After doing this, we shall resume trying to analyze the convolution equation as a Fredholm kernel equation.

(c) Comparing the regular Fourier Transform with the Fourier Transform on a Group

Here is a comparison:

Fourier Transform:

$A(g) = \Sigma_i A_i \varphi_i(g)$	expansion	$\langle g\|A\rangle = \Sigma_i \langle g\|i\rangle\langle i\|A\rangle$
$A_i = \int dg\, A(g)\varphi_i(g)$	projection	$\langle i\|A\rangle = \int dg \langle i\|g\rangle\langle g\|A\rangle$
$\int dg\, \varphi_i(g)\varphi_j(g) = \delta_{ij}$	orthogonality	$\langle i\|j\rangle = \delta_{ij}$
$\Sigma_i \varphi_i(g)\, \varphi_i(g') = \delta(g-g')$	completeness	$1 = \Sigma_i \|i\rangle\langle i\|$

Fourier Transform on a Group:

$A(g) = \Sigma_\sigma d^\sigma \mathrm{tr}[\, A^\sigma D^\sigma(g)] = \Sigma_\sigma\ d^\sigma \Sigma_{k,k'} A^\sigma_{kk'} D^\sigma_{k'k}(g)$	expansion
$A^\sigma_{kk'} = \int dg\, A(g)\, D^\sigma_{kk'}(g^{-1})$	projection
$\int dg\ D^{\sigma'}(g)_{nm} D^\sigma(g)_{k'k}{}^* = (1/d^\sigma)\, \delta_{\sigma\sigma'}\ \delta_{nk'}\ \delta_{mk}$	orthogonality
$\Sigma_\sigma d^\sigma \mathrm{tr}[D^\sigma(g'^{-1})D^\sigma(g)] = \Sigma_\sigma d^\sigma \Sigma_{k,k'} D^\sigma(g'^{-1})_{kk'} D^\sigma(g)_{k'k} = \delta(g-g')$	completeness

I don't have any instant analogous bra-ket representation to write on the right hand side here, but let's try to construct something. Let's *try* these forms:

$$D^\sigma_{k'k}(g) \equiv \langle g \mid \sigma,k',k\rangle$$

$$D^\sigma_{kk'}(g^{-1}) = D^\sigma_{kk'}{}^{-1}(g) = D^\sigma_{kk'}{}^{T*}(g) = D^\sigma_{k'k}{}^*(g) = \langle g \mid \sigma,k',k\rangle^* = \langle \sigma,k',k \mid g\rangle$$

$$\langle g|g'\rangle = \delta(g-g') \qquad 1 = \int dg\, |g\rangle\langle g|$$

$$\langle \sigma,a,b|\ \sigma',a',b'\rangle = (1/d^\sigma)\, \delta_{\sigma\sigma'}\delta_{aa'}\delta_{bb'} \qquad 1 = \Sigma_{\sigma,k,k'}\ |\sigma,k,k'\rangle\, d^\sigma\, \langle \sigma,k,k'|$$

$A^{\sigma}_{k,k'} = \langle \sigma,k',k|A\rangle$ // note order change! $\quad A(g) = \langle g|A\rangle$

Using these rules, I will now try to "translate" each of the four lines in our group transform:

expansion: $\langle g|A\rangle = \Sigma_{\sigma,k,k'}\, d^{\sigma} \langle \sigma,k',k|A\rangle \langle g \mid \sigma,k',k\rangle = \Sigma_{\sigma,k,k'} \langle g \mid \sigma,k',k\rangle\, d^{\sigma} \langle \sigma,k',k|A\rangle$

And we can then use 1 = ... to recover $\langle g|A\rangle$, so the expansion line looks good. Next:

projection: $A^{\sigma}_{k,k'} = \int dg\, A(g)\, D^{\sigma}_{k,k'}(g^{-1})$

$\langle \sigma,k',k|A\rangle = \int dg \langle g|A\rangle \langle \sigma,k',k \mid g\rangle = \int dg \langle \sigma,k',k \mid g\rangle\langle g|A\rangle$ = OK

