Confusion about Completeness Relations: A Paradox PhL 5.23.10

Well, this is just one of those things. You make a mistake early on, or some bad assumptions, or allow yourself to be misled, and you then tumble down a sad tunnel to disaster. You then have to find the problem, thread backwards out of the disaster, and make sure everything is OK again. A lot of things have to be screwed back down that came loose in the process.

Overview: I was simply updating the Fourier Cosine Transform section of my transforms.doc (and also another doc where I actually did the work), and I started noticing inconsistencies, factors of 2 wrong, completely illogical results. I began thinking that somehow the distributional equalities for delta functions are dependent on the choice of Hilbert Space (eg, the space of *even* L_2 functions), rather than being general mathematical statements. My disaster pathway and recovery was this:

1. I was "pretty darn sure" that

$$\int_0^\infty dk \cos(kx') \cos(kx) = (\pi/2)\, \delta(x' - x) \qquad \text{// wrong!!} \qquad (*)$$

$$\int_0^\infty dk \sin(kx') \sin(kx) = (\pi/2)\, \delta(x' - x) \qquad \text{// wrong!!}$$

I saw the first (*) *clearly stated* on page 293 of Stak (4.70) and I knew for 100% sure that

$$\int_0^\infty dk \, [\, \cos(kx') \cos(kx) + \sin(kx') \sin(kx) \,] = \pi\, \delta(x' - x) \qquad \text{// this is correct}$$

so given the first line above, I felt the second had to also be true. But then when I inserted projection into expansion and assumed f(x) happened to be even (the cosine case), I kept being off by a factor of 2, which I interpreted as saying that (*) "sometimes" has a different constant depending on "your interval" or "your problem" etc etc. I kept trying different examples, trying to get the simplest possible case that showed "the paradox" (hence all the examples below, which I have now fixed up). I was thinking maybe somehow "completeness" is specific to the interval, and so on (It is in a sense, but not in the sense I was thinking about.) In fact, I *thought* I carefully *derived* (*) "from first principles" so I was very sure it was true, especially with Stakgold's blatant p 293 confirmation. Notice the combination of making an error AND being misled, or reinforced, in that error. The *failure* of this derivation is now documented in Appendix B. I was so amazed about the bad factor of (1/2) that I did a specific transform of an even function (Appendix A), but that example (gaussian) did not show up the factor of (1/2) "error"! This bad situation persisted from maybe noon 5/23 to noon 5/24. I then happened to notice that the above δ relation has different symmetry on the two sides. The left side is even in x, but the right side is not even in x, so I finally realized "something is wrong here".

2. I did not know the resolution at this point, so I went off and tried to do a rigorous proof of (*) using distribution theory (see "an exercise in distribution theory.doc"**)**. I used a Gaussian test function φ(x) even though it is not quite kosher, and I attempted to prove the distributional equation (*) by integrating both sides against the test function. To my great interest, my proof failed, so the plot thickened. I then studied this distributional theory example more, and I finally realized that the correct results are these

$$\int_0^\infty dk \cos(kx') \cos(kx) = (\pi/2) [\delta(x'- x) + \delta(x'+ x)] \qquad \text{// this is correct}$$

$$\int_0^\infty dk \sin(kx') \sin(kx) = (\pi/2) [\delta(x'- x) - \delta(x'+ x)] \qquad \text{// this is correct}$$

These are the "general mathematic results" and are not dependent on any particular ODE problem you might be thinking about. These results have all the correct symmetry properties. Then, adding insult to injury, I realized you can derive the above equations in about 10 seconds by just expanding cos in exponentials everywhere. The Stakgold statement of (*) was technically correct because his problem was using x in $(0,\infty)$ so the second delta function cannot get "a hit" and can be omitted. [But he issued no comments on this.]

Once I included the second term in the cosine result above, all the paradoxes went away, and no mystery was left. The mysterious factor of (1/2) for even functions always gets resolved by that second delta function and you get $(1/2) \{ f(x) + f(-x) \} = f(x)$ because you assumed f(x) was even.

3. The reader should have no reason to look any further in this document apart from the numbered results stated in Section 1, and perhaps the "quick review" of transform theory in Section 3. I just keep all the rest around in case I get confused again some day and start wandering down this same path. One detail to note is that in this world we are dealing with a singular boundary value problem so the eigenfunctions and their eigenvalues don't strictly even exist. But we know how to handle this, and we know how to normalize even the singular eigenfunctions in a manner that makes the transform theory work perfectly.

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1. First, derive some "facts" from the exponential delta function statement.

Using only the fact that $\int_{-\infty}^{\infty} ds\, e^{-ist} = 2\pi \delta(t)$, we do the following derivation:

$$\int_{-\infty}^{\infty} dx \cos(k'x)\cos(kx) = (1/4) \int_{-\infty}^{\infty} dx \left[e^{ik'x} + e^{-ik'x}\right]\left[e^{ikx} + e^{-ikx}\right]$$

$$= (1/4) \int_{-\infty}^{\infty} dx \left[e^{i(k'+k)x} + e^{i(k'-k)x} + e^{i(-k'+k)x} + e^{i(-k'-k)x} \right]$$

$$= (1/4)\, 2\pi \left[\delta(k'+k) + \delta(k'-k) + \delta(-k'+k) + \delta(-k'-k)\right]$$

$$= \pi \left[\delta(k'-k) + \delta(k'+k)\right]$$

Thus we may say that

$$(1/\pi) \int_{-\infty}^{\infty} dx \cos(k'x)\cos(kx) = \delta(k'-k) + \delta(k'+k) \qquad (1)$$

$$(2/\pi) \int_{0}^{\infty} dx \cos(k'x)\cos(kx) = \delta(k'-k) + \delta(k'+k) \qquad (3)$$

