An Exercise in Distribution Theory PhL 5.24.10

How would you verify (or disprove) this claimed fact?

$$\int_{-\infty}^{\infty} dx \cos(k'x) \cos(kx) = \pi \delta(k-k')$$

In distribution theory, we could come up with a test function $\varphi(k)$ being some kind of smooth thing with finite support, and we would then say

$$\int_{-\infty}^{\infty} dx \cos(k'x) \cos(kx) = \pi \delta(k-k') \quad \Rightarrow$$

$$\varphi(k') \int_{-\infty}^{\infty} dx \cos(k'x) \cos(kx) = \pi \delta(k-k') \varphi(k')$$

$$\int dk' \varphi(k') \int_{-\infty}^{\infty} dx \cos(k'x) \cos(kx) = \int dk' \pi \delta(k-k') \varphi(k')$$

so this last line is the "meaning" of the distributional statement of the first line. At this point it would be good to ask about the integration range of k. Whatever it is, we will assume that we have chosen k' to be within the range. Then we have

$$\int dk' \varphi(k') \int_{-\infty}^{\infty} dx \cos(k'x) \cos(kx) = \pi \varphi(k)$$

We might then continue along and try to say

$$\text{LHS} = \int_{-\infty}^{\infty} dx \cos(kx) \int dk' \varphi(k') \cos(k'x)$$

Now the range of the dk' integration seems perhaps more important.

We might follow page 33 Stak and choose the following test function

$$\varphi_a(k) = \exp(a^2/(k^2 - a^2) \qquad |k| < a \qquad \text{and 0 outside}$$

This function is maximal when the exponent is maximal which is when $k^2 = 0$ and we have $e^{-1} = 1/e$. So this function peaks at k = 0 and has half width a. Let's now move this so the center is somewhere else

$$\varphi_{a;k_0}(k) = \exp(a^2/([k-k_0]^2 - a^2) \qquad |k-k_0| < a \qquad \text{and 0 outside}$$

Now assume for the moment that the range of dk' is $(-\infty,\infty)$. Then we can consider:

$$\int dk' \, \varphi(k') \cos(k'x) \;=\; \int_{k_0-a}^{k_0+a} dk \, \exp(a^2/([k-k_0]^2 - a^2)) \cos(kx)$$

Is this a doable integral? I don't recall Stak every trying an integral like this. And I don't see anything in GR that involves something like exp[1/(ax^2+ bx + c)] but I'm sure you could do fractions and relate to simpler integrals, but it sounds unpleasant. I would rather use a Gaussian test function even though it does not have finite support, hope to get away with this. So let's try this Gaussian form

$$\varphi_{a;k_0}(k) = \exp(- [k-k_0]^2/a^2)$$

I am then trying to show this is true

$$\int_{-\infty}^{\infty} dx \cos(kx) \int_{-\infty}^{\infty} dk' \, \varphi_{a;k_0}(k') \cos(k'x) \;=\; \pi \, \varphi_{a;k_0}(k)$$

which is to say, I want to show this is true:

$$\int_{-\infty}^{\infty} dx \cos(kx) \int_{-\infty}^{\infty} dk' \exp(- [k'-k_0]^2/a^2) \cos(k'x) \;=\; \pi \exp(- [k-k_0]^2/a^2)$$

So let's go compute this integral

$$\int_{-\infty}^{\infty} dk' \exp(- [k'-k_0]^2/a^2) \cos(k'x)$$

Shift integration variable to k" = k'-k_0 to get

$$= \int_{-\infty}^{\infty} dk'' \exp(-k''^2/a^2) \cos[k''x + k_0 x]$$

I find this on GR7 page 488

$$2. \qquad \int_{-\infty}^{\infty} e^{-q^2x^2} \cos[p(x+\lambda)]\, dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \cos p\lambda$$

so set q = 1/a and x = k" and p = x and pλ = xλ = k_0x so λ = k_0 so we have

$$= \sqrt{\pi}\, a \;\exp(-x^2a^2/4) \cos(k_0 x)$$

So we now want to show that this is true:

$$\int_{-\infty}^{\infty} dx \cos(kx) \sqrt{\pi}\, a \;\exp(-x^2a^2/4) \cos(k_0 x) \;=\; \pi \exp(- [k-k_0]^2/a^2)$$

