

Solving Frontier Problems of Physics: The Decomposition Method

by

George Adomian

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Fundamental Theories of Physics

Solving Frontier Problems of Physics: The Decomposition Method

Fundamental Theories of Physics

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IN MEMORY OF MY FATHER AND MOTHER
HAIG AND VARTUHI ADOMIAN

دانشگاه صنعتی امیرکبیر
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EARLIER WORKS BY THE AUTHOR

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Stochastic Systems, Academic Press, 1983; also Russian transl. ed.

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Partial Differential Equations with R. E. Bellman, D. Reidel Publishing Co 1985.

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PREFACE

I discovered the very interesting Adomian method and met George Adomian himself some years ago at a conference held in the United States. This new technique was very surprising for me, an applied mathematician, because it allowed solution of exactly nonlinear functional equations of various kinds (algebraic, differential, partial differential, integral,...) without discretizing the equations or approximating the operators. The solution when it exists is found in a rapidly converging series form, and time and space are not discretized. At this time an important question arose: why does this technique, involving special kinds of polynomials (Adomian polynomials) converge? I worked on this subject with some young colleagues at my research institute and found that it was possible to connect the method to more well-known formulations where classical theorems (fixed point theorem, substituted series, ...) could be used. A general framework for decomposition methods has even been proposed by Lionel Gabet, one of my researchers who has obtained a Ph.D. thesis on this subject. During this period a fruitful cooperation has been developed between George Adomian and my research institute. We have frequently discussed advances and difficulties and we exchange ideas and results.

With regard to this new book, I am very impressed by the quality and the importance of the work, in which the author uses the decomposition method for solving frontier problems of physics. Many concrete problems involving differential and partial differential equations (including Navier-Stokes equations) are solved by means of the decomposition technique developed by Dr. Adomian. The basic ideas are clearly detailed with specific physical examples so that the method can be easily understood and used by researchers of various disciplines. One of the main objectives of this method is to provide a simple and unified technique for solving nonlinear functional equations.

Of course some problems remain open. For instance, practical convergence may be ensured even if the hypotheses of known methods are not satisfied. That means that there still exist opportunities for further theoretical studies to be done by pure or applied mathematicians, such as proving convergence in more general situations. Furthermore, it is not always easy to take into account the boundary conditions for complex domains.

In conclusion, I think that this book is a fundamental contribution to the theory and practice of decomposition methods in functional analysis. It

completes and clarifies the previous book of the author published by Kluwer in 1989. The decomposition method has now lost its mystery but it has won in seriousness and power. Dr. Adomian is to be congratulated for his fundamental contribution to functional and numerical analysis of complex systems.

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(Paris VI)
Paris, France
September 9, 1993

FOREWORD

This book is intended for researchers and (primarily graduate) students of physics, applied mathematics, engineering, and other areas such as biomathematics and astrophysics where mathematical models of dynamical systems require quantitative solutions. A major part of the book deals with the necessary theory of the decomposition method and its generalizations since earlier works. A number of topics are not included here because they were dealt with previously. Some of these are delay equations, integro-differential equations, algebraic equations and large matrices, comparisons of decomposition with perturbation and hierarchy methods requiring closure approximation, stochastic differential equations, and stochastic processes [1]. Other topics had to be excluded due to time and space limitations as well as the objective of emphasizing utility in solving physical problems.

Recent works, especially by Professor Yves Cherruault in journal articles and by Lionel Gabet in a dissertation, have provided a rigorous theoretical foundation supporting the general effectiveness of the method of decomposition. The author believes that this method is relevant to the field of mathematics as well as physics because mathematics has been essentially a linear operator theory while we deal with a nonlinear world. Applications have shown that accurate and easily computed quantitative solutions can be determined for nonlinear dynamical systems without assumptions of "small" nonlinearity or computer-intensive methods.

The evolution of the research has suggested a theory to unify linear and nonlinear, ordinary or partial differential equations for solving initial or boundary-value problems efficiently. As such, it appears to be valuable in the background of applied mathematicians and theoretical or mathematical physicists. An important objective for physics is a methodology for solution of dynamical systems—which yields verifiable and precise quantitative solutions to physical problems modelled by nonlinear partial differential equations in space and time. Analytical methods which do not require a change of the model equation into mathematically more tractable, but necessarily less realistic representation, are of primary concern. Improvement of analytical methods would in turn allow more sophisticated modelling and possible further progress. The final justification of theories of physics is in the correspondence of predictions with nature rather than in rigorous proofs which may well

restrict the stated problem to a more limited universe. The broad applicability of the methodology is a dividend which may allow a new approach to mathematics courses as well as being useful for the physicists who will shape our future understanding of the world.

Recent applications by a growing community of users have included areas such as biology and medicine, hydrology, and semiconductors. In the author's opinion this method offers a fertile field for pure mathematicians and especially for doctoral students looking for dissertation topics. Many possibilities are included directly or indirectly. Some repetition of objectives and motivations (for research on decomposition and connections with standard methods) was believed to be appropriate to make various chapters relatively independent and permit convenient design of courses for different specialties and levels.

Partial differential equations are now solved more efficiently, with less computation, than in the author's earlier works. The Duffing oscillator and other generic oscillators are dealt with in depth. The last chapter concentrates on a number of frontier problems. Among these are the Navier-Stokes equations, the N-body problem, and the Yukawa-coupled Klein-Gordon-Schrödinger equation. The solutions of these involve no linearization, perturbation, or limit on stochasticity. The Navier-Stokes solution [2] differs from earlier analyses [3]. The system is fully dynamic, considering pressure changing as the velocity changes. It now allows high velocity and possible prediction of the onset of turbulence.

The references listed are not intended to be an exhaustive or even a partial bibliography of the valuable work of many researchers in these general areas. Only those papers are listed which were considered relevant to the precise area and method treated. (New work is appearing now at an accelerating rate by many authors for submission to journals or for dissertations and books. A continuing bibliography could be valuable to future contributors and reprints received by the author will be recorded for this purpose.)

The author appreciates the advice, questions, comments, and collaboration of early workers in this field such as Professors R.E. Bellman, N. Bellomo, Dr. R. McCarty, and other researchers over the years, the important work by Professor Yves Cherruault on convergence and his much appreciated review of the entire manuscript, the support of my family, and the editing and valuable contributions of collaborator and friend, Randolph Rach, whose insights and willingness to share his time and knowledge on difficult problems have been an important resource. The book contains work originally typeset by Arlette

Revells and Karin Haag. The camera-ready manuscript was prepared with the dedicated effort of Karin Haag, assisted by William David. Laura and William David assumed responsibility for office management so that research results could be accelerated. Computer results on the Duffing equation were obtained by Dr. McLowery Elrod with the cooperation of the National Science Center Foundation headed by Dr. Fred C. Davison, who has long supported this work. Gratitude is due to Ronald E. Meyers, U.S. Army Research Laboratories, White Sands Missile Range, who supported much of this research and also contributed to some of the development. Thanks are also due to the Office of Naval Research, Naval Research Laboratories, and Paul Palo of the Naval Civil Engineering Laboratories, who have supported work directed toward applications as well as intensive courses at NRL and NCEL. The author would also like to thank Professor Alwyn Van der Merwe of the University of Denver for his encouragement that led to this book. Most of all, the unfailing support by my wife, Corinne, as well as her meticulous final editing, is deeply appreciated.

G. Adomian

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ON MODELLING PHYSICAL PHENOMENA

Our use of the term “mathematical model” or “model” will refer to a set of consistent equations intended to describe the particular features or behavior of a physical system which we seek to understand. Thus, we can have different models of the system dependent on the questions of interest and on the features relevant to those questions. To derive an adequate mathematical description with a consistent set of equations and relevant conditions, we clearly must have in mind a purpose or objective and limit the problem to exclude factors irrelevant to our specific interest. We begin by considering the pertinent physical principles which govern the phenomena of interest along with the constitutive properties of material with which the phenomena may interact.

Depending on the problem, a model may consist of algebraic equations, integral equations, or ordinary, partial, or coupled systems of differential equations. The equations can be nonlinear and stochastic in general with linear or deterministic equations being special cases. (In some cases, we may have delays as well.) Combinations of these equations such as integro-differential equations also occur.