Now repeat the above for two sines and edit the changes { $\sin = [e^{+} - e^{-}]/(2i)$ }

$$\int_{-\infty}^{\infty} dx \sin(k'x)\sin(kx) = (-1/4) \int_{-\infty}^{\infty} dx \left[e^{ik'x} - e^{-ik'x}\right]\left[e^{ikx} - e^{-ikx}\right]$$

$$= (-1/4) \int_{-\infty}^{\infty} dx \left[e^{i(k'+k)x} - e^{i(k'-k)x} - e^{i(-k'+k)x} + e^{i(-k'-k)x} \right]$$

$$= (-1/4)\, 2\pi \left[\delta(k'+k) - \delta(k'-k) - \delta(-k'+k) + \delta(-k'-k)\right]$$

$$= \pi \left[\delta(k'-k) - \delta(k'+k)\right]$$

Thus we may say that

$$(1/\pi) \int_{-\infty}^{\infty} dx \sin(k'x)\sin(kx) = \delta(k'-k) - \delta(k'+k) \qquad (2)$$

$$(2/\pi) \int_{0}^{\infty} dx \sin(k'x)\sin(kx) = \delta(k'-k) - \delta(k'+k) \qquad (4)$$

If we try $\int_{-\infty}^{\infty} dx \sin(k'x)\cos(kx)$, say, the integrand is odd in x and we just get 0.

So here is a summary of the above results expressed "both ways" :

$$(1/\pi) \int_{-\infty}^{\infty} dx \cos(k'x)\cos(kx) = \delta(k'-k) + \delta(k'+k) \qquad (1)$$

$$(1/\pi) \int_{-\infty}^{\infty} dx \sin(k'x)\sin(kx) = \delta(k'-k) - \delta(k'+k) \qquad (2)$$

$$(2/\pi) \int_0^\infty dx \cos(k'x) \cos(kx) = \delta(k'- k) + \delta(k'+ k) \quad (3)$$

$$(2/\pi) \int_0^\infty dx \sin(k'x) \sin(kx) = \delta(k'- k) - \delta(k'+ k) \quad (4)$$

$$(1/\pi) \int_{-\infty}^\infty dk \cos(kx') \cos(kx) = \delta(x'- x) + \delta(x'+ x) \quad (5)$$

$$(1/\pi) \int_{-\infty}^\infty dk \sin(kx') \sin(kx) = \delta(x'- x) - \delta(x'+ x) \quad (6)$$

$$(2/\pi) \int_0^\infty dk \cos(kx') \cos(kx) = \delta(x'- x) + \delta(x'+ x) \quad (7)$$

$$(2/\pi) \int_0^\infty dk \sin(kx') \sin(kx) = \delta(x'- x) - \delta(x'+ x) \quad (8)$$

If our entire universe consists of x in $(0,\infty)$, then we can drop the second delta function. You might worry about what happens in this case if x = 0 *and* x' = 0. We can see that LHS(8) = 0, so in this case we must keep both terms to get $\delta(0)-\delta(0) = 0$. And I guess we conclude RHS(7) = $2\,\delta(0)$ in this case.

We can add the following to the above list

$$\int_0^\infty dk \cos[k(x-x')] = \int_0^\infty dk \, [\, \cos(kx') \cos(kx) + \sin(kx') \sin(kx) \,] = \pi\, \delta(x'- x) \quad (9)$$

which is easy to verify from (7) and (8). This appears in Jackson 3.139 page 84.

2. Thinking about the Fourier Cosine Integral Transform

Let's start with the complex transform from above which was this

$$f(x) = \int_{-\infty}^\infty dk \, f^{\mathbf{k}}\, e^{\mathbf{-ikx}} \quad \text{// expansion}$$

$$f^{\mathbf{k}} = (1/2\pi) \int_{-\infty}^\infty dx \, f(x)\, e^{\mathbf{+ikx}} \quad \text{// projection}$$

$$\int_{-\infty}^\infty dx \, e^{\mathbf{i(k-k')x}} = \int_{-\infty}^\infty dx \cos[(k-k')x] = 2\pi\, \delta(k-k') \quad \text{// orthogonality}$$

$$\int_{-\infty}^\infty dk \, e^{\mathbf{-ik(x'-x)}} = \int_{-\infty}^\infty dk \cos[k(x-x')] = 2\pi\, \delta(x-x') \quad \text{// completeness}$$

(0) Let's first verify that the above is correct in our "direction" of interest:

If we insert the projection into the expansion, we get

$$f(x) = \int_{-\infty}^\infty dk \, f^{\mathbf{k}}\, e^{\mathbf{-ikx}} = \int_{-\infty}^\infty dk \, \{ (1/2\pi) \int_{-\infty}^\infty dx' \, f(x')\, e^{\mathbf{+ikx'}} \}\, e^{\mathbf{-ikx}}$$

$$= \int_{-\infty}^{\infty} dx' f(x')\{ (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{+ikx'} e^{-ikx} \}$$

from which we conclude that

$$\int_{-\infty}^{\infty} dk\, e^{+ikx'} e^{-ikx} = 2\pi\, \delta(x-x')$$

which agrees with our section 1 above. And of course the other direction is the same idea. The other direction is a statement of orthogonality for these non-normalizable functions.