So this also looks good. Next:

orthogonality: $\int dg\ D^{\sigma'}(g)_{nm}\, D^{\sigma}(g)^{*}_{k',k} = \int dg \langle g \mid \sigma',n,m\rangle \langle g \mid \sigma,k',k\rangle^{*}$

$= \int dg \langle \sigma,k',k \mid g\rangle\langle g \mid \sigma',n,m\rangle = \langle \sigma,k',k \mid \sigma',n,m\rangle = (1/d^{\sigma})\, \delta_{\sigma\sigma'}\delta_{k'n}\delta_{km}$ = OK

One more line to check out

completeness: $\Sigma_{\sigma}\, d^{\sigma}\, \Sigma_{k,k'} D^{\sigma}(g'^{-1})_{kk'} D^{\sigma}(g)_{k'k} = \Sigma_{\sigma,k,k'}\, d^{\sigma} \langle \sigma,k',k \mid g'\rangle \langle g \mid \sigma,k',k\rangle$

$= \Sigma_{\sigma,k,k'} \langle g \mid \sigma,k',k\rangle\, d^{\sigma} \langle \sigma,k',k \mid g'\rangle = \langle g \mid g'\rangle = \delta(g-g')$

so we are cooking with gas now. Here is a summary:

<u>Fourier Transform:</u>

$A(g) = \Sigma_i A_i \varphi_i(g)$	expansion	$\langle g|A\rangle = \Sigma_i \langle g|i\rangle\langle i|A\rangle$
$A_i = \int dg\, A(g)\varphi_i(g)$	projection	$\langle i|A\rangle = \int dg \langle i|g\rangle\langle g|A\rangle$
$\int dg\, \varphi_i(g)\varphi_j(g) = \delta_{ij}$	orthogonality	$\langle i|j\rangle = \delta_{ij}$
$\Sigma_i\, \varphi_i(g)\, \varphi_i(g') = \delta(g-g')$	completeness	$1 = \Sigma_i\, \|i\rangle\langle i\|$

<u>Fourier Transform on a Group:</u>

$A(g) = \Sigma_{\sigma}\, d^{\sigma}\, \Sigma_{k,k'}\, A^{\sigma}_{k,k'}\, D^{\sigma}_{k'k}(g)$	expansion	$\langle g|A\rangle = \Sigma_{\sigma,k,k'} \langle g \mid \sigma,k',k\rangle\, d^{\sigma} \langle \sigma,k',k|A\rangle$
$A^{\sigma}_{k,k'} = \int dg\, A(g)\, D^{\sigma}_{k,k'}(g^{-1})$	projection	$\langle \sigma,k',k|A\rangle = \int dg \langle \sigma,k',k \mid g\rangle\langle g|A\rangle$
$\int dg\ D^{\sigma'}(g)_{nm}\, D^{\sigma}(g)^{*}_{k',k} = (1/d^{\sigma})\, \delta_{\sigma\sigma'}\, \delta_{nk'}\, \delta_{mk}$	orthogonality	$= \int dg \langle \sigma,k',k \mid g\rangle\langle g \mid \sigma',n,m\rangle$
$\Sigma_{\sigma}\, d^{\sigma}\, \Sigma_{k,k'} D^{\sigma}(g'^{-1})_{kk'} D^{\sigma}(g)_{k'k} = \delta(g-g')$	completeness	$= \Sigma_{\sigma,k,k'} \langle g \mid \sigma,k',k\rangle\, d^{\sigma} \langle \sigma,k',k \mid g'\rangle$

$D^{\sigma}_{k',k}(g) \equiv \langle g \mid \sigma,k',k \rangle$

$D^{\sigma}_{kk'}(g^{-1}) = D^{\sigma}_{kk',}{}^{-1}(g) = D^{\sigma}_{kk',}{}^{T*}(g) = D^{\sigma}_{k',k}{}^{*}(g) = \langle g \mid \sigma,k',k \rangle^{*} = \langle \sigma,k',k \mid g \rangle$

$\langle g|g' \rangle = \delta(g-g')$ $\qquad 1 = \int dg\, |g\rangle\langle g|$

$\langle \sigma,a,b| \sigma',a',b' \rangle = (1/d^{\sigma})\, \delta_{\sigma\sigma'}\delta_{aa'}\delta_{bb'}$ $\qquad 1 = \Sigma_{\sigma,k,k'} \mid \sigma,k,k' \rangle\, d^{\sigma}\, \langle \sigma,k,k' \mid$