A model using differential equations must also include the initial/boundary conditions. Since nonlinear and nonlinear stochastic equations are extremely sensitive to small changes in inputs, parameters, or initial conditions, solutions may change rather radically with such changes. Consequently, exact specification of the model is sometimes not a simple matter. Prediction of future behavior is therefore limited by the precision of the initial state. When significant nonlinearity is present, small changes (perhaps only 1%) in the system may make possible one or many different solutions. If small but appreciable randomness, or, possibly, accumulated round-off error in iterative calculation is present, we may observe a random change from one solution to another—an apparently chaotic behavior.

To model the phenomena, process, or system of interest, we first isolate the relevant parameters. From experiments, observations, and known relationships, we seek mathematical descriptions in the form of equations which we can then solve for desired quantities. This process is neither universal nor can it take everything into account; we must tailor the model to fit

the questions to which we need answers and neglect extraneous factors. Thus a model necessarily excludes the universe external to the problem and region of interest to simplify as much as possible, and reasonably retain only factors relevant to the desired solution.

Modelling is necessarily a compromise between physical realism and our ability to solve the resulting equations. Thus, development of understanding based on verifiable theory involves both modelling and analysis. Any incentive for more accurate or realistic modelling is limited by our ability to solve the equations; customary modelling uses restrictive assumptions so that well-known mathematics can be used. Our objective is to minimize or avoid altogether this compromise for mathematical tractability which requires linearization and superposition, perturbation, etc., and instead, to model the problem with its inherent nonlinearities and random fluctuation or uncertain data.

We do this because the decomposition method is intended to solve nonlinear and/or stochastic ordinary or partial differential equations, integro-differential equations, delay equations, matrix equations, etc., avoiding customary restrictive assumptions and methods, to allow solutions of more realistic models. If the deductions resulting from solution of this model differ from accurate observation of physical reality, then this would mean that the model is a poor one and we must re-model the problem. Hence, modelling and the solution procedure ought to be applied interactively. Since we will be dealing with a limited region of space-time which is of interest to the problem at hand, we must consider conditions on the boundaries of the region to specify the problem completely. If we are interested in dynamical problems such as a process evolving over time, then we must consider solutions as time increases from some initial time: i.e., we will require initial conditions. We will be interested generally in differential equations which express relations between functions and derivatives. These equations may involve use of functions, ordinary or partial derivatives, and nonlinearities and even stochastic processes to describe reality. Also, of course, initial and boundary conditions must be specified to make the problem completely determinable.

If the solution is to be valid, it must satisfy the differential equation and the properly specified conditions, so appropriate smoothness must exist. We have generally assumed that nonlinearities are analytic but will discuss some exceptions in a later chapter. An advantage, other than the fact that problems are considered more realistically than by customary constraints, is that

solutions are not obtained here by discretized methods: solutions are continuous and computationally much more efficient as we shall see. If we can deal with a physical problem as it is, we can expect a useful solution, i.e., one in which the mathematical results correspond to reality. If our model is poor because the data are found from measurements which have some error, it is usual to require that a small change in the data must lead to a small change in the solution. This does not apply to nonlinear equations because small changes in initial data can cause significant changes in the solution, especially in stochastic equations. This is a problem of modelling. If the data are correct and the equation properly describes the problem, we expect a correct and convergent solution.

The initial/boundary conditions for a specific partial differential equation, needless to say, cannot be arbitrarily assigned: they must be consistent with the physical problem being modelled.

Suppose we consider a solid body where $u(x,y,z,t)$ represents a temperature at x,y,z at time t . If we consider a volume V within the body which is bounded by a smooth closed surface S and consider the change of heat in V during an interval (t_1, t_2) , we have, following the derivation of N.S. Koshlyakov, M.M. Smirnov, and E.B. Gliner [1]

$$Q_1 = \int_{t_1}^{t_2} dt \iint_S k(x,y,z) \frac{\partial u}{\partial n} dS$$

where n is the normal to S in the direction of decreasing temperatures and k is the internal heat conductivity, a positive function independent of the direction of the normal. The amount of heat to change the temperature of V is

$$Q_2 = \int_{t_1}^{t_2} dt \iiint_V c\rho \frac{\partial u}{\partial t} dV$$

where $c(x,y,z)$ is the specific heat and $\rho(x,y,z)$ is the density. If heat sources with density $g(x,y,z,t)$ exist in the body, we have

$$Q_3 = \int_{t_1}^{t_2} dt \iiint_V g(x,y,z,t) dV$$

Since $Q_2 = Q_1 + Q_3$, it follows that

$$c\rho \frac{\partial u}{\partial t} = \operatorname{div}(k \operatorname{grad} u) + g$$

If $c\rho$ and k are constants, we can write $a^2 = k/c\rho$ and $f(x, y, z, t) = g(x, y, z, t)/c\rho$. Then

$$\frac{\partial u}{\partial t} = a^2 \nabla^2 u + f$$

(which neglects heat exchange between S and the surrounding medium).

Now to determine a solution, we require the temperature at an initial instant $u(x, y, z, t = 0)$ and either the temperatures at every point of the surface or the heat flow on the surface. These are constraints or commonly, the boundary conditions. If we do not neglect heat exchange to the surrounding medium which is assumed to have uniform temperature u_m , a third boundary condition can be written as $\alpha(u - u_m) = -k \partial u / \partial n|_S$ (if we assume the coefficient of exchange is uniform for all of S).

Thus the solution must satisfy the equation, the initial condition, and one of the above boundary conditions or constraints which make the problem specific. We have assumed a particular model which is formulated using fundamental physical laws such as conservation of energy, so the initial distribution must be physically correct and not arbitrary. If it is correct, it leads to a specific physically correct solution. The conditions and the equation must be consistent and physically correct. The conditions must be smooth, bounded, and physically realizable. The initial conditions must be consistent with the boundary conditions and the model. The derived "solution" is verified to be consistent with the model equation and the conditions and is therefore the solution.

NOTE: Koshlyakov, et. al. [1] state that we *must specify* $u(t = 0)$ *within the body and one of the boundary conditions* such as u on S . However S is not insulated from the body. The initial condition $u(t = 0)$ fixes u on S *also* if surroundings are ignored. It seems that *either* one or the other should be enough in a *specific* problem and if you give both, they must be consistent with each other *and* the model (equation). The same situation arises when, e.g., in a square or rectangular domain, we assign boundary conditions on the four sides, which means that physically we have discontinuity at the corners.

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THE DECOMPOSITION METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS

A critically important problem in frontier science and technology is the physically correct solution of nonlinear and/or stochastic systems modelled by differential or partial differential equations for general initial/boundary conditions.

The usual procedures of analysis necessarily change such problems in essential ways in order to make them mathematically tractable by established methods. Unfortunately these changes necessarily change the solutions; therefore, they can deviate, sometimes seriously, from the actual physical behavior. These procedures include linearization techniques, perturbation methods, and restrictions on the nature and magnitude of stochastic processes. The avoidance of these limitations so that physically correct solutions can be obtained would add in an important way to our insights into the natural behavior of complex systems and would offer a potential for advances in science and technology.

The prior art in mathematical analysis as seen in the literature necessarily relies on such limiting procedures. Thus it may well be said that physics is usually perturbative theory and mathematics is essentially linear operator theory. Of course there are some methods of solving nonlinear equations, but not general methods. For example, clever transformation of variables sometimes results in a linear equation; however, this rarely works.

The objective of the decomposition method is to make possible physically realistic solutions of complex systems without the usual modelling and solution compromises to achieve tractability. A bonus is that it essentially combines the fields of ordinary and partial differential equations. This chapter will summarize the method and will briefly discuss applications and consequences for analysis and computation.

Suppose we think about physical systems described by nonlinear partial differential equations. In the more complicated problems, we ordinarily must resort to discretized methods and numerical computation. An appropriate example is fluid flow and "computational fluid dynamics" (C.F.D.), an area of

intensive research in attempting to develop codes for study of transonic and hypersonic flow. Because of the symbiosis between such existing methodology and supercomputers, as well as the complexity, these methods are computationally intensive. Massive printouts are the result and functional dependences are difficult to see. We have a constant demand for faster computers, superconductivity, parallelism, etc., because of the necessity to cut down computation time. Thus a continuous solution and considerably decreased computation is evidently a desirable goal.