(1) We first write out the expansion:

$$f(x) = \int_{-\infty}^{\infty} dk\, f^{k} \cos(kx) + i \int_{-\infty}^{\infty} dk\, f^{k} \sin(kx)$$

If f(x) is even in x, the second term must vanish since it is odd in x. Thus, for even f(x),

$$f(x) = \int_{-\infty}^{\infty} dk\, f^{k} \cos(kx)$$

(2) We next write out the projection:

$$f^{k} = (1/2\pi) \int_{-\infty}^{\infty} dx\, f(x) \cos(kx) + (i/2\pi) \int_{-\infty}^{\infty} dx\, f(x) \sin(kx)$$

If f(x) is even in x, the second term must vanish since its *integrand* is odd in x. Thus, for even f(x),

$$f^{k} = (1/2\pi) \int_{-\infty}^{\infty} dx\, f(x) \cos(kx)$$

and this tells us that f^{k} is even in k. At this point, we have said nothing about things being *real*. If even f(x) happens to be real, then f^{k} is also real, but let's not make this restriction!

(3) At this point, merely assuming that f(x) is even, we have shown that our Full FT can simply be rewritten in this manner,

$$f(x) = \int_{-\infty}^{\infty} dk\, f^{k} \cos(kx)$$

$$f^{k} = (1/2\pi) \int_{-\infty}^{\infty} dx\, f(x) \cos(kx)$$

If we insert the projection into the expansion, we get

$$f(x) = \int_{-\infty}^{\infty} dk\, f^{k} \cos(kx) = \int_{-\infty}^{\infty} dk\, \{(1/2\pi) \int_{-\infty}^{\infty} dx' f(x') \cos(kx')\} \cos(kx)$$

$$= (1/2) \int_{-\infty}^{\infty} dx' f(x') \{ (1/\pi) \int_{-\infty}^{\infty} dk \cos(kx') \cos(kx) \}$$

$$= (1/2) \int_{-\infty}^{\infty} dx' f(x') \{ \delta(x'- x) + \delta(x'+ x) \} \qquad \text{// Section 1 result (5)}$$

$$= (1/2) [f(x) + f(-x)]$$

$$= f(x) \qquad \text{// using the fact that f(x) was assumed to be even.}$$

When I first did this, I didn't know about the second δ term, so I was getting a "paradox" f(x) = (1/2) f(x), but that paradox has now gone away!

3. Quick Review of Transform Theory.

What exactly is the *meaning* of one of these "transforms"? As Stakgold explains (p 268 circa), you start with a differential operator L, throw in some homo boundary conditions, then solve $L\varphi_\lambda = s\lambda\varphi_\lambda$ and determine the spectrum for λ and the eigenfunctions φ_λ. You know from general theory that eigenfunctions corresponding to different eigenvalues will be orthogonal (perhaps with s(x) weighting function). You solve (L-sλ)g = δ for the Green's function g(x|ξ λ) and you then can show that the completeness delta function is given by

$$-(1/2\pi i) \oint d\lambda\, g(x|\xi; \lambda) = \delta(x-\xi)/s(x) = \Sigma_\lambda \varphi_\lambda(x) \bar{\varphi}_\lambda(\xi) \qquad \text{// completeness}$$

$$\int dx\, s(x)\varphi_{\lambda'}(x)^* \varphi_\lambda(x) = \delta_{\lambda\lambda'} \qquad \text{// orthonormality}$$

where the φ_n must be to be normalized. *Now* you can talk about a transform. Try this

$$f(x) = \Sigma_\lambda f^\lambda \varphi_\lambda(x) \qquad \text{// expansion}$$

To "get" the transform, you need to *solve* this expansion for the projection f^λ. To do that you apply the operator $\int dx\, s(x)\, \varphi_{\lambda'}(x)^*$ to both sides (integral is over your "interval"),

$$\int dx\, s(x)\varphi_{\lambda'}(x)^* f(x) = \int dx\, s(x)\varphi_{\lambda'}(x) \{\Sigma_\lambda f^\lambda \varphi_\lambda(x) \}$$

$$= \Sigma_\lambda f^\lambda \{\int dx\, s(x)\varphi_{\lambda'}(x)^* \varphi_\lambda(x)\} = \Sigma_\lambda f^\lambda \delta_{\lambda\lambda'} = f^{\lambda'}$$

so only (weighted) orthogonality was needed to find the projection. Now we have a "transform"

$$f(x) = \Sigma_\lambda f^\lambda \varphi_\lambda(x) \qquad \text{// expansion}$$

$f^{\lambda} = \int dx\, s(x)\, \varphi_{\lambda}(x)^*\, f(x)$ // projection

I have been deliberately vague about the meaning of Σ_{λ}. It means the sum over the spectrum. Each eigenfunction of the complete basis is supposed to be included once (not two or three times....).

So we were able to obtain our "transform" without ever considering the "completeness relation". However, we verify that now by inserting the projection into the expansion

$$f(x) = \Sigma_{\lambda}\, f^{\lambda}\, \varphi_{\lambda}(x) = \Sigma_{\lambda}\, \{ \int dx'\, s(x')\, \varphi_{\lambda}(x')^*\, f(x') \}\, \varphi_{\lambda}(x) = \int dx'\, f(x')\, s(x') \{ \Sigma_{\lambda}\, \varphi_{\lambda}(x')^*\, \varphi_{\lambda}(x) \}$$

$$= \int dx'\, f(x')\, \delta(x-x') = f(x)$$

Now go back to the development of Section 2 above. We start with the full FT, but then we restrict our domain to functions which are even in x. We end up with this "translation" or "rewrite" of the full FT:

$$f(x) = \int_{-\infty}^{\infty} dk\, f^{k}\, \cos(kx)$$

$$f^{k} = (1/2\pi) \int_{-\infty}^{\infty} dx\, f(x) \cos(kx)$$

What is the corresponding EV problem here? Full x interval. No weight function. Stakgold approaches this issue on page 293. Oops. There are no "true" eigenvalues of the EV equation because EF's are not normalizable on our infinite interval. But the Green's problem does have a solution which he writes down, and we can then apply our rule $-(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) = \delta(x-\xi)/s(x)$ and this he claims gives the same result as I show in Section 1 above [where we drop the second δ since working on $(0,\infty)$ only]. So Stak gets this thing and the resulting transform by actually computing g, he does not do it as a specialization of the full complex FT.