$A^{\sigma}_{kk'} = \langle \sigma,k',k|A \rangle$ $\qquad A(g) = \langle g|A \rangle$

Now, finally, let's try to identify the composite index i with σ,k',k so that $|i\rangle = |\, \sigma,k',k\, \rangle$ which we mean as just a shorthand notation. Then $\Sigma_{\sigma,k,k'}$ becomes just Σ_i and we can try to write our four lines for Fourier Transform on a group. Start with the expansion:

$$\langle g|A \rangle = \Sigma_{\sigma,k,k'} \langle g \mid \sigma,k',k \rangle\, d^{\sigma}\, \langle \sigma,k',k|A \rangle = \Sigma_i \langle g|i \rangle\, d^{i}\, \langle i|A \rangle = \Sigma_i\, d^{i} A_i\, \varphi_i(g)$$

so this looks like our regular Fourier expansion except we have the dimension $d^i = d^{\sigma}$ sitting in there. Next, the projection,

$\langle \sigma,k',k|A \rangle = \int dg\, \langle \sigma,k',k \mid g \rangle \langle g|A \rangle$ or $\langle i|A \rangle = \int dg\, \langle i|g \rangle \langle g|A \rangle$ or $A_i = \int dg\, A(g)\, \varphi_i(g)$

and this exactly matches our Fourier projection with no adjustment. Next, the orthogonality:

$$(1/d^{\sigma})\, \delta_{\sigma\sigma'}\, \delta_{nk'}\, \delta_{mk} = \int dg\, \langle \sigma,k',k\, |g\rangle\langle g \mid \sigma',n,m \rangle = \int dg\, \langle i\, |g\rangle\langle g \mid i' \rangle = \int dg\, \varphi_i(g)^{*}\, \varphi_{i'}(g)$$

$$= \langle i|i' \rangle = (1/d^{i})\, \delta_{ii'}$$

So in this case we do have an adjustment, and we write

$$\int dg\, \varphi_i(g) \varphi_j(g) = (1/d^{i})\, \delta_{ij}$$

where of course δ_{ij} is shorthand for $\delta_{\sigma\sigma'}\delta_{aa'}\delta_{bb'}$. Finally we come to completeness:

$$\Sigma_{\sigma,k,k'} \langle g \mid \sigma,k',k \rangle\, d^{\sigma}\, \langle \sigma,k',k \mid g' \rangle = \Sigma_i \langle g|i \rangle\, d^{i}\, \langle i \mid g' \rangle = \langle g|g' \rangle = \delta(g-g')$$

So now let's do a comparison between our Fourier Transform and our Group version with this compacted notation:

<u>Fourier Transform:</u>

$A(g) = \Sigma_i\, A_i\, \varphi_i(g)$	expansion	$\langle g|A \rangle = \Sigma_i \langle g|i \rangle \langle i|A \rangle$
$A_i = \int dg\, A(g) \varphi_i(g)$	projection	$\langle i|A \rangle = \int dg\, \langle i|g \rangle \langle g|A \rangle$
$\int dg\, \varphi_i(g) \varphi_j(g) = \delta_{ij}$	orthogonality	$\langle i|j \rangle = \delta_{ij}$

$\Sigma_i \varphi_i(g)\, \varphi_i(g') = \delta(g-g')$	completeness	$1 = \Sigma_i \, \|i\rangle\langle i\|$
Fourier Transform on a Group:	$\|i\rangle = \|\, \sigma,k',k\,\rangle \quad d^i = d^\sigma$	$\Sigma_i = \Sigma_{\sigma,k,k'}$
$A(g) = \Sigma_i \, d^i A_i \, \varphi_i(g)$	expansion	$\langle g\|A\rangle = \Sigma_i \, d^i \langle g\|i\rangle\langle i\|A\rangle$
$A_i = \int dg \; A(g)\varphi_i(g)$	projection	$\langle i\|A\rangle = \int dg \; \langle i\|g\rangle\langle g\|A\rangle$
$\int dg \; \varphi_i(g)\varphi_j(g) \;=\; (1/d^i)\delta_{ij}$	orthogonality	$\langle i\|j\rangle = (1/d^i)\delta_{ij}$
$\Sigma_i \; \varphi_i(g) \; d^i \; \varphi_i(g') = \delta(g-g')$	completeness	$1 = \Sigma_i \, \|i\rangle d^i \langle i\|$