Notice how the gaussian is "controlling" the integrals, making them doable. So now we face a more complicated integral

$$\int_{-\infty}^{\infty} dx\ \exp(-x^2a^2/4)\cos(kx)\cos(k_0x) = 2\int_{0}^{\infty} dx\ \exp(-x^2a^2/4)\cos(kx)\cos(k_0x)$$

Luckily this also appears in GR7 page 489

$$2.\qquad \int_0^\infty e^{-\beta x^2}\cos ax\cos bx\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left\{e^{-\frac{(a-b)^2}{4\beta}} + e^{-\frac{(a+b)^2}{4\beta}}\right\} \qquad [\mathrm{Re}\,\beta > 0]$$

so we have x = x, $\beta = (a/2)^2 = a^2/4$ a = k, b = k_0 so we find // $1/(4\beta) = 1/a^2$

$$\int_{0}^{\infty} dx\ \exp(-x^2a^2/4)\cos(kx)\cos(k_0x) = (1/4)\sqrt{\pi}\,(2/a)\,\{\exp(-[k-k_0]^2/a^2) + \exp(-[k+k_0]^2/a^2)\}$$

$$= \sqrt{\pi}\,(1/2a)\,\{\exp(-[k-k_0]^2/a^2) + \exp(-[k+k_0]^2/a^2)\}$$

So we now have

$$\int_{-\infty}^{\infty} dx\ \cos(kx)\sqrt{\pi}\,a\ \exp(-x^2a^2/4)\cos(k_0x) =$$

$$= \sqrt{\pi}\,a\ *\ 2\ *\sqrt{\pi}\,(1/2a)\,\{\exp(-[k-k_0]^2/a^2) + \exp(-[k+k_0]^2/a^2)\}$$

$$= \pi\,\{\exp(-[k-k_0]^2/a^2) + \exp(-[k+k_0]^2/a^2)\}$$

$$= \pi\,\{\varphi(k) + \varphi(-k)\}$$

which is NOT the right answer, but it is sort of "close". We have an extra second term that we don't want to be there! Is this an artifact because I failed to use a proper test function? I don't think so.

Let's go back to our "claimed fact"

$$\int_{-\infty}^{\infty} dx\ \cos(k'x)\cos(kx) = \pi\,\delta(k-k')$$

Mystery question: the LHS is obviously symmetric under $k \to -k$, but that is not true for the RHS! So something is amiss here! Stak has lots of delta function stuff involving sums of trig functions. On page 284 Stak gets into the singular BC world and on page 284 he considers the sine transform situation which he defines on x in $(0,\infty)$ only. Then on page 286 this leads to the $\pi/2$ delta function completeness. He shows in 4.58 how the issue of the "measure" works out and why you get just dk = dv for him. He claims that the δ completeness "holds rigorously in distribution theory".

Now, from the work I just did above, I think I have in fact proved this result:

$$\int_{-\infty}^{\infty} dx \cos(k'x) \cos(kx) = \pi \delta(k-k') + \pi \delta(k+k')$$

where both terms must be present if your "range" for k is (-∞,∞). This then has the proper symmetries under negating one or both k factors. How can I show this? In the above work I had πδ(k-k') on the RHS. This caused the integrated RHS to be π φ(k). If instead I had πδ(k+k') = πδ(-k-k') on the RHS, the integrated RHS would have been π φ(-k). If I then added the two RHS's, I would have gotten the integrated RHS to be π [φ(k) + φ(-k)]. And this is what I got! So I think I can say that I have used distribution theory to prove the above fact. And there would be a corresponding other fact

$$\int_{-\infty}^{\infty} dk \cos(kx) \cos(kx') = \pi \delta(x-x') + \pi \delta(x+x')$$

which you would verify in the same manner. Then of course if your problem is x in (0,∞), you could delete the second term.

I don't think I have ever seen these integrals with the two terms anywhere. So this throws a new monkey wrench into things.