Commented added: In the cosine problem on $(0,\infty)$, you do get $\Sigma_{\lambda}\, \varphi_{\lambda}(x)\, \bar{\varphi}_{\lambda}(x') = \delta(x-x')$ as noted above. But when you "extend" this problem to the full x interval $(-\infty,\infty)$, you pick up that second δ term which seems to conflict with our "theory" which, as you see, only shows $\delta(x-x')$. In this extended problem, our BC is $u'(0) = 0$ so is not really at an endpoint, so I think we have to say this does not really fit into the formalism. We are making it fit by doing this little extension. This could surely be put on a solid basis, but I don't see the need to do it since everything seems OK now.

4. Write out the full FT in real and imaginary parts:

$f(x) = \int_{-\infty}^{\infty} dk\, f^{k}\, e^{-ikx}$ // expansion

$f^{k} = (1/2\pi) \int_{-\infty}^{\infty} dx\, f(x)\, e^{+ikx}$ // projection

$f(x) = \int_{-\infty}^{\infty} dk\, f^{k}\, [\cos(kx) - i \sin(kx)]$ // expansion

$$f^{k} = (1/2\pi) \int_{-\infty}^{\infty} dx\, f(x) [\cos(kx) + i \sin(kx)] \qquad \text{// projection}$$

What happens when we put the projection into the expansion?

$$f(x) = \int_{-\infty}^{\infty} dk\, f^{k} [\cos(kx) - i \sin(kx)]$$

$$= \int_{-\infty}^{\infty} dk \{(1/2\pi) \int_{-\infty}^{\infty} dx' f(x') [\cos(kx') + i \sin(kx')]\} [\cos(kx) - i \sin(kx)]$$

$$= \int_{-\infty}^{\infty} dx' f(x') \{ (1/2\pi) \int_{-\infty}^{\infty} dk [\cos(kx') + i \sin(kx')] [\cos(kx) - i \sin(kx)]$$

$$= \int_{-\infty}^{\infty} dx' f(x') \{ (1/2\pi) \int_{-\infty}^{\infty} dk [\cos(kx') \cos(kx) + \sin(kx') \sin(kx)] \}$$

We threw out integrals with odd integrands. Now, there are two terms remaining, both are even, the cos cos and the sin sin. We know from Section 1 that these add up to $2\pi\, \delta(x-x')$ and then we get the right answer.

Now look what happens when we apply this to an even function:

$$f(x) = \int_{-\infty}^{\infty} dk\, f^{k} \cos(kx) \quad + \text{vanishing term} \qquad \text{// expansion}$$

$$f^{k} = (1/2\pi) \int_{-\infty}^{\infty} dx\, f(x) \cos(kx) \quad + \text{vanishing term} \qquad \text{// projection}$$

For even functions, the second terms are not there, so we don't get that sin(kx') sin(kx) contribution. In this case we get just

$$\int_{-\infty}^{\infty} dx' f(x') \{ (1/2\pi) \int_{-\infty}^{\infty} dk [\cos(kx') \cos(kx) \}$$

$$= (1/2) \int_{-\infty}^{\infty} dx' f(x') \{ (1/\pi) \int_{-\infty}^{\infty} dk [\cos(kx') \cos(kx) \}$$

$$= (1/2) \int_{-\infty}^{\infty} dx' f(x') \{ [\delta(x' - x) + \delta(x' + x)] \} \qquad \text{// Section 1 (5)}$$

$$= (1/2) \{ f(x) + f(-x) \} = f(x) \qquad \text{// since f(x) assumed even}$$

and all is well. When I did not know about the presence of the $\delta(x'+ x)$ term, things were not "well" and I kept getting $f(x) = (1/2) f(x)$ and this was the big "paradox".

5. Stak with full intervals. Let's look at Stak p 293 yet again and just convert his results to *full* intervals. So here is then his transform (4.71) -- I am doing *nothing more than rewriting it* --

$$f(x) = (2/\pi)^{1/2}\,(1/2)\ \int_{-\infty}^{\infty} dk\ f^{k}\cos(kx) \ = \ (1/2\pi)^{1/2}\ \int_{-\infty}^{\infty} dk\ f^{k}\cos(kx)$$

$$f^{k} = (2/\pi)^{1/2}\,(1/2)\ \int_{-\infty}^{\infty} dx\ f(x)\cos(kx) \ = \ (1/2\pi)^{1/2}\int_{-\infty}^{\infty} dx\ f(x)\cos(kx)$$

Now, what happens if we put the projection into the expansion:

$$f(x) = (1/2\pi)^{1/2}\ \int_{-\infty}^{\infty} dk\ f^{k}\cos(kx)$$

$$= (1/2\pi)^{1/2}\ \int_{-\infty}^{\infty} dk\ \{(1/2\pi)^{1/2}\int_{-\infty}^{\infty} dx'\ f(x')\cos(kx')\}\cos(kx)$$

$$= \int_{-\infty}^{\infty} dx'\ f(x')\ \{\ (1/2\pi)\ \int_{-\infty}^{\infty} dk\ \cos(kx')\}\cos(kx)\ \}$$

$$= (1/2)\ \int_{-\infty}^{\infty} dx'\ f(x')\ \{\ (1/\pi)\ \int_{-\infty}^{\infty} dk\ \cos(kx')\}\cos(kx)\ \}$$

