Jim,

Summary: This doc (but without Section (3), and without this summary) was emailed to Jim on 1.8.11. I had referred him to a 2009 PDF paper which did numerical work on the Love equation and I thought maybe Jim could improve their methods using his quadrature stuff.

[This subject is completely separate from my Lk=δ work with Jim, although the Love equation is one you would like to convert to an ODE with some L of this type. I now know this is not possible using a constant-coefficient L, and also not possible using a 2nd order form with function coefficients.]

Jim had suggested by phone that maybe the Love equation could be diagonalized since it had convolution form. In this doc I showed why that cannot be done. The reason is that the kernel, though of convolution form, is not periodic on the interval which is the Love equation region of integration.

In Section 3 added later I go on to show how this fact does not stop you from doing a partial wave analysis of the equation which converts it to a (non-diagonal) matrix equation. This is what the PDF paper was above, and various Si and Ci functions appear:

$$(1)\quad Si(x)=\int_0^x \frac{\sin(t)}{t}dt$$

$$(2)\quad Ci(x)=\int_0^x \frac{\cos(t)-1}{t}dt+ln(x)+\gamma$$

---

(1) The Love Equation has this form (and it happens that u(x) is known to be even)

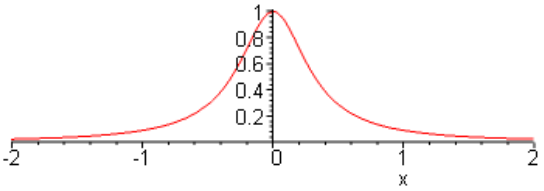
$$u(x) - \int_{-1}^{1} dt\, u(t)\, (\kappa/\pi) / [\, \kappa^2 + (x-t)^2] = 1 \qquad\qquad x \text{ in } (-1,1)$$

where the kernel is

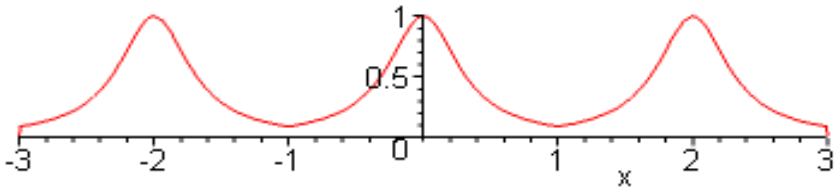
$$k(x,t) = g(x-t) = (\kappa/\pi) / [\, \kappa^2 + (x-t)^2] \qquad \text{so} \qquad g(x) = (\kappa/\pi) / [\, \kappa^2 + x^2]$$

and here is a plot of g(x) for $\kappa = 1/\pi$:

Figure 1



For example, in the integration if we have a point t = - 3/4 and we select x = + 3/4, then we are evaluating the function g(x) at argument 1.5 which is out on the right side tail. This g(x) not a periodic function! We could *consider* the following function which *is* periodic,



but this is *not* the function which appears in the Love equation ! For example, the point in the tail just mentioned at x = 1.5 would have the wrong value.

(2) We can convert the Love equation to a familiar form using $t = \theta/\pi$ and redefine k and g,

$$U(\theta) - \int_{-\pi}^{\pi} d\theta' \, U(\theta') \, \kappa / [(\kappa\pi)^2 + (\theta-\theta')^2] = 1 \qquad \theta \text{ in } (-\pi,\pi)$$

$$k(\theta,\theta') = g(\theta-\theta') = \kappa / [(\kappa\pi)^2 + (\theta-\theta')^2] \qquad \text{so} \qquad g(\theta) = \kappa / [(\kappa\pi)^2 + \theta^2]$$

and we can then write (as you suggested, where $U(\theta) = u(x(\theta))$)

$$U(\theta) - 1 = \int_{-\pi}^{\pi} d\theta' \, U(\theta') \, g(\theta-\theta')$$

If $g(\theta)$ were a periodic function, meaning $g(\theta) = g(\theta+2\pi)$, then this would be a convolution equation on the parameter space of the group SO(2) which could be diagonalized into the form

$$U_n - \delta_{n,0} = U_n \, g_n$$

which has solution (assuming $g_n \neq 0$)

$$U_n = \delta_{n,0} / (1-g_0) \qquad \Rightarrow \qquad U(\theta) = U_0 = 1/(g_0-1) \quad = \text{a constant !}$$

using the complex Fourier series transform (sum n is over all integers)

$$U(\theta) = \Sigma_n \, U_n \, e^{-in\theta} \qquad \text{// expansion}$$

$$U_n = (1/2\pi) \int_{-\pi}^{\pi} d\theta \, U(\theta) \, e^{+in\theta} \qquad \text{// inversion}$$

$$(1/2\pi) \int_{-\pi}^{\pi} d\theta \, e^{in\theta} \, e^{-in'\theta} = \delta_{nn'} \qquad \text{// orthogonality}$$

$$(1/2\pi) \; \Sigma_n \, e^{-in\theta'} \, e^{in\theta} = \delta(\theta-\theta') \qquad \text{// completeness}$$

BUT, since our $g(\theta)$ is not periodic, this diagonalization is invalid, and so we are not concerned about the bogus conclusion that $U(\theta)$ = constant.

So the upshot is that the Fourier Series diagonalization idea,

$$A(\theta) = \int_{-\pi}^{\pi} d\theta'/2\pi \, B(\theta') \, C(\theta-\theta') \qquad \text{// convolution equation}$$

$A_n = B_n C_n$ // after diagonalization

is only valid if all three functions A,B,C are periodic on $(-\pi,\pi)$.

(3) Something we *can* do is make g(x) be a replication of the portion of g(x) shown above in Figure 1 on the interval (-2,+2). That is to say, imagine replacing g(x) with Figure 1 stamped an infinite number of times across the x axis. This does not affect the integral equation because the largest difference between x and t is 2, and this will be in the range of this replicated g(x). In this case, g(x) is a periodic function, albeit with period 4 units. Since the integral does not cover this entire range, diagonalization as described does not work because the entire group parameter space is not covered by the integral. But, we can still do a cosine series expansion of this replicated g(x) with period 4 units. That is to say, we can define

$\varphi_n(x) = \cos(nx\pi/2)$ // which has period 4 units

and then we can expand our replicated kernel g(x) and the Love solution u(x)in this way

$$g(x) = \Sigma_{n=0}^{\infty} g_n \cos(nx\pi/2) \quad \Rightarrow \quad g_n = (1/2)(1+\delta_{n,0}) \int_{-2}^{2} dx \cos(nx\pi/2)g(x)$$

$$u(x) = \Sigma_{n=0}^{\infty} u_n \cos(nx\pi/2) \quad \Rightarrow \quad u_n = (1/2)(1+\delta_{n,0}) \int_{-2}^{2} dx \cos(nx\pi/2)u(x)$$

Our integral equation was

$$\int_{-1}^{1} dt\, g(x-t)\, u(t) = u(x) - 1 \qquad x \text{ in } (-1,1) \qquad g(x) = (\kappa/\pi) / [\kappa^2 + x^2]$$

which we can write in suggestive form as

$$\Sigma_t G_{xt} U_t = [U - 1]_x$$

If we change from the coordinate representation to the cosine representation, this becomes

$$\Sigma_{m=0}^{\infty} G_{nm}\, u_m = [u - 1]_n$$

$$[u - 1]_n = (1/2)(1+\delta_{n,0}) \int_{-2}^{2} dx \cos(nx\pi/2)[u(x) - 1]$$

This does not have the nice diagonalized form mentioned above because G_{nm} is not diagonal, but it is still something that might be usable. We have

$$G_{nm} = \langle\varphi_n|G|\varphi_m\rangle = \int_{-2}^{2} \cos(nx\pi/2) \cos(mx\pi/2)\, g(x)$$

$$= (\kappa/\pi) \int_{-2}^{2} \cos(nx\pi/2) \cos(mx\pi/2) / [\kappa^2 + x^2]$$

According to Maple, this integral is a mess of Si, Ci, sh and ch functions:

```
> restart;
> assume(n,integer, m,integer);
> Int(cos(n*x*Pi/2)*cos(m*x*Pi/2)/(kappa^2 + x^2),x=-2..2);
```

$$\int_{-2}^{2} \frac{\cos\left(\frac{1}{2} n x \pi\right) \cos\left(\frac{1}{2} m x \pi\right)}{\kappa^2 + x^2} dx$$

```
> int(cos(n*x*Pi/2)*cos(m*x*Pi/2)/(kappa^2 + x^2),x=-2..2): simplify(%);
```

$$-\frac{1}{2}\left(\mathrm{Si}\left(-n\pi + m\pi - \frac{1}{2} I\pi\kappa m + \frac{1}{2} I\pi\kappa n\right)\sinh(\%2) - I\,\mathrm{Ci}\left(-n\pi - \frac{1}{2} I\pi\kappa n + m\pi + \frac{1}{2} I\pi\kappa m\right)\cosh(\%2)\right.$$
$$+\mathrm{Si}\left(n\pi + \frac{1}{2} I\pi\kappa m + m\pi + \frac{1}{2} I\pi\kappa n\right)\sinh(\%1) + I\,\mathrm{Ci}\left(-n\pi + \frac{1}{2} I\pi\kappa m - m\pi + \frac{1}{2} I\pi\kappa n\right)\cosh(\%1)$$
$$+\mathrm{Si}\left(-n\pi - \frac{1}{2} I\pi\kappa n + m\pi + \frac{1}{2} I\pi\kappa m\right)\sinh(\%2) - \mathrm{Si}\left(-n\pi + \frac{1}{2} I\pi\kappa m - m\pi + \frac{1}{2} I\pi\kappa n\right)\sinh(\%1)$$
$$\left.+I\,\mathrm{Ci}\left(-n\pi + m\pi - \frac{1}{2} I\pi\kappa m + \frac{1}{2} I\pi\kappa n\right)\cosh(\%2) - I\,\mathrm{Ci}\left(-n\pi - \frac{1}{2} I\pi\kappa m - m\pi - \frac{1}{2} I\pi\kappa n\right)\cosh(\%1)\right)/\kappa$$

$$\%1 := \frac{1}{2}\pi\,(n+m)\,\kappa$$

$$\%2 := \frac{1}{2}\pi\,(m-n)\,\kappa$$