(d) Resume our attempted interpretation of a convolution equation as a Fredholm equation

Earlier we got things rolling by writing the convolution equation in this Fredholm manner

$$A(g) = \int dg_1 B(g_1) C(g_1^{-1} g) \;=\; \int dg_1 \; K(g, g_1) \; B(g_1) \qquad\qquad K(g, g_1) \;\equiv C(g_1^{-1} g)$$

We know that the diagonalized version of this equation has this form within each σ subspace,

$$A^\sigma_{kk'} \;=\; \Sigma_{k''} \; C^\sigma_{kk''} \; B^\sigma_{k''k'}$$

$$\langle \sigma,k',k|A\rangle \;= \Sigma_{k''} \; \langle \sigma,k'',k|C\rangle\langle \sigma,k',k''|B\rangle$$

$$\langle i_1|A\rangle = \Sigma_{k''} \; \langle i_2|C\rangle \; \langle i_3|B\rangle$$

I seem to be unable to force this diagonalized equation into any kind of reasonable notation using the composite index $|i\rangle = |\, \sigma,k',k\,\rangle$. The diagonalized equation involves three different composite indices which I have called $i_{1,2,3}$ all of which have the same σ. The sum is then somehow only over a part of the composite index.

Perhaps we would like to see the kernel K object in there instead of C. We know that

$$C^\sigma_{kk''} = \int dg' \; C(g') \; D^\sigma_{kk''}(g'^{-1})$$

As we did in earlier sections, we can use the invariant measure idea. Suppose we set $g' = g_1^{-1} g$ so that then $g'^{-1} = (g_1^{-1} g)^{-1} = g^{-1} \, g_1$ and we can then write

$$C^\sigma_{kk''} = \int d(g_1^{-1} g) \; C(g_1^{-1} g) \; D^\sigma_{kk''}(\, g^{-1} \, g_1)$$

$$= \int dg \; C(g_1^{-1} g) \; D^\sigma_{kk''}(\, g^{-1} \, g_1) \qquad\qquad \text{// here we use the invariant measure idea!}$$

$= \int dg\, K(g, g_1)\ D^{\sigma}_{kk"}(\, g^{-1}\, g_1)$ // and now K appears

$= \int dg\, K(g, g_1)\ \Sigma_{k'}\, D^{\sigma}_{kk'}(g^{-1})\, D^{\sigma}_{k'k"}(g_1)$ // "the group property" of representations

$= \Sigma_{k'}\ \{\ \int dg\, K(g, g_1)\ D^{\sigma}_{kk'}(g^{-1})\}\ \ D^{\sigma}_{k'k"}(g_1)$

$= \Sigma_{k'}\ \{\, K^{\sigma}_{kk'}(g_1)\,\}\ \ D^{\sigma}_{k'k"}(g_1)$ // treat g_1 as a parameter in the K projection

I don't know whether this is buying us anything, but we then have

$C^{\sigma}_{kk"} = \Sigma_{k'}\, K^{\sigma}_{kk'}(g_1)\, D^{\sigma}_{k'k"}(g_1)$ // which is somehow independent of g_1

Then we can get K into our diagonalized equation

$A^{\sigma}_{kk'} = \Sigma_{k"}\, C^{\sigma}_{kk"}\, B^{\sigma}_{k"k'} = \Sigma_{k"}\, \{\, \Sigma_n\, K^{\sigma}_{kn}(g_1)\, D^{\sigma}_{nk"}(g_1)\}\, B^{\sigma}_{k"k'}$

$= \Sigma_{n,k"}\, K^{\sigma}_{kn}(g_1)\, D^{\sigma}_{nk"}(g_1)\, B^{\sigma}_{k"k'}$

This does nothing for me whatsoever!