$$= (1/2)\ \int_{-\infty}^{\infty} dx'\ f(x')\ \{\ \delta(x'- x) + \delta(x'+ x)\ \}$$ // Section 1 (5)

$$= (1/2)\ \{f(x) + f(-x)\} \ = f(x)$$ // all is well

6. Stak with half intervals. OK, let's do it again with things exactly as he has them stated.

$$f(x) = (2/\pi)^{1/2}\ \int_{0}^{\infty} dk\ f^{k}\ \cos(kx)$$

$$f^{k} \ = (2/\pi)^{1/2}\ \int_{0}^{\infty} dx\ f(x)\cos(kx)$$

Now put the projection into the expansion:

$$f(x) = (2/\pi)^{1/2}\ \int_{0}^{\infty} dk\ f^{k}\cos(kx)$$

$$= (2/\pi)^{1/2}\ \int_{0}^{\infty} dk\{(2/\pi)^{1/2}\ \int_{0}^{\infty} dx'\ f(x')\cos(kx')\}\cos(kx)$$

$$= \int_{0}^{\infty} dx'\ f(x')\ \ [\ (2/\pi)\ \int_{0}^{\infty} dk\{\cos(kx')\}\cos(kx)\ ]$$

$= \int_0^\infty dx' \, f(x') \;\; [\, \delta(x' - x) + \delta(x' + x) \,]$ // Section 1 result (7)

$= f(x)$ // since don't get a hit on the second delta function since $x > 0$

Let's go back to our "theory" which is this (set $s(x) = 1$ to keep things clearer)

$\Sigma_{\boldsymbol{\lambda}} \, \varphi_{\boldsymbol{\lambda}}(x) \, \overline{\varphi}_{\boldsymbol{\lambda}}(\xi) = \delta(x-\xi)$ // completeness

$\int dx \, \varphi_{\boldsymbol{\lambda}'}(x) \, \overline{\varphi}_{\boldsymbol{\lambda}}(x) = \delta_{\boldsymbol{\lambda}'\boldsymbol{\lambda}}$ // orthogonality

If λ is a continuous spectrum, we would I think write these as

$\int d\lambda \, \varphi_{\boldsymbol{\lambda}}(x) \, \overline{\varphi}_{\boldsymbol{\lambda}}(\xi) = \delta(x-\xi)$ // completeness

$\int dx \, \varphi_{\boldsymbol{\lambda}'}(x) \, \overline{\varphi}_{\boldsymbol{\lambda}}(x) = \delta(\lambda-\lambda')$ // orthogonality

where $d\lambda$ integrates over the continuous spectrum, and dx integrates over the interval. The $\varphi_{\boldsymbol{\lambda}}(x)$ are the eigenfunctions of the Laplace equation in 1D with some unspecified BC's, and λ is in the spectrum. The transform is then going to be

$$f(x) = \int d\lambda \, f^{\boldsymbol{\lambda}} \, \varphi_{\boldsymbol{\lambda}}(x)$$

$$f^{\boldsymbol{\lambda}} = \int dx \, f(x) \, \overline{\varphi}_{\boldsymbol{\lambda}}(x)$$

If you "put either into the other" you will get an identity according to the completeness and orthogonality relations shown above.

Example 1: Consider $(0,\infty)$ as the interval for x, and I guess $u'(0) = 0$ as the boundary condition. We know that the eigenfunctions have the form $\sim \cos(kx)$ where $k = \sqrt{\lambda}$.

First question: what is "the measure" going to be in the $d\lambda$ integral? Note that $d\lambda \neq dk$. I will just assume for the moment that the correct measure is dk. Then we have

$\int dk \, \varphi_{\boldsymbol{k}}(x) \, \overline{\varphi}_{\boldsymbol{k}}(\xi) = \delta(x-\xi)$ // completeness

$\int_0^\infty \varphi_{\boldsymbol{k}'}(x) \, \overline{\varphi}_{\boldsymbol{k}}(x) = \delta(k-k')$ // orthogonality

I can see that the spectrum for k (without duplication) would be the positive real k axis. If I include the negative axis, I am including the same eigenfunction twice and that must be wrong. So I will refine the above and say this:

$$\int_0^\infty dk\ \varphi_k(x)\ \bar{\varphi}_k(\xi) = \delta(x-\xi) \qquad \text{// completeness}$$

$$\int_0^\infty dx\ \varphi_{k'}(x)\ \bar{\varphi}_k(x) = \delta(k-k') \qquad \text{// orthogonality}$$

From Section 1 (3) we may write

$$\int_0^\infty dx\ \cos(kx)\cos(k'x) = (\pi/2)\ \delta(k-k')$$

where we omit the $\delta(k'+ k)$ term since we are only using $k > 0$. It then follows that the "correctly normalized" eigenfunctions $\varphi_k(x)$ are these

$$\varphi_k(x) = (2/\pi)^{1/2}\ \cos(kx)$$

$$\Rightarrow \int_0^\infty dx\ \varphi_{k'}(x)\ \bar{\varphi}_k(x) = \delta(k-k')$$

We can then write the correct transform for this problem as follows

$$f(x) = \int_0^\infty dk\ f^k\ \varphi_k(x) = (2/\pi)^{1/2} \int_0^\infty dk\ f^k \cos(kx)$$

$$f^k = \int_0^\infty dx\ f(x)\ \bar{\varphi}_k(x) = (2/\pi)^{1/2} \int_0^\infty dx\ f(x) \cos(kx)$$

and this is exactly how it appears on Stak p 293. At this point, I could "move the constant" so it is all in the projection by writing:

$$g(x) = \int_0^\infty dk\ g^k \cos(kx)$$

$$g^k = (2/\pi) \int_0^\infty dx\ g(x) \cos(kx)$$

Example 2: Consider $(-\infty,\infty)$ as the interval for x, and I guess $u'(0) = 0$ as the boundary condition. We know that the eigenfunctions have the form $\sim \cos(kx)$ where $k = \sqrt{\lambda}$.