Would the same thing be true for sines? My guess would be this

$$\int_{-\infty}^{\infty} dx \sin(k'x) \sin(kx) = \pi \delta(k-k') - \pi \delta(k+k')$$

It then has the right symmetries. If I were to redo the above analysis, I would have some different integrals. First, we had this one from page 488

$$2. \qquad \int_{-\infty}^{\infty} e^{-q^2x^2} \cos[p(x+\lambda)]\, dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \cos p\lambda$$

and the one for sines is very similar, same page,

$$1. \qquad \int_{-\infty}^{\infty} e^{-q^2x^2} \sin[p(x+\lambda)]\, dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \sin p\lambda$$

the only difference being the appearance of the letters "sin" in place of "cos" on the right. We would then need to look up an integral with two sines. We had this one from page 489

$$2. \qquad \int_0^{\infty} e^{-\beta x^2} \cos ax \cos bx\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}} \left\{ e^{-\frac{(a-b)^2}{4\beta}} + e^{-\frac{(a+b)^2}{4\beta}} \right\} \qquad [\mathrm{Re}\,\beta > 0]$$

and the sine one would be this

1. $$\int_0^\infty e^{-\beta x^2} \sin ax \sin bx \, dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}} \left\{ e^{-\frac{(a-b)^2}{4\beta}} - e^{-\frac{(a+b)^2}{4\beta}} \right\} \qquad [\mathrm{Re}\,\beta > 0]$$

and there you see the minus sign we want to see on the second term.

So here are my conclusions:

$\int_{-\infty}^{\infty}$ dx cos(k'x) cos(kx) = π δ(k-k') + π δ(k+k')

$\int_{-\infty}^{\infty}$ dx sin(k'x) sin(kx) = π δ(k-k') - π δ(k+k')

and you can then write these in various other ways, such as

$\int_{0}^{\infty}$ dx cos(k'x) cos(kx) = (π/2) δ(k-k') + (π/2) δ(k+k')

$\int_{0}^{\infty}$ dx sin(k'x) sin(kx) = (π/2) δ(k-k') - (π/2) δ(k+k')

and other forms obtained by doing x ↔ k and x'↔k'

Now let's go back to the classic Fourier idea

$\int_{-\infty}^{\infty}$ dk e^{-ikx} = 2π δ(x)

$\int_{-\infty}^{\infty}$ dx e^{-ikx} = 2π δ(k)

Can I relate these to my dual -delta expression above? The first could be written

$\int_{-\infty}^{\infty}$ dk $e^{-ik(x-x')}$ = 2π δ(x-x')

or

$\int_{-\infty}^{\infty}$ dk cos[k(x-x')] = 2π δ(x-x')

or

$\int_{-\infty}^{\infty}$ dk [cos(kx) cos(kx') + sin(kx) sin(kx')] = 2π δ(x-x')

Now using the rules given above,

$$\int_{-\infty}^{\infty} dk \cos(kx) \cos(kx') = \pi \delta(x-x') + \pi \delta(x+x')$$

$$\int_{-\infty}^{\infty} dk \sin(kx) \sin(kx') = \pi \delta(x-x') - \pi \delta(x+x')$$

we can add and the first term doubles and the second cancels, QED! So maybe I have resolved my mystery finally.

Here us another way to verify my claims:

$$\int_{-\infty}^{\infty} dx \cos(k'x) \cos(kx) = (1/4) \int_{-\infty}^{\infty} dx \left[e^{ik'x} + e^{-ik'x}\right] \left[e^{ikx} + e^{-ikx}\right]$$

$$= (1/4) \int_{-\infty}^{\infty} dx \left[e^{i(k'+k)x} + e^{i(k'-k)x} + e^{i(-k'+k)x} + e^{i(-k'-k)x} \right]$$

$$= (1/4)\, 2\pi \left[\delta(k'+k) + \delta(k'-k) + \delta(-k'+k) + \delta(-k'-k)\right]$$

$$= \pi \left[\delta(k'+k) + \delta(k'-k)\right]$$

and that was a lot easier than what I did above. Live and learn! Same idea for the sine result.