(e) Conclusions

I have tried to make the Fourier on a Group situation "look like" a regular Fourier case by using the composite index $|i\rangle = |\, \sigma; k', k\,\rangle$. But this just does not cut the mustard. We need more than one symbol. We need σ to specify a subspace σ, and then within that subspace we need two more indices to specify our representation function D, and you cannot just sweep this all under the rug. In the group case, we end up with a finite matrix equation $A^{\sigma}_{kk'} = \Sigma_{k"}\, C^{\sigma}_{kk"}\, B^{\sigma}_{k"k'}$ within each subspace. We can in principle invert this equation to find the $B^{\sigma}_{k"k'}$ in each finite subspace, and then we have a theoretical solution to the overall problem of solving the integral equation. In the Fredholm case, we don't have these subspaces. The group method aligns with the Fredholm method only if all representations are 1D, and then there are no k type indices and it all works out.

So we really have two distinct "worlds" here. A convolution equation is a very special case of an integral equation. It has to have the invariant measure of some group, and it has to have the special convolution form. Stak uses * to indicate such a form I think. When you "diagonalize" the convolution equation, you end up with an infinite number of finite dimensional matrix equations. You can in theory solve all of these finite matrix equations A = BC to get B, and in that way you can say you have solved the integral equation. Maybe there is a perturbation approach when you do that.

The Fredholm integral equation (meaning inhomo Fred 1) is a much more general animal. If you can find a set of eigenfunctions of the kernel operator K, you can solve such a Fred equation as outlined above, where you have in effect "diagonalized" the matrix K.

For the convolution equation case, you can *think of* a composite index $|i\rangle = |\, \sigma; k', k\,\rangle$ and you can think then of vector components A_i, but one finds that the coefficients only "interact" with each other in σ subspaces. Perhaps one should think of $A(g) = \Sigma_{\sigma}\, A^{(\sigma)}(g)$ so the function has pieces in the different

subspaces, and we similarly have our Hilbert Space $H = \Sigma_\sigma H^{(\sigma)}$. We could talk then about $\langle g|A^{(\sigma)}\rangle$ as one of the function pieces. Then

$$A^{(\sigma)}(g) = \langle g|A^{(\sigma)}\rangle = d^\sigma \Sigma_{k,k'} A^\sigma_{kk'} D^\sigma_{k'k}(g) = d^\sigma \text{ trace}[A^\sigma D^\sigma(g)]$$

This says then that the piece of A(g) inside subspace σ is a linear combinations of the harmonics of that subspace with coefficients $A^\sigma_{kk'}$. You can write the harmonics as $\langle g|\ \sigma,k',k\rangle$ if you want, and then you can think of $A^\sigma_{kk'} = \langle\sigma,k',k|A\rangle$ as being the amount of that harmonic contained in $A^{(\sigma)}(g)$. I was looking for some object of the form $\langle\sigma,k\ |\ A\ |\ \sigma,k'\rangle$ but I don't see this as having any meaning because A is a ket, not an operator. We CAN think of $D^\sigma_{k'k}(g)$ as $\langle\sigma,k'|D(g)|\sigma,k\rangle$ where D(g) is an operator. So

$$A^{(\sigma)}(g) = d^\sigma \Sigma_{k,k'} A^\sigma_{kk'} \langle\sigma,k'|D(g)|\sigma,k\rangle$$

but there is no operator A that makes any sense to me. $A^\sigma_{kk'}$ is just a coefficient, a number. Yes, you can write it as $\langle\sigma,k',k|A\rangle$, but all you mean is that it is the amount of $D^\sigma_{k'k}(g) = \langle g|\sigma,k',k\rangle$ in A(g).

We could have started off writing this

$$A(g) = \Sigma_{\sigma,\sigma'} A^{\sigma,\sigma'}_{k,k'} \langle\sigma',k'|D(g)|\sigma,k\rangle$$

but we quickly find that if we use the right kets, the matrix $\langle\sigma',k'|D(g)|\sigma,k\rangle$ has block diagonal form. So THIS is the matrix that we should be thinking of in terms of diagonalization into block form.