First question: what is "the measure" going to be in the $d\lambda$ integral? Note that $d\lambda \neq dk$. I will just assume for the moment that the correct measure is dk. Then we have

$$\int dk\ \varphi_k(x)\ \bar{\varphi}_k(\xi) = \delta(x-\xi) \qquad \text{// completeness}$$

$$\int_{-\infty}^\infty \varphi_{k'}(x)\ \bar{\varphi}_k(x) = \delta(k-k') \qquad \text{// orthogonality}$$

I can see that the spectrum for k (without duplication) would be the positive real k axis. If I include the negative axis, I am including the same eigenfunction twice and that must be wrong. So I will refine the above and say this:

$$\int_0^\infty dk\ \varphi_k(x)\ \bar{\varphi}_k(\xi) = \delta(x-\xi) \qquad \text{// completeness}$$

$$\int_{-\infty}^\infty dx\ \varphi_{k'}(x)\ \bar{\varphi}_k(x) = \delta(k-k') \qquad \text{// orthogonality}$$

The corresponding transform should be this:

$$f(x) = \int_0^\infty dk\ f^k\ \varphi_k(x)$$

$$f^k = \int_{-\infty}^\infty dx\ f(x)\ \bar{\varphi}_k(x)$$

Let's just verify to make sure we are still error-free at this point:

$$f(x) = \int_0^\infty dk\ f^k\ \varphi_k(x) = \int_0^\infty dk\ \{ \int_{-\infty}^\infty dx'\ f(x')\ \bar{\varphi}_k(x')\}\ \varphi_k(x)$$

$$= \int_{-\infty}^\infty dx'\ f(x')\ \{ \int_0^\infty dk\ \bar{\varphi}_k(x')\ \varphi_k(x)\ \} = \int_{-\infty}^\infty dx'\ f(x')\ \delta(x-x') = f(x)$$

so yes, we are still OK.

So now let's find our normalized eigenfunctions $\varphi_k(x)$. I will assume that the following Section 1 fact derived above is a mathematical fact that is independent of the theory of ODE's and all that stuff. It is something I could prove rigorously in distribution theory by taking finite models for delta functions. That fact is this:

$$\int_{-\infty}^\infty dx\ \cos(kx)\cos(k'x) = \pi\ \delta(k-k') \qquad \text{// Section 1 result (1) restricting to } k,k' > 0$$

It then follows that the "correctly normalized" eigenfunctions $\varphi_k(x)$ are these

$$\varphi_k(x) = (1/\pi)^{1/2}\ \cos(kx)$$

$$\Rightarrow \int_{-\infty}^\infty dx\ \varphi_{k'}(x)\ \bar{\varphi}_k(x) = \delta(k-k')$$

We can then write the correct transform for this problem as follows

$$f(x) = \int_0^\infty dk\ f^k\ \varphi_k(x) = (1/\pi)^{1/2} \int_0^\infty dk\ f^k\ \cos(kx)$$

$$f^k = \int_{-\infty}^\infty dx\ f(x)\ \bar{\varphi}_k(x) = (1/\pi)^{1/2} \int_{-\infty}^\infty dx\ f(x)\ \cos(kx)$$

and this form does *not* appear anywhere in Stak. If we now put the second into the first we get

$$f(x) = (1/\pi)^{1/2} \int_0^\infty dk\, f^k \cos(kx)$$

$$= (1/\pi)^{1/2} \int_0^\infty dk \{ (1/\pi)^{1/2} \int_{-\infty}^\infty dx' f(x') \cos(kx')\} \cos(kx)$$

$$= (1/2) \int_{-\infty}^\infty dx' f(x') \{ (2/\pi) \int_0^\infty dk \{ \cos(kx') \cos(kx) \}$$

$$= (1/2) \int_{-\infty}^\infty dx' f(x') \{ \delta(x' - x) + \delta(x' + x) \}$$

$$= (1/2) \{ f(x) + f(-x)\}$$

$$= f(x)$$ // since f(x) assumed even (see the BC), and all is well

Example 3. Our example 2 which came out fine was this:

$$f(x) = \int_0^\infty dk\, f^k \varphi_k(x) = (1/\pi)^{1/2} \int_0^\infty dk\, f^k \cos(kx)$$

$$f^k = \int_{-\infty}^\infty dx\, f(x)\, \bar{\varphi}_k(x) = (1/\pi)^{1/2} \int_{-\infty}^\infty dx\, f(x) \cos(kx)$$

We could "move the constant" to get this alternate form

$$g(x) = \int_0^\infty dk\, g^k \cos(kx)$$

$$g^k = (1/\pi) \int_{-\infty}^\infty dx\, g(x) \cos(kx)$$

Now, I can see from the second line that g^k is even in k if I choose to interpret it as existing for negative k (that is, I am *extending* the meaning of g^k). Then the integrand in the first line is even in k, and we could then write the first transform above in this way:

$$g(x) = (1/2) \int_{-\infty}^\infty dk\, g^k \cos(kx)$$

$$g^k = (1/\pi) \int_{-\infty}^\infty dx\, g(x) \cos(kx)$$

and I could then again "move the constant" and obtain

$$h(x) = \int_{-\infty}^\infty dk\, h^k \cos(kx)$$

$$h^k = (1/2\pi) \int_{-\infty}^\infty dx\, h(x) \cos(kx)$$

Now let's insert the second into the first and see what happens:

$$h(x) = \int_{-\infty}^{\infty} dk \{(1/2\pi) \int_{-\infty}^{\infty} dx' h(x') \cos(kx')\} \cos(kx)$$

$$= (1/2) \int_{-\infty}^{\infty} dx' h(x') \{ (1/\pi) \int_{-\infty}^{\infty} dk \cos(kx') \cos(kx) \}$$

$$= (1/2) \int_{-\infty}^{\infty} dx' h(x') \{ \delta(x-x') + \delta(x+x') \}$$

$$= h(x) \qquad \text{// as usual, and all is well}$$

Conclusion: all paradoxes have now gone away!