Closed-form analytical solutions are considered ideal when possible. However, they may necessitate changing the actual or real-life problem to a more tractable *mathematical* problem. Except for a small class of equations in which clever transformations can result in linear equations, it becomes necessary to resort to linearization or statistical linearization techniques, or assumptions of "weak nonlinearity," etc. What we get then is solution of the *simpler mathematical problem*. The resulting solution can deviate significantly from the solution of the actual problem; nonlinear systems can be extremely sensitive to small changes. These small changes can occur because of inherent stochastic effects or computer errors; the resulting solutions (especially in strongly nonlinear equations) can show violent, erratic (or "chaotic") behavior. Of course, it is clear that considerable progress has been made with the generally used procedures, and, in many problems, these methods remain adequate. Thus, in problems which are close to linear, or where perturbation theory is adequate, excellent solutions are obtained.

In many frontier problems, however, we have strong nonlinearities or stochasticity in parameters, so that it becomes important to find a new approach and that is our subject here.

✓ We begin with the (deterministic) form $Fu = g(t)$ where F is a nonlinear ordinary differential operator with linear and nonlinear terms. We could represent the linear term by Lu where L is the linear operator. In this case L must be easily invertible which may not be the case, i.e., we may have a difficult Green's function and a consequently difficult integration. Instead, we write the linear term as $Lu + Ru$ where we choose L as the highest-ordered derivative. Now L^{-1} is simply an n -fold integration for an n th order L . The remainder of the linear operator is R . (In cases where stochastic terms are present in the linear operator, we can include a stochastic operator term $\mathcal{R}u$.) The nonlinear term is represented by Nu . Thus $Lu + Ru + Nu = g$ and we write

$$Lu = g - Ru - Nu$$

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu$$

For initial-value problems we conveniently define L^{-1} for $L = d^n/dt^n$ as the n -fold definite integration operator from 0 to t . For the operator $L = d^2/dt^2$, for example, we have $L^{-1}Lu = u - u(0) - tu'(0)$ and therefore

$$u = u(0) + tu'(0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu$$

For the same operator equation but now considering a boundary value problem, we let L^{-1} be an indefinite integral and write $u = A + Bt$ for the first two terms and evaluate A, B from the given conditions. The first three terms are identified as u_0 in the assumed decomposition $u = \sum_{n=0}^{\infty} u_n$. Finally, assuming Nu is analytic, we write $Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n)$ where the A_n are specially generated (Adomian) polynomials for the specific nonlinearity. They depend only on the u_0 to u_n components and form a rapidly convergent series. The A_n are given as

$$A_0 = f(u_0)$$

$$A_1 = u_1(d/du_0)f(u_0)$$

$$A_2 = u_2(d/du_0)f(u_0) + (u_1^2/2!)(d^2/du_0^2)f(u_0)$$

$$A_3 = u_3(d/du_0)f(u_0) + u_1u_2(d^2/du_0^2)f(u_0) + (u_1^3/3!)(d^3/du_0^3)f(u_0)$$

⋮

and can be found from the formula (for $n \geq 1$)

$$A_n = \sum_{v=1}^n c(v, n) f^{(v)}(u_0)$$

In the linear case where $f(u) = u$, the A_n reduce to u_n . Otherwise $A_n = A_n(u_0, u_1, \dots, u_n)$. For $f(u) = u^2$, for example, $A_0 = u_0^2$, $A_1 = 2u_0u_1$, $A_2 = u_1^2 + 2u_0u_2$, $A_3 = 2u_1u_2 + 2u_0u_3, \dots$. It is to be noted that in this scheme, the sum of the subscripts in each term of the A_n are equal to n . The $c(v, n)$ are products (or sums of products) of v components of u whose subscripts sum to n , divided by the factorial of the number of repeated subscripts. Thus $c(1, 3)$ can only be u_3 , $c(2, 3)$ is u_1u_2 and $c(3, 3) = (1/3!)u_1^3$. For a nonlinear equation in

u, one may express any given function $f(u)$ in the A_n by $f(u) = \sum_{n=0}^{\infty} A_n$.

We have previously pointed out that the A_n polynomials are not unique, e.g., for $f(u) = u^2$, $A_0 = u_0^2$, $A_1 = 2u_0u_1$, $A_2 = u_1^2 + 2u_0u_2, \dots$. But A_1 could also be $2u_0u_1 + u_1^2$, i.e., it could include the first term of A_2 since u_0 and u_1 are known when u_2 is to be calculated.

It is now established that the sum of the series $\sum_{n=0}^{\infty} A_n$ for Nu is equal to the sum of a generalized Taylor series about $u_0(x)$, that $\sum_{n=0}^{\infty} u_n$ is equal to a generalized Taylor series about the function u_0 , and that the series terms approach zero as $1/(mn)!$ if m is the order of the highest linear differential operator. Since the series converges (in norm) and does so very rapidly, the n -

term partial sum $\phi_n = \sum_{i=0}^{n-1} u_i$ can serve as a practical solution for synthesis and design. The $\lim_{n \rightarrow \infty} \phi_n = u$.

Other convenient algorithms have been developed for composite and multidimensional functions as well as for particular functions of interest. As an example for solution of the Duffing equation, we use the notation $A_m[f(u)] = A_m(u^3)$.

If we write $f(u) = \sum_{n=0}^{\infty} A_n[f(u)]$ or more simply $f(u) = \sum_{n=0}^{\infty} A_n$ and let $f(u) = u$, we have $u = \sum_{n=0}^{\infty} u_n$ since then $A_0 = u_0$, $A_1 = u_1, \dots$. Thus we can say u and $f(u)$, i.e., the solution and any nonlinearity, are written in terms of the A_n , or, that we do this for the nonlinearity and think of u as simply decomposed into components u_i to be evaluated such that the n -term approximation $\phi_n = \sum_{i=0}^{n-1} u_i$ approaches $u = \sum_{n=0}^{\infty} u_n$ as $n \rightarrow \infty$. The solution can now be written as:

$$u = \sum_{n=0}^{\infty} u_n = u_0 - L^{-1}R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} A_n$$

so that

$$u_1 = -L^{-1}Ru_0 - L^{-1}A_0$$

$$u_2 = -L^{-1}Ru_1 - L^{-1}A_1$$

etc. All components are determinable since A_0 depends only on u_0 , A_1 depends on u_0, u_1 , etc. The practical solution will be the n -term approximation or approximant to u , sometimes written $\phi_n[u]$ or simply ϕ_n .

$$\varphi_n = \sum_{i=0}^{n-1} u_i$$

$$\lim_{n \rightarrow \infty} \varphi_n = \sum_{i=0}^{\infty} u_i = u$$

Convergence has been rigorously established by Professor Yves Cherruault [1]. Also further rigorous re-examination has most recently been done by Lionel Gabet [2]. The rapidity of this convergence means that few terms are required as shown in examples, e.g., [1].

BASIS FOR THE EFFECTIVENESS OF DECOMPOSITION:

Let's consider the physical basis for the accuracy and rapid rate of convergence. The initial term u_0 is an optimal first approximation containing essentially all *a priori* information about the system. Thus, $u_0 = \Phi + L^{-1}g$ contains the given input g (which is bounded in a physical system) and the initial or boundary conditions included in Φ which is the solution of $L\Phi = 0$. Furthermore, the following terms converge for bounded t as $1/(mn)!$ where n is the order of L and m is the number of terms in the approximant ϕ_m . Hence even with very small m , the ϕ_m will contain most of the solution. Of the following derived terms, u_1 is particularly simple, since A_0 , the first of the polynomials representing the nonlinearity $f(u)$, is $f(u_0)$ which of course is also known. The $f(u)$ need not even be analytic [3] but could be piecewise-differentiable (Sobolev space) and represented by a series of analytic functions. We do require bounded Φ and g , which is physically reasonable and the u_m terms must be (Lebesgue) integrable. For n th order differential operators L , we must have an n -fold differentiability/continuity requirement on solutions in order to assure existence and uniqueness. The development was based on a decomposition in calculable terms of the solution to be found rather than an expansion in a series. We will see that the method works for initial-value or boundary-value problems and for linear or nonlinear, ordinary or partial differential equations and even for stochastic systems. We prove now that the $\sum A_n$ is a rearranged Taylor series about this optimal first approximation $\phi_1 = u_0$.