So in a convolution equation we are dealing with a diagonalization of the group operators D(g) into a matrix which is in block diagonal form, where each block is associated with a subspace σ. In each subspace, we have a finite matrix problem of dimension d^σ to deal with.

In a Fred 1 equation, if we use the eigenfunction method described above, we are in effect diagonalizing the operator K into a fully diagonal (infinite discrete) matrix $\langle j|K|i\rangle = \lambda_i\langle j|i\rangle = \lambda_i\delta_{ij}$. These two concepts of diagonalization are just not closely connected with each other, that is the upshot of this entire section!

7. Application of Section 6 to Green's Functions in Potential Theory

Suppose we have this integral equation in regular potential theory (g is symmetric)

$$a(x) = \int g(x,x')b(x')dx' \qquad\qquad g(x,x') = \Sigma_i \varphi_i(x)\,\varphi_i(x')^*/\lambda_i$$

where the φ_i and λ_i come from the problem $-\nabla^2\varphi_i(x) = \lambda_i\,\varphi_i(x)$ with $\varphi_i(x \text{ on } \sigma) = 0$. We can write our integral equation this way

$$a_x = \Sigma_x G_{x,x'} b_{x'}$$

and obviously our kernel is not diagonal in the x basis. But, we can show that G is diagonal in the "i" basis:

$$G_{ij} = \int dx \int dx' \langle i|x\rangle\langle x|G|x'\rangle\langle x'|j\rangle = \int dx \int dx' \, \varphi_i(x)^* g(x;x') \varphi_j(x')$$

$$= \int dx \int dx' \, \varphi_i(x)^* \{ \Sigma_k \, \varphi_k(x) \varphi_k(x')^* / \lambda_k \} \varphi_j(x')$$

$$= \int dx' \, \varphi_i(x')^* \, \varphi_j(x') / \lambda_i = \delta_{ij} / \lambda_i = \delta_{ij} \mu_i$$

and indeed, in this "i basis", operator G is diagonal with diagonal elements equal to the eigenvalues of the integral equation $G\varphi_i = \mu_i \varphi_i$ which we know are the same as those of $-\nabla^2 \varphi_i = \lambda_i \varphi_i$.

Now back to the integral equation which was in the coordinate basis

$$\langle x|a\rangle = \int dx' \langle x|G|x'\rangle\langle x'|b\rangle$$

We slice off $\langle x|$ and change the left sides to the i basis,

$$\langle i|a\rangle = \int dx' \langle i|G|x'\rangle\langle x'|b\rangle$$

Then we insert completeness $\Sigma_j |j\rangle\langle j| = 1$ on the RHS after the G

$$\langle i|a\rangle = \int dx' \langle i|G \Sigma_j |j\rangle\langle j|x'\rangle\langle x'|b\rangle = \Sigma_j G_{ij} \int dx' \langle j|x'\rangle\langle x'|b\rangle = \Sigma_j G_{ij} \langle j|b\rangle$$

and no surprise we end up with

$$a_i = \Sigma_j G_{ij} b_j = G_{ii} b_i = (1/\lambda_i) \, b_i$$

So here we have diagonalized an integral equation in which the kernel is a potential theory Green's Function. We did not use the "group theory method" since the Green's Function is not of convolution form. We simply used the "find a basis in which the integral operator G is diagonal" method.

8. Repeat Section 7 in Heat Conduction Theory

Suppose we have this integral equation in heat conduction theory where $t > t'$:

$$a(x,t) = \int g(x,t;x',t') b(x',t') dx' \qquad g(x,t;x',t') = \theta(t-t') \, \Sigma_i \, \varphi_i(x) \, \varphi_i(x')^* \, e^{-\lambda_i (t-t')}$$

where the φ_i and λ_i come from the problem $-\nabla^2 \varphi_i(x) = \lambda_i \, \varphi_i(x)$ with $\varphi_i(x \text{ on } \sigma) = 0$. That is to say, these are the same φ_i we had in Section 7 above.