Appendix A: Notes from my failed attempt to show that $\int_{-\infty}^{\infty} dk \sin(kx)\sin(kx') = \pi \delta(x-x')$.

When I first did this, I thought I had a proof of this fact, but now that I know that the result is *wrong*, I went back and found the error in my proof. So here is the logic of that proof with the error corrected and we see how the proof leads to the conclusion that the above is correct only for $x \neq -x'$. If we are working with x only in $(0,\infty)$, then the quoted fact can be regarded as true, but not for x in $(-\infty,\infty)$.

Let's now look at the full transform completeness statement

$$\int_{-\infty}^{\infty} dk\, e^{-ik(x'-x)} = 2\pi \delta(x-x') \qquad \text{// completeness}$$

We can write this out the LHS as follows:

$$\int_{-\infty}^{\infty} dk\, e^{-ik(x'-x)} = \int_{-\infty}^{\infty} dk \{\cos[k(x-x')] - i \sin[k(x-x')] \} = 2\pi \delta(x-x')$$

But we know that $\int_{-\infty}^{\infty} dk \sin[k(x-x')] = 0$ because the integrand is an odd function of k. Thus, we arrive at these fairly obvious facts

$$\int_{-\infty}^{\infty} dk \cos[k(x-x')] = 2\pi \delta(x-x')$$

$$\int_{-\infty}^{\infty} dk \sin[k(x-x')] = 0$$

We can write

$\cos[k(x-x')] = \cos(kx)\cos(kx') + \sin(kx)\sin(kx')$ // Schaum p 15

$\sin[k(x-x')] = \sin(kx)\cos(kx') - \cos(kx)\sin(kx')$ // Schaum p 14

Inserting these into the above we get

$$\int_{-\infty}^{\infty} dk \{ \cos(kx)\cos(kx') + \sin(kx)\sin(kx')\} = 2\pi\,\delta(x-x') \quad \text{// this is true!}$$

$$\int_{-\infty}^{\infty} dk \{ \sin(kx)\cos(kx') - \cos(kx)\sin(kx')\} = 0 \quad \text{// and so is this.}$$

Each term in the second equation is 0 since the integrand is odd. Neither term in the first line vanishes due to oddness of the integrand in k.

Now, is there some way we can show that the two terms in the first line are the same? Try this:

$$\int_{-\infty}^{\infty} dk \cos(kx)\cos(kx') = \int_{-\infty}^{\infty} dk \sin(kx+\pi/2)\sin(kx' + \pi/2)$$

We are free to change variables to $k' = k + a$ for any fixed a, so write this as

$$= \int_{-\infty}^{\infty} dk' \sin([k'-a]x+\pi/2)\sin([k'-a]x' + \pi/2)$$

and now remove the prime

$$= \int_{-\infty}^{\infty} dk \sin([k-a]x+\pi/2)\sin([k-a]x' + \pi/2)$$

$$= \int_{-\infty}^{\infty} dk \sin(kx + [\pi/2-ax]) \sin(kx' + [\pi/2-ax'])$$

If we now choose a such that $ax = \pi/2$ this becomes [so from now on, $a = (\pi/2)/x$]

$$= \int_{-\infty}^{\infty} dk \sin(kx) \sin(kx' + [ax-ax'])$$

Now expand

$\sin(kx' + [ax-ax']) = \sin(kx') \cos[ax-ax'] + \cos(kx') \sin[ax-ax']$

Our expression above then becomes

$$= \int_{-\infty}^{\infty} dk \sin(kx) \{ \sin(kx') \cos[ax-ax'] + \cos(kx') \sin[ax-ax'] \}$$

$$= \cos[ax-ax'] \int_{-\infty}^{\infty} dk \sin(kx) \sin(kx') + \sin[ax-ax'] \int_{-\infty}^{\infty} dk \sin(kx) \cos(kx')$$

In the second term the integrand is odd so it vanishes. The first term then gives

$$= \cos[ax-ax'] \int_{-\infty}^{\infty} dk \sin(kx) \sin(kx')$$

But a is not arbitrary it is equal to a = (π/2)/x so this result is then

$$= \cos[x(\pi/2)/x - x'(\pi/2)/x] \int_{-\infty}^{\infty} dk \sin(kx) \sin(kx')$$

$$= \cos[(\pi/2) - (\pi/2)x'/x] \int_{-\infty}^{\infty} dk \sin(kx) \sin(kx')$$

$$= \sin[(\pi/2)(x'/x)] \int_{-\infty}^{\infty} dk \sin(kx) \sin(kx')$$

So we have now proven this fact:

$$\int_{-\infty}^{\infty} dk \cos(kx)\cos(kx') = \sin[(\pi/2)(x'/x)] \int_{-\infty}^{\infty} dk \sin(kx) \sin(kx')$$

And we have this starting fact from above:

$$\int_{-\infty}^{\infty} dk \cos(kx)\cos(kx') + \int_{-\infty}^{\infty} dk \sin(kx)\sin(kx')\} = 2\pi \delta(x-x')$$

Inserting the previous line we get

$$\{ \sin[(\pi/2)(x'/x)] + 1\} \int_{-\infty}^{\infty} dk \sin(kx)\sin(kx') = 2\pi \delta(x-x')$$

Now, as long as we don't have x = -x', we can divide both sides by {} to get

$$\int_{-\infty}^{\infty} dk \sin(kx)\sin(kx') = 2\pi \delta(x-x') / \{\sin[(\pi/2)(x'/x)] + 1\} = 2\pi \delta(x-x') / \{1 + 1\} = \pi \delta(x-x')$$

If x = -x', then {} = 0 and we conclude that $\int_{-\infty}^{\infty} dk \sin(kx)\sin(kx') = 0/0$ so we have no conclusion in this case. There could be an extra δ(x+x') term sitting on the right and we would never know it.

Appendix B: An example of transform of an even function f(x)

Let's take f(x) = exp(-q²x²) as an example. The integral in question is this: (GR7 p 488)

2. $$\int_{-\infty}^{\infty} e^{-q^2x^2}\cos[p(x+\lambda)]\,dx = \frac{\sqrt{\pi}}{q}e^{-\frac{p^2}{4q^2}}\cos p\lambda$$

Thus we get, setting p = k and λ = 0

$f^k = (1/2\pi) \int_{-\infty}^{\infty} dx\, \exp(-q^2x^2)\cos(kx) = (1/2\pi)(\sqrt{\pi}/q)\exp(-k^2/4q^2)$

Fine, now put this back into the expansion:

$f(x) = \int_{-\infty}^{\infty} dk\, f^k \cos(kx)$

$= \int_{-\infty}^{\infty} dk\{ (1/2\pi)(\sqrt{\pi}/q)\exp(-k^2/4q^2)\} \cos(kx)$

$= (1/2\pi)(\sqrt{\pi}/q) \int_{-\infty}^{\infty} dk\, \exp(-k^2/4q^2)\} \cos(kx)$ $\qquad Q = (2q)^{-1}$

$= (1/2\pi)(\sqrt{\pi}/q) \{(\sqrt{\pi}/Q)\exp(-x^2/4Q^2)$ $\qquad 1/Q = 2q,\ \ 1/(4Q^2) = 4q^2/4$

$= (1/2\pi)(\sqrt{\pi}/q) \{(2q\sqrt{\pi})\exp(-x^2/4Q^2)$

$= \exp(-x^2q^2) = f(x)$ $\qquad$ QED and all is well

The following sections were added on 4.22.11, well after this doc was written.

7. Does the Fourier Transform apply to piecewise continuous functions?

I think the answer is yes, but with a caveat mentioned below. Generically, we know we can fit a step or triangle function, say, using a Fourier Transform. The box gives the sinc, and the box is "piecewise continuous". This works because the basis functions form a complete set in some sense. I am looking now on the web. One guys says this, where he opens by saying f(x) and f'(x) are both piecewise continuous. Notice that he indicates L_1 convergence and not L_2. Notice also that the LHS of his theorem allows as there might be a discontinuity at x ! So this theorem has a lot to say about our questions of today.

Theorem 8 (The Fourier integral)

1. If $f(x)$ and $f'(x)$ are piecewise continuous in every finite interval
2. and $\int_{-\infty}^{\infty} |f(x)|dx$ converges, i.e $f(x)$ is absolutely integrable in $(-\infty, \infty)$

Then

$$\frac{1}{2}[f(x-) + f(x+)] = \frac{1}{2\pi} \lim_{\Omega \to \infty} \int_{-\Omega}^{\Omega} e^{i\omega x} \left(\int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt \right) d\omega \qquad (12)$$

Another author makes a similar statement for Fourier series and has a proof

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nx + \sum_{n=1}^{N} b_n \sin nx$$

Question: What is $\lim_{N \to \infty} S_N(x)$?

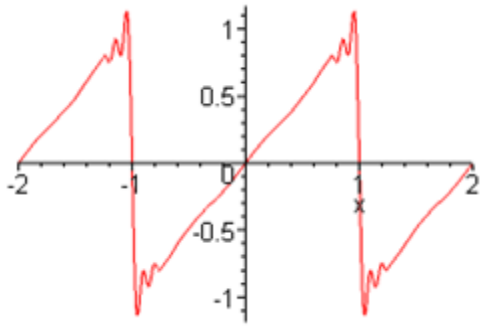
Answer: $S_N(x) \to \frac{1}{2}[f(x^+) + f(x^-)]$ as $N \to \infty$.

And another author says

"Complex exponentials (or sines and cosines) are periodic functions, and the set of complex exponentials is complete and orthogonal. Thus the Fourier transform can represent **any piecewise continuous function** and minimizes the least-square error between the function and its representation."

Here then is that caveat. If you compute a Fourier projection and then insert it back into the expansion, you are supposed to recreate the original function from which you got the projections. BUT, even an full (infinite number of terms) Fourier Series or a Fourier Integral expansion cannot be discontinuous. So, if your starting function had a (finite) discontinuity, your recovered function will have a nearly vertical interpolating line such that *right at* the point of discontinuity, your expansion will create a value that is exactly the average of the function value just to either side of the discontinuity.

A standard example with Fourier Series is reconstructing a sawtooth. If you take a finite but reasonable number of terms, your reconstruction looks like this (from Maple doc)



Here you see those slightly tipped vertical lines which cross in this case right at the value x = 0, so in the reconstruction here (which happens to be from a Laplace transform), you get f(1) = (1/2)[1 + (-1)] = 0. I think the same idea happens in all "Fourier related" transforms and probably others as well.

I don't know what this theorem is called, and I have decided not to track it down right now, but to accept the result.