To see that the series in A_n polynomials forms a generalized Taylor series about a function (rather than a point), we write

$$f(u) = \sum_{n=0}^{\infty} A_n = f(u_0) + u_1 f^{(1)}(u_0) + (u_1^2/2!) f^{(2)}(u_0) \\ + u_2 f^{(1)}(u_0) + u_3 f^{(1)}(u_0) + u_1 u_2 f^{(2)}(u_0) + (u_1^3/3!) f^{(3)}(u_0) + \dots$$

which can be rearranged as

$$f(u) = f(u_0) + (u_1 + u_2 + \dots) f^{(1)}(u_0) + [(u_1^2/2!) + u_1 u_2 + \dots] f^{(2)}(u_0) + \dots \\ = f(u_0) + [(u - u_0)/1!] f^{(1)}(u_0) + [(u - u_0)^2/2!] f^{(2)}(u_0) + \dots \\ = \sum_{n=0}^{\infty} [(u - u_0)^n/n!] f^{(n)}(u_0)$$

A REFERENCE LIST OF THE ADOMIAN POLYNOMIALS:

$$\begin{aligned} A_0 &= f(u_0) \\ A_1 &= u_1 f^{(1)}(u_0) \\ A_2 &= u_2 f^{(1)}(u_0) + (1/2!) u_1^2 f^{(2)}(u_0) \\ A_3 &= u_3 f^{(1)}(u_0) + u_1 u_2 f^{(2)}(u_0) + (1/3!) u_1^3 f^{(3)}(u_0) \\ A_4 &= u_4 f^{(1)}(u_0) + [(1/2!) u_2^2 + u_1 u_3] f^{(2)}(u_0) \\ &\quad + (1/2!) u_1^2 u_2 f^{(3)}(u_0) + (1/4!) u_1^4 f^{(4)}(u_0) \\ A_5 &= u_5 f^{(1)}(u_0) + [u_2 u_3 + u_1 u_4] f^{(2)}(u_0) \\ &\quad + [(1/2!) u_1 u_2^2 + (1/2!) u_1^2 u_3] f^{(3)}(u_0) \\ &\quad + (1/3!) u_1^3 u_2 f^{(4)}(u_0) + (1/5!) u_1^5 f^{(5)}(u_0) \\ A_6 &= u_6 f^{(1)}(u_0) + [(1/2!) u_3^2 + u_2 u_4 + u_1 u_5] f^{(2)}(u_0) \\ &\quad + [(1/3!) u_2^3 + u_1 u_2 u_3 + (1/2!) u_1^2 u_4] f^{(3)}(u_0) \\ &\quad + [(1/2!) u_1^2 (1/2!) u_2^2 + (1/3!) u_1^3 u_3] f^{(4)}(u_0) \\ &\quad + (1/4!) u_1^4 u_2 f^{(5)}(u_0) + (1/6!) u_1^6 f^{(6)}(u_0) \\ A_7 &= u_7 f^{(1)}(u_0) + [u_3 u_4 + u_2 u_5 + u_1 u_6] f^{(2)}(u_0) \\ &\quad + [(1/2!) u_2^2 u_3 + u_1 (1/2!) u_3^2 + u_1 u_2 u_4 + (1/2!) u_1^2 u_5] f^{(3)}(u_0) \\ &\quad + [u_1 (1/3!) u_2^3 + (1/2!) u_1^2 u_2 u_3 + (1/3!) u_1^3 u_4] f^{(4)}(u_0) \end{aligned}$$

$$+[(1/3!)u_1^3(1/2!)u_2^2 + (1/4!)u_1^4u_3]f^{(5)}(u_0)$$

$$+(1/5!)u_1^5u_2f^{(6)}(u_0) + (1/7!)u_1^7f^{(7)}(u_0)$$

$$\begin{aligned} A_8 = & f^{(1)}(u_0)u_8 + f^{(2)}(u_0)[(1/2!)u_4^2 + u_3u_5 + u_2u_6 + u_1u_7] \\ & + f^{(3)}(u_0)[u_2(1/2!)u_3^2 + (1/2!)u_2^2u_4 + u_1u_3u_4 + u_1u_2u_5 + (1/2!)u_1^2u_6] \\ & + f^{(4)}(u_0)[(1/4!)u_2^4 + u_1(1/2!)u_2^2u_3 + (1/2!)u_1^2(1/2!)u_3^2 \\ & + (1/2!)u_1^2u_2u_4 + (1/3!)u_1^3u_5] + f^{(5)}(u_0)[(1/2!)u_1^2(1/3!)u_2^3 + (1/3!)u_1^3u_2u_3 \\ & + (1/4!)u_1^4u_4] + f^{(6)}(u_0)[(1/4!)u_1^4(1/2!)u_2^2 + (1/5!)u_1^5u_3] \\ & + f^{(7)}(u_0)(1/6!)u_1^6u_2 + f^{(8)}(u_0)(1/8!)u_1^8 \end{aligned}$$

$$\begin{aligned} A_9 = & f^{(1)}(u_0)u_9 + f^{(2)}(u_0)[u_4u_5 + u_3u_6 + u_2u_7 + u_1u_8] \\ & + f^{(3)}(u_0)[(1/3!)u_3^3 + u_2u_3u_4 + (1/2!)u_2^2u_5 + u_1(1/2!)u_4^2 \\ & + u_1u_3u_5 + u_1u_2u_6 + (1/2!)u_1^2u_7] \\ & + f^{(4)}(u_0)[(1/3!)u_2^3u_3 + u_1u_2(1/2!)u_3^2 + u_1(1/2!)u_2^2u_4 \\ & - (1/2!)u_1^2u_3u_4 + (1/2!)u_1^2u_2u_5 + (1/3!)u_1^3u_6] \\ & + f^{(5)}(u_0)[u_1(1/4!)u_2^4 + (1/2!)u_1^2(1/2!)u_2^2u_3 \\ & + (1/3!)u_1^3(1/2!)u_3^2 + (1/3!)u_1^3u_2u_4 + (1/4!)u_1^4u_5] \\ & + f^{(6)}(u_0)[(1/3!)u_1^3(1/3!)u_2^3 + (1/4!)u_1^4u_2u_3 + (1/5!)u_1^5u_4] \\ & + f^{(7)}(u_0)[(1/5!)u_1^5(1/2!)u_2^2 + (1/6!)u_1^6u_3] \\ & + f^{(8)}(u_0)(1/7!)u_1^7u_2 + f^{(9)}(u_0)(1/9!)u_1^9 \end{aligned}$$

$$\begin{aligned} A_{10} = & f^{(1)}(u_0)u_{10} + f^{(2)}(u_0)[(1/2!)u_5^2 + u_4u_6 + u_3u_7 + u_2u_8 + u_1u_9] \\ & + f^{(3)}(u_0)[(1/2!)u_3^2u_4 + u_2(1/2!)u_4^2 + u_2u_3u_5 + (1/2!)u_2^2u_6 \\ & + u_1u_4u_5 + u_1u_3u_6 + u_1u_2u_7 + (1/2!)u_1^2u_8] \\ & + f^{(4)}(u_0)[(1/2!)u_2^2(1/2!)u_3^2 + (1/3!)u_2^3u_4 + u_1(1/3!)u_3^3 \\ & + u_1u_2u_3u_4 + u_1(1/2!)u_2^2u_5 + (1/2!)u_1^2(1/2!)u_4^2 \\ & + (1/2!)u_1^2u_3u_5 + (1/2!)u_1^2u_2u_6 + (1/3!)u_1^3u_7] \\ & + f^{(5)}(u_0)[(1/5!)u_2^5 + u_1(1/3!)u_2^3u_3 + (1/2!)u_1^2u_2(1/2!)u_3^2 \\ & + (1/2!)u_1^2(1/2!)u_2^2u_4 + (1/3!)u_1^3u_3u_4 \end{aligned}$$

$$\begin{aligned}
& + (1/3!)u_1^3u_2u_3 + (1/4!)u_1^4u_4 \\
& + f^{(6)}(u_0)\left[(1/2!)u_1^2(1/4!)u_2^4 + (1/3!)u_1^3(1/2!)u_2^2u_3\right. \\
& \left. + (1/4!)u_1^4(1/2!)u_2^2 + (1/4!)u_1^4u_2u_4 + (1/5!)u_1^5u_5\right] \\
& + f^{(7)}(u_0)\left[(1/4!)u_1^4(1/3!)u_2^2 + (1/5!)u_1^5u_2u_3 + (1/6!)u_1^6u_4\right] \\
& + f^{(8)}(u_0)\left[(1/6!)u_1^6(1/2!)u_2^2 + (1/7!)u_1^7u_3\right] \\
& + f^{(9)}(u_0)(1/8!)u_1^8u_2 + f^{(10)}(u_0)(1/10!)u_1^{10}
\end{aligned}$$

EXAMPLE: $Nu = u^5$.

$$A_0 = u_0^5$$

$$A_1 = 5u_0^4u_1$$

$$A_2 = 5u_0^4u_2 + 10u_0^3u_1^2$$

$$A_3 = 5u_0^4u_3 + 20u_0^3u_1u_2 + 10u_0^2u_1^3$$

$$A_4 = 5u_0^4u_4 + 5u_1^4u_0 + 10u_0^3u_2^2 + 20u_0^3u_1u_3 + 30u_0^2u_1^2u_2$$

Notice that for u^m each individual term is the product of m factors. Each term of A_n has five factors—the sum of superscripts is m (or 5 in this case). The sum of subscripts is n . The second term of A_4 , as an example, is $5u_1u_1u_1u_1u_0$ and the sum of subscripts is 4. A very convenient check on the numerical coefficients in each term is the following. Each coefficient is $m!$ divided by the product of factorials of the superscripts for a given term. Thus, the second term of $A_3(u^5)$ has the coefficient $5!/(3!)(1!)(1!) = 20$. The last term of A_4 has the coefficient $5!/(2!)(2!)(1!) = 30$. Continuing with the A_n , for u^5 we have

$$\begin{aligned}
A_5 = & u_1^5 + 5u_0^4u_5 + 20u_0^3u_1u_4 + 20u_0^3u_2u_3 + 20u_1^3u_0u_2 \\
& + 30u_0^2u_2^2u_1 + 30u_0^2u_1^2u_3
\end{aligned}$$

$$\begin{aligned}
A_6 = & 5u_0^4u_6 + 5u_1^4u_2 + 10u_0^3u_3^2 + 10u_0^2u_2^3 + 20u_0^3u_1u_5 + 20u_0^3u_2u_4 \\
& + 20u_1^3u_0u_3 + 30u_0^2u_1^2u_4 + 30u_1^2u_2^2u_0 + 60u_0^2u_1u_2u_3
\end{aligned}$$

$$\begin{aligned}
A_7 = & 5u_0^4u_7 + 5u_1^4u_3 + 10u_1^3u_2^2 + 20u_0^3u_1u_6 + 20u_0^3u_2u_5 + 20u_0^3u_3u_4 \\
& + 20u_2^3u_1u_0 + 20u_1^3u_0u_4 + 30u_0^2u_2^2u_3 + 30u_0^2u_3^2u_1 + 30u_0^2u_1^2u_5 \\
& + 60u_0^2u_1u_2u_4 + 60u_1^2u_0u_2u_3
\end{aligned}$$

$$\begin{aligned}
A_8 = & 5u_0^4u_8 + 5u_1^4u_0 + 5u_1^4u_4 + 10u_0^3u_4^2 + 10u_1^2u_2^3 + 20u_0^3u_3u_5 + 20u_0^3u_2u_6 \\
& + 20u_0^3u_1u_7 + 20u_1^3u_3u_0 + 20u_1^3u_2u_3 + 30u_0^2u_2u_3^2 + 30u_0^2u_2^2u_4 \\
& + 20u_0^3u_1u_7 + 20u_1^3u_3u_0 + 20u_1^3u_2u_3 + 30u_0^2u_2u_3^2 + 30u_0^2u_2^2u_4 \\
& + 30u_0^2u_1^2u_6 + 30u_1^2u_3^2u_0 + 60u_1^2u_2u_4u_0 + 60u_1u_2^2u_3u_0 + 60u_0^2u_1u_3u_4
\end{aligned}$$

EXAMPLE: $Nu = u^3$

$$A_0 = u_0^3$$

$$A_1 = 3u_0^2u_1$$

$$A_2 = 3u_0^2u_2 + 3u_1^2u_0$$

$$A_3 = u_1^3 + 3u_0^2u_3 + 6u_0u_1u_2$$

$$A_4 = 3u_0^2u_4 + 3u_1^2u_2 + 3u_2^2u_0 + 6u_0u_1u_3$$

$$A_5 = 3u_0^2u_5 + 3u_1^2u_3 + 3u_2^2u_1 + 6u_0u_1u_4 + 6u_0u_2u_3$$

$$A_6 = u_2^3 + 3u_0^2u_6 + 3u_1^2u_4 + 3u_2^2u_0 + 6u_0u_1u_5 + 6u_0u_2u_4 + 6u_1u_2u_3$$

$$A_7 = 3u_0^2u_7 + 3u_1^2u_5 + 3u_2^2u_3 + 3u_3^2u_1 + 6u_0u_1u_6 + 6u_0u_2u_5 + 6u_0u_3u_4$$

$$\begin{aligned}
A_8 = & 3u_0^2u_8 + 3u_1^2u_6 + 3u_2^2u_4 + 3u_3^2u_2 + 3u_4^2u_0 + 6u_0u_1u_7 + 6u_0u_2u_6 \\
& - 6u_0u_3u_5 + 6u_1u_2u_5 + 6u_1u_3u_4
\end{aligned}$$

$$\begin{aligned}
A_9 = & u_3^3 + 3u_0^2u_9 + 3u_1^2u_7 + 3u_2^2u_5 + 3u_4^2u_1 + 6u_0u_1u_8 + 6u_0u_2u_7 \\
& - 6u_0u_3u_6 + 6u_0u_4u_5 + 6u_1u_2u_6 + 6u_1u_3u_5 + 6u_2u_3u_4
\end{aligned}$$

$$\begin{aligned}
A_{10} = & 3u_0^2u_{10} + 3u_1^2u_8 + 3u_2^2u_6 + 3u_3^2u_4 + 3u_4^2u_2 + 3u_5^2u_0 + 6u_0u_1u_9 \\
& - 6u_0u_2u_8 + 6u_0u_3u_7 + 6u_0u_4u_6 + 6u_1u_2u_7 + 6u_1u_3u_6 \\
& + 6u_1u_4u_5 + 6u_2u_3u_5
\end{aligned}$$

EXAMPLE: $Nu = u^2$

$$A_0 = u_0^2$$

$$A_1 = 2u_0u_1$$

$$A_2 = u_1^2 + 2u_0u_2$$

$$A_3 = 2u_1u_2 + 2u_0u_3$$

$$A_4 = u_2^2 + 2u_1u_3 + 2u_0u_4$$

$$A_5 = 2u_2u_3 + 2u_1u_4 + 2u_0u_5$$

EXAMPLE: $N\theta = \sin \theta$

$$\begin{aligned}
A_0 &= \sin \theta_0 \\
A_1 &= \theta_1 \cos \theta_0 \\
A_2 &= -(\theta_1^2/2) \sin \theta_0 + \theta_2 \cos \theta_0 \\
A_3 &= -(\theta_1^3/6) \cos \theta_0 - \theta_1 \theta_2 \sin \theta_0 + \theta_3 \cos \theta_0 \\
&\vdots
\end{aligned}$$

EXAMPLE: $f(x) = \sinh(x/2)$

$$\begin{aligned}
A_1 &= \frac{1}{2} x_1 \cosh(x_0/2) \\
A_2 &= \frac{1}{2} x_2 \cosh(x_0/2) + \frac{1}{2} x_1^2 \left(\frac{1}{2}\right)^2 \sinh\left(\frac{x_0}{2}\right) \\
A_3 &= \frac{1}{2} x_3 \cosh(x_0/2) + \left(\frac{1}{2}\right)^2 x_1 x_2 \sinh(x_0/2) + \frac{1}{6} x_1^3 \left(\frac{1}{2}\right)^3 \cosh(x_0/2) \\
A_4 &= \frac{1}{2} x_4 \cosh(x_0/2) + \left[\frac{1}{2} x_2^2 + x_1 x_3\right] \left(\frac{1}{2}\right)^2 \sinh(x_0/2) \\
&\quad + \frac{1}{2} x_1^2 x_2 \left(\frac{1}{2}\right)^3 \cosh(x_0/2) + \frac{1}{24} \left(\frac{1}{2}\right)^4 x_1^4 \sinh(x_0/2)
\end{aligned}$$

EXAMPLE: $f(u) = u^{-m}$, $m > 0$

$$\begin{aligned}
A_0 &= u_0^{-m} \\
A_1 &= -m u_0^{-(m+1)} u_1 \\
A_2 &= \frac{1}{2} m(m+1) u_0^{-(m+2)} u_1^2 - m u_0^{-(m+1)} u_2 \\
A_3 &= -\frac{1}{6} m(m+1)(m+2) u_0^{-(m+3)} u_1^3 + m(m+1) u_0^{-(m+2)} u_1 u_2 - m u_0^{-(m+1)} u_3 \\
&\vdots
\end{aligned}$$

EXAMPLE: $f(u) = u^\gamma$ where γ is a decimal number.

$$\begin{aligned}
A_0 &= u_0^\gamma \\
A_1 &= \gamma u_0^{\gamma-1} u_1 \\
A_2 &= \gamma u_0^{\gamma-1} u_2 + \frac{1}{2} \gamma(\gamma-1) u_0^{\gamma-2} u_1^2 \\
A_3 &= \gamma u_0^{\gamma-1} u_3 + \gamma(\gamma-1) u_0^{\gamma-2} u_1 u_2 + \frac{1}{6} \gamma(\gamma-1)(\gamma-2) u_0^{\gamma-3} u_1^3 \\
A_4 &= \gamma u_0^{\gamma-1} u_4 + \gamma(\gamma-1) u_0^{\gamma-2} \left(\frac{1}{2} u_2^2 + u_1 u_3\right) + \frac{1}{2} \gamma(\gamma-1)(\gamma-2) u_0^{\gamma-3} u_1^2 u_2 \\
&\quad + \frac{1}{24} \gamma(\gamma-1)(\gamma-2)(\gamma-3) u_0^{\gamma-4} u_1^4
\end{aligned}$$

EXAMPLE: Consider the linear (deterministic) ordinary differential equation $d^2u/dx^2 - kx^p u = g$ with $u(1) = u(-1) = 0$. Write $L = d^2/dx^2$ and $Lu = g + kx^p u$. Operating with L^{-1} , we have $L^{-1}Lu = L^{-1}g + L^{-1}kx^p u$. Then

$$u = c_1 + c_2 x + gx^2/2 + L^{-1}kx^p u$$

Let $\sum_{n=0}^{\infty} u_n$ with $u_0 = c_1 + c_2 x + gx^2/2$. Then $u_{m+1} = L^{-1}kx^p u_m$ with $m \geq 0$. Thus

$$\begin{aligned} u &= \sum_{m=0}^{\infty} (L^{-1}kx^p)^m u_0 \\ u &= \sum_{m=0}^{\infty} (L^{-1}kx^p)^m c_1 + \sum_{m=0}^{\infty} (L^{-1}kx^p)^m c_2 x \\ &\quad + \sum_{m=0}^{\infty} (L^{-1}kx^p)^m gx^2/2 \\ u &= c_1 \phi_1(x) + c_2 \phi_2(x) + \Gamma(x) \end{aligned}$$

where

$$\begin{aligned} \phi_1(x) &= \sum_{m=0}^{\infty} k^m x^{mp+2m} / (mp+2m-1)(mp+2m) \\ \phi_2(x) &= \sum_{m=0}^{\infty} k^m x^{mp+2m+1} / (mp+2m)(mp+2m-1) \\ \Gamma(x) &= \sum_{m=0}^{\infty} (1/2)gk^m x^{mp+2m+2} / (mp+2m+1)(mp+2m+1) \end{aligned}$$

Since $u(1) = u(-1) = 0$, we have

$$\begin{aligned} c_1 \phi_1(1) + c_2 \phi_2(1) + \Gamma(1) &= 0 \\ c_1 \phi_1(-1) + c_2 \phi_2(-1) + \Gamma(-1) &= 0 \end{aligned}$$

Hence c_1 and c_2 are determined.

Suppose that in the above example, we let $k = 40$, $p = 1$, $g = 2$. Thus we consider the equation $d^2u/dx^2 - 40xu = 2$ with $u(-1) = u(1) = 0$. This is the one-dimensional case of the elliptic equation $\nabla^2 u = f(x, y, z) + k(x, y, z)u$ arising in problems of physics and engineering. Here $L = d^2/dx^2$ and we have $Lu = 2 + 40xu$. This is a relatively stiff case because of the large coefficient of

*Numerous other examples appear in the literature.

u , and the non-zero forcing function which yields an additional Airy-like function. Operating with L^{-1} yields $u + A + Bx + L^{-1}(2) + L^{-1}(40xu)$. Let $u_0 + A + Bx + L^{-1}(2) = A + Bx + x^2$ and let $u = \sum_{n=0}^{\infty} u_n$ with the components to be determined so that the sum is u . We identify $u_{n+1} = L^{-1}(40xu_n)$. Then all components can be determined, e.g.,

$$u_1 = (20/3)Ax^3 + (10/3)Bx^4 + 2x^5$$

$$u_2 = (80/9)Ax^6 + (200/63)Bx^7 + (10/7)x^8$$

⋮

An n -term approximant $\phi_n = \sum_{i=0}^{n-1} u_i$ with $n = 12$ for $x = 0.2$ is given by -0.135649 , for $x = 0.4$ is given by -0.113969 , for $x = 0.6$ is given by -0.083321 , for $x = 0.8$ is given by -0.050944 , and for $x = 1.0$ is, of course, zero. These easily obtained results are correct to seven digits. We see that a better solution is obtained and much more easily than by variational methods. The solution is found just as easily for nonlinear versions without linearization.

ANALYTIC SIMULANTS OF DECOMPOSITION SOLUTIONS:

We now introduce the convenient concept of "simulants" to solutions by decomposition. The m -term "approximant" ϕ_m to the solution u , indicated by $\phi_m[u]$, will mean m terms of the convergent series $\sum_{n=0}^{\infty} u_n$ which represents u in the decomposition method. If we have an equation $\Gamma u = g(t)$ where Γ is a general differential operator such as, for example,

$$\frac{d^2}{dt^2}(\cdot) + \alpha \frac{d}{dt}(\cdot) + \beta(\cdot)$$

and we write $g(t) = \sum_{n=0}^{\infty} g_n t^n$ but only use m terms of the series, we have the m -term approximant

$$\phi_m[g] = \sum_{n=0}^{m-1} g_n t^n$$

The corresponding solution of the equation is the simulant of the solution u , thus

$$\Gamma \sigma_m[u] = \phi_m[g]$$

Analogous to the limit $m \rightarrow \infty$ of $\phi_m[g] = g$, the limit as $m \rightarrow \infty$ of $\sigma_m[u] = u$.

Possible stopping rules arise in computerized calculation when the last computed simulant $\sigma_{n+1}[u]$ corresponds to $\sigma_n(u)$ to the number of decimal places of interest to us.

We can also conceive of using the entire series for g , for example, writing the sum of the infinite series but using a sequence of approximants to parameters $\alpha, \beta, \dots$, in Γ and a corresponding sequence of simulants $\sigma_n[u]$ as parametrized by $\phi_n[\alpha]$ or $\phi_n[\beta]$. In solving a partial differential equation by decomposition, we may develop a sequence of simulants $\sigma_m(u)$ for the solution u by concurrently improving the level of approximation of the coefficients, or the given conditions, or the input functions. For example in $L_1 \sigma_m[u] \div R_1 \sigma_m[u] = \phi_m[g]$ where $g(t, x) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} g_{\ell,m} t^{\ell} x^m$ we can compute each $\sigma_m(u)$ for given approximations of g , and of the initial conditions, or finally of coefficients such as

$$\alpha(t, x) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \alpha_{\ell,m} t^{\ell} x^m$$

We can also use the concept of simulants with asymptotic decomposition which is discussed in [4]. Consider the equation

$$d^2 u / dx^2 + u = g(x) = \sum_{n=0}^{\infty} g_n x^n \quad u(0) = c_0 \text{ and } u'(0) = c_1$$

By the asymptotic decomposition technique [5], the equation is written as

$$u = g(x) - d^2 u / dx^2$$

Then $u_1 = g(x) = \sum_{n=0}^{\infty} g_n x^n$ and $u_m = -(d^2 / dx^2) u_{m-1}$ for $m > 1$. We use an approximant of g or

$$\phi_m[g] = \sum_{n=0}^{m-1} g_n x^n$$

Then the simulant, or analytic simulant, of u is the solution of

$$\frac{d^2 \sigma_m[u]}{dx^2} + \sigma_m[u] = \phi_m[g]$$

or

$$\sigma_m[u] = \phi_m[g] - \frac{d^2 \sigma_m[u]}{dx^2}$$

$\sigma_m[u]$ is a series which we write as $\sum_{n=0}^{\infty} \sigma_n^{(m)}$ where

$$\sigma_0^{(m)} = \phi_m[g]$$

$$\sigma_n^{(m)} = -\frac{d^2}{dx^2} \sigma_{n-1}^{(m)} \quad (n \geq 1)$$

$$\sum_{n=0}^{\infty} \sigma_n^{(m)} = \sigma_m = -\frac{d^2}{dx^2} \sigma_{m-1}^{(m)}$$

It is straightforward enough that if we don't use all of g , we have only $\sigma_m[u]$ which approaches u in the limit as $m \rightarrow \infty$ in $\phi_m = \sum_{i=0}^{m-1} u_i$. In the same equation $u = g(x) - (d^2/dx^2)u$

$$u_0 = g(x) \quad u_m = -\frac{d^2}{dx^2} u_{m-1}$$

we can write

$$u_0 = \sum_{n=0}^{\infty} a_n^{(0)} x^n \quad \dots a_n^{(0)} = g_n$$

$$u_m = \sum_{n=0}^{\infty} a_n^{(m)} x^n$$

$$a_n^{(m)} = -(n+1)(n+2)a_{n+2}^{(m-1)}$$

so that

$$u = \sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_n^{(m)} x^n = \sum_{n=0}^{\infty} x^n \sum_{m=0}^{\infty} a_n^{(m)} = \sum_{n=0}^{\infty} a_n x^n$$

where $a_n = \sum_{m=0}^{\infty} a_n^{(m)}$. Therefore

$$u = \sum_{m=0}^{\infty} a_m x^m$$

is the solution for asymptotic decomposition. We make two observations:

- 1) The method works very well for nonlinear equations where we solve for the nonlinear term and express it in the above polynomials.
- 2) Ordinary differential equations with singular coefficients offer no special difficulty with asymptotic decomposition, e.g.,

$$(1-x)\left(d^2/dx^2\right)u + u = g(x)$$

$$u = g(x) + (x-1)d^2u/dx^2$$

$$u_0 = g(x)$$

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THE DECOMPOSITION METHOD IN SEVERAL DIMENSIONS

Mathematical physics deals with physical phenomena by modelling the phenomena of interest, generally in the form of nonlinear partial differential equations. It then requires an effective analysis of the mathematical model, such that the processes of modelling and of analysis yield results in accordance with observation and experiment. By this, we mean that the mathematical solution must conform to physical reality, i.e., to the real world of physics. Therefore, we must be able to solve differential equations, in space and time, which may be nonlinear and often stochastic as well, without the concessions to tractability which have been customary both in graduate training and in research in physics and mathematics. Nonlinear partial differential equations are very difficult to solve analytically, so methods such as linearization, statistical linearization, perturbation, quasi-monochromatic approximations, white noise representation of actual stochastic processes, etc. have been customary resorts. Exact solutions in closed form are not a necessity. In fact, for the world of physics only a sufficiently accurate solution matters. All modelling is approximation, so finding an improved general method of analysis of models also contributes to allowing development of more sophisticated modelling [1, 2].

Our objective in this chapter is to see how to use the decomposition method for partial differential equations. (In the next chapter, we will also introduce double decomposition which offers computational advantages for nonlinear ordinary differential equations and also for nonlinear partial differential equations.) These methods are applicable in problems of interest to theoretical physicists, applied mathematicians, engineers, and other disciplines and suggest developments in pure mathematics.

We now consider some generalizations for partial differential equations. Just as we solved for the linear differential operator term Lu and then operated on both sides with L^{-1} , we can now do the same for highest-ordered linear operator terms in all independent variables. If we have differentiations for example, with respect to x and t , represented by $L_x u$ and $L_t u$, we obtain equations for either of these. We can operate on each with the appropriate inverse. We begin by considering some illuminating examples.

Consider the example $\partial u / \partial t + \partial u / \partial x + f(u) = 0$ with $u(t = 0) = 1/2x$ and $u(x = 0) = -1/t$. For simplicity assume $f(u) = u^2$. By decomposition,

writing $L_t u = -(\partial/\partial x)u - u^2$, then writing $u = \sum_{n=0}^{\infty} u_n$ and representing u^2 by $u = \sum_{n=0}^{\infty} A_n$ derived for the specific function, we have

$$L_t u = -(\partial/\partial x) \sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n$$

$$u = u_0 - L_t^{-1}(\partial/\partial x) \sum_{n=0}^{\infty} u_n - L_t^{-1} \sum_{n=0}^{\infty} A_n$$

Consequently,

$$u_0 = u(x, 0) = 1/2x$$

$$u_1 = -L_t^{-1}(\partial/\partial x)u_0 - L_t^{-1}A_0$$

$$u_2 = -L_t^{-1}(\partial/\partial x)u_1 - L_t^{-1}A_1$$

$$\vdots$$

Substituting the $A_n \{u^2\}$ and summing, we have

$$u = \frac{1}{2x} + \frac{t}{4x^2} + \frac{t^2}{8x^3} + \frac{t^3}{16x^4} + \dots$$

$$u = \frac{1}{2x} [1 + \frac{t}{2x} + \frac{t^2}{4x^2} + \dots]$$

which converges if $(t/2x) < 1$ to $u = 1/(2x - t)$. If we solve for $L_x u$, we have

$$L_x u = -(\partial/\partial t)u - f(u) \quad \dots$$

$$u = u_0 - L_x^{-1}(\partial/\partial t) \sum_{n=0}^{\infty} u_n - L_x^{-1} \sum_{n=0}^{\infty} A_n$$

$$u_0 = -1/t$$

$$u_1 - L_x^{-1}(\partial/\partial t)u_0 - L_x^{-1}A_0 = -2x/t^2$$

$$\vdots$$

or $u = -(1/t)[1 + 2x/t + \dots]$ which converges near the initial condition if $2x/t < 1$ to $u = 1/(2x - t)$.

Both operator equations yield distinct series which converge to the same function with different convergence regions and different convergence rates. It is interesting to observe that convergence can be changed by the choice of the operator equations. In earlier work, the solutions (called "partial solutions") of each operator equation were combined to ensure use of all the

given conditions. We see that partial differential equations are solvable by looking at each dimension separately, and thus our assertion holds about the connection between the fields of ordinary and partial differential equations. There is, of course, much more to be said about this and we must leave further discussion to the growing literature, perhaps beginning with the introduction represented by the given references.

Consider the equation $u_{tt} - u_{xx} + (\partial/\partial t)f(u) = 0$ where $f(u(x,t))$ is an analytic function. Let L_t represent $\partial^2/\partial t^2$ and let L_x represent $\partial^2/\partial x^2$. We now write the equation in the form

$$L_t u - L_x u = -(\partial/\partial t)f(u)$$

Using the decomposition method, we can solve for either linear term; thus,

$$L_t u = L_x u - (\partial/\partial t)f(u)$$

$$L_x u = L_t u + (\partial/\partial t)f(u)$$

Operating with the inverse operators, we have

$$u = \Phi_t + L_t^{-1} L_x u - L_t^{-1} (\partial/\partial t)f(u)$$

$$u = \Phi_x + L_x^{-1} L_t u + L_x^{-1} (\partial/\partial t)f(u)$$

where the Φ_t, Φ_x are evaluated from the given initial/boundary conditions. Generally, either can be used to get a solution, so solving a partial differential equation is analogous to solving an ordinary differential equation with the L_x operator in the first equation and the L_t operator in the second equation assuming the role of the remainder operator R in an ordinary differential equation. The $(\partial/\partial t)f(u)$ is a nonlinearity handled as before. The solution depends on the explicit $f(u)$ and the specified conditions on the wave equation.

To illustrate the procedure, we first consider the case with $f(u) = 0$ in order to see the basic procedure most clearly. We have, therefore, $u_{tt} - u_{xx} = 0$ and we will take as given conditions $u(0, x) = 0$, $u(t, 0) = 0$, $u(\pi/2, x) = \sin x$, $u(t, \pi/2) = \sin t$.

Let $L_t = \partial^2/\partial t^2$ and $L_x = \partial^2/\partial x^2$ and write the equation as $L_t u = L_x u$. Following our procedure, we can write either $u = c_1 k_1(x) + c_2 k_2(x)t + L_t^{-1} L_x u$ or $u = c_3 k_3(t) + c_4 k_4(t)x + L_x^{-1} L_t u$.

Define $\Phi_t = c_1 k_1(x) + c_2 k_2(x)t$ and $\Phi_x = c_3 k_3(t) + c_4 k_4(t)x$ to rewrite the above as $u = \Phi_t + L_t^{-1} L_x u$ and $u = \Phi_x + L_x^{-1} L_t u$.

The first approximant ϕ_1 is $u_0 = \Phi_t$. The two-term approximant ϕ_2 is $u_0 + u_1$ where $u_1 = L_t^{-1} L_x u_0$. Applying the t conditions $u(0, x) = 0$ and $u(\pi/2, x) = \sin x$ to the one-term approximant $\phi_1 = u_0 = c_1 k_1(x) + c_2 k_2(x)t$ we have

$$\begin{aligned} c_1 k_1(x) &= 0 \\ c_2 k_2(x) \pi/2 &= \sin x \end{aligned}$$

or $c_2 = 2/\pi$ and $k_2(x) = \sin x$.

The next term is $u_1 = L_t^{-1} L_x u_0 = L_t^{-1} L_x [c_2 t \sin x]$, and we continue in the same manner to obtain $u_2, u_3, \dots, u_n$ for some n . Clearly, for any n ,

$$u_n = (L_t^{-1} L_x)^n u_0 = c_2 (\sin x) (-1)^n t^{2n+1} / (2n+1)!$$

If we write for the m -term approximant, we have for the two cases:

$$\Phi_m = c_2 \sin x \sum_{n=0}^{m-1} (-1)^n t^{2n+1} / (2n+1)!$$

Since $\Phi_m(\pi/2, x) = \sin x$

$$\begin{aligned} c_1 k_1(x) &= 0 \\ c_2 \sin x \sum_{n=0}^{m-1} (\pi/2)^{2n+1} / (2n+1)! &= \sin x \end{aligned}$$

As $m \rightarrow \infty$, $c_2 \rightarrow 1$. The sum approaches $\sin t$ in the limit. Hence our approximation becomes an exact solution $u = \sin x \sin t$. (The same result can be found from the other operator equation.)

Thus, in this case, the series is summed. In general, it is not and we get a series with a convergence region in which numerical solutions stabilize quickly to a solution within the range of accuracy needed. Adding the nonlinear term does not change this; the A_n converge rapidly and the procedure amounts to a generalized Taylor expansion for the solution about the function u_0 rather than about a point. We call the solution an approximation because it is usually not a closed form solution; however, we point out that all modelling is approximation, and a closed form which

necessarily changes the physical problem by employing linearization is not more desirable and is, in fact, generally less desirable in that the problem has been changed to a different one. Recent work by Y. Cherruault and L. Gabet on the mathematical framework has provided the rigorous basis for the decomposition method. The method is clearly useful to physicists and other disciplines in which real problems must be mathematically modelled and solved. The method is also adaptable to systems of nonlinear, stochastic, or coupled boundary conditions (as shown in the author's earlier books). The given conditions must be consistent with the physical problem being solved.

Consider the same example $u_t - u_{xx} = 0$ with $0 \leq x \leq \pi$ and $t \geq 0$, assuming now the conditions which yield an interesting special case for the methodology.

$$\begin{aligned} u(x, 0) &= \sin x & u(0, t) &= 0 \\ u_t(x, 0) &= 0 & u(\pi, t) &= 0 \end{aligned}$$

Decomposition results in the equations

$$\begin{aligned} u &= c_1 k_1(t) + c_2 k_2(t)x + L_x^{-1} L_t u \\ u &= c_3 k_3(x) + c_4 k_4(x)t + L_x^{-1} L_t u \end{aligned}$$

The one-term approximant $\varphi_1 = u_0$ in the first equation is

$$u_0 = c_1 k_1(t) + c_2 k_2(t)x$$

Satisfying conditions on x we have $c_1 k_1(t) = 0$ and $c_2 k_2(t)\pi = 0$. Hence $u_0 = 0$. The first equation clearly does not contribute in this special case; we need only the second. Thus,

$$u_0 = c_3 k_3(x) + c_4 k_4(x)t$$

Applying conditions on t , $c_3 k_3(x) = \sin x$ and $c_4 k_4(x)t = 0$. Hence

$$\begin{aligned} u_0 &= \sin x \\ u_1 &= L_t^{-1} L_x u_0 = (-t^2/2!) \sin x \\ u_2 &= L_t^{-1} L_x u_1 = (t^4/4!) \sin x \\ &\vdots \end{aligned}$$

or

$$u = (1 - t^2/2! + t^4/4! - \dots) \sin x = \sin x \cos t$$

We are dealing with a methodology for solution of physical systems which have a solution, and we seek to find this solution without changing the problem to make it tractable. The conditions must be known; otherwise, the model is not complete. If the solution is known, but the initial conditions are not, they can be found by mathematical exploration and consequent verification.

Finally, we consider the general form

$$L_x u = L_t u + (\partial/\partial t)f(u)$$

$$L_t u = L_x u - (\partial/\partial t)f(u)$$

or

$$u = c_1 k_1(t) + c_2 k_2(t)x + L_x^{-1} L_t u + L_x^{-1} (\partial/\partial t)f(u)$$

$$u = c_3 k_3(x) + c_4 k_4(x)t + L_t^{-1} L_x u - L_t^{-1} (\partial/\partial t)f(u)$$

We now let $u = \sum_{n=0}^{\infty} u_n$ and $f(u) = \sum_{n=0}^{\infty} A_n$ and note this is equivalent to letting u as well as $f(u)$ be equal to $\sum_{n=0}^{\infty} A_n$ where the A_n are generated for the specific $f(u)$. If $f(u) = u$ we obtain $A_0 = u_0$, $A_1 = u_1, \dots$, i.e., $\sum_{n=0}^{\infty} A_n \{u\} = \sum_{n=0}^{\infty} u_n$. Now

$$u = u_0 + L_x^{-1} L_t \sum_{n=0}^{\infty} u_n + L_x^{-1} (\partial/\partial t) \sum_{n=0}^{\infty} A_n$$

$$u = u_0 + L_t^{-1} L_x \sum_{n=0}^{\infty} u_n - L_t^{-1} (\partial/\partial t) \sum_{n=0}^{\infty} A_n$$

To go further we must have the conditions on u . Suppose we choose

$$u(0, t) = 0$$

$$u(x, 0) = f(x)$$

$$u(\pi, t) = 0$$

$$u_t(x, 0) = 0$$

Satisfying the conditions, we have $c_1 k_1(t) = c_2 k_2(t) = 0$ or $u_0 = 0$. Therefore the equation involving L_x^{-1} does not contribute. In the remaining equation we get $c_3 k_3(x) = f(x)$ and $c_4 k_4(x)t = 0$. Hence,

$$\begin{aligned}
u_0 &= f(x) \\
u_1 &= L_t^{-1} L_x u_0 - L_t^{-1} (\partial / \partial t) A_0 \\
u_2 &= L_t^{-1} L_x u_1 - L_t^{-1} (\partial / \partial t) A_1 \\
&\vdots
\end{aligned}$$

Thus, components of u are determined and we can write $\varphi_n = \sum_{m=0}^{n-1} u_m$ as an n -term approximation converging to u as $m \rightarrow \infty$. To complete the problem, $f(u)$ must be explicitly given so that we can generate the A_n . We see that the solution depends both on the specified $f(x)$ and on the given conditions.

RESULTS AND POTENTIAL APPLICATIONS:

The decomposition method provides a single method for linear and nonlinear multidimensional problems and has now been applied to a wide variety of problems in many disciplines. One application of the decomposition method is in the development of numerical methods for the solution of nonlinear partial differential equations. Decomposition permits us to have an essentially unique numeric method tailored individually for each problem. In a preliminary test of this notion, a numerical decomposition was performed on Burger's equation. It was found that the same degree of accuracy could be achieved in two percent of the time required to compute a solution using Runge-Kutta procedures. The reasons for this are discussed in [2].

EXAMPLE: Consider the dissipative wave equation

$$u_{tt} - u_{xx} + (\partial / \partial t)(u^2) = g = -2 \sin^2 x \sin t \cos t$$

with specified conditions $u(0,t) = u(\pi,t) = 0$