We can write our integral equation this way, where we suppress the time labels on things,

$$a_x = \Sigma_x G_{x,x'} b_{x'}$$

and obviously our kernel is not diagonal in the x basis. But, we can show that G is diagonal in the "i" basis. Again, we just regard t and t' as parameters which don't change and we could put them in if we wanted, but we suppress them:

$$G_{ij} = \int dx \int dx' \langle i|x\rangle\langle x|G|x'\rangle\langle x'|j\rangle = \int dx \int dx' \varphi_i(x)^* g(x;x') \varphi_j(x')$$

$$= \int dx \int dx' \varphi_i(x)^* \{ \Sigma_k \varphi_k(x) \varphi_k(x')^* e^{-\lambda_k (t-t')} \} \varphi_j(x')$$

$$= \int dx' \varphi_i(x')^* \varphi_j(x') e^{-\lambda_i (t-t')} = \delta_{ij} e^{-\lambda_i (t-t')}$$

and indeed, in this "i basis", operator G is diagonal with diagonal elements as shown.

Now back to the integral equation which was in the coordinate basis (and we continue to suppress time labels)

$$\langle x|a\rangle = \int dx' \langle x|G|x'\rangle\langle x'|b\rangle$$

We slice off $\langle x|$ and change the left sides to the i basis,

$$\langle i|a\rangle = \int dx' \langle i|G|x'\rangle\langle x'|b\rangle$$

Then we insert completeness $\Sigma_j |j\rangle\langle j| = 1$ on the RHS after the G

$$\langle i|a\rangle = \int dx' \langle i|G \Sigma_j |j\rangle\langle j|x'\rangle\langle x'|b\rangle = \Sigma_j G_{ij} \int dx' \langle j|x'\rangle\langle x'|b\rangle = \Sigma_j G_{ij} \langle j|b\rangle$$

and no surprise we end up with

$$a_i(t) = \Sigma_j G_{ij} b_j = G_{ii} b_i = e^{-\lambda_i (t-t')} b_i(t')$$

So here we have diagonalized an integral equation in which the kernel is a heat conduction theory Green's Function. We did not use the "group theory method" since the Green's Function is not of convolution form in spatial coordinates. We simply used the "find a basis in which the integral operator G is diagonal" method.

Example: In the discussion of Stak page 229-230, we have our finite length ℓ rod with u = 0 on both ends, and we have a set of $\varphi_n(x) = \sqrt{2/\ell} \sin(n\pi x/\ell)$ so that

$$g(x,t;x',t') = \theta(t-t') \Sigma_n \varphi_n(x) \varphi_n(x')^* e^{-\lambda_n (t-t')} = \theta(t-t') (2/\ell) \Sigma_n \sin(n\pi x/\ell) \sin(n\pi x'/\ell) e^{-\lambda_n (t-t')}$$

where in this problem $\lambda_n = (n\pi/\ell)^2$. Our integral equation $b(x,t) = \int g(x,t;x',t')a(x',t')dx'$ is stated as 7.90 (notice we have reversed b and a relative to discussion above) where he uses t' = 0 and t = 1 and suppresses the time indices in a and b. We know from above that our diagonalized equation will be this:

$$b_n(t) = e^{-\lambda_n(t-t')} a_n(t')$$

or

$$b_n(1) = e^{-\lambda_n} a_n(0)$$

or

$$b_n = \exp[-(n\pi/\ell)^2] a_n$$

or

$$a_n = \exp[(n\pi/\ell)^2] b_n$$

which we see agrees with page 230 B and C. The projections are given of course by

$$a_n = \int dx\ \varphi_n(x)\ a(x) = \sqrt{2/\ell} \int_0^\ell dx\ \sin(n\pi x/\ell)\ a(x)$$

and these are just Fourier Sine Series projections, so we are talking this transform. If we attempt to solve for a(x) given b(x) we get

$$a(x) = \sqrt{2/\ell}\ \Sigma_n\ a_n \sin(n\pi x/\ell) = \sqrt{2/\ell}\ \Sigma_n \exp[(n\pi/\ell)^2] b_n \sin(n\pi x/\ell) \qquad \text{// page 230 D}$$

and Stak comments that it is "unlikely" that this Σ_n converges. He knows and we know that the only chance this has of converging is if b(x) is infinitely differentiable, see page 230. But assuming Stak's condition E is met by the projections of b(x), there will be a unique solution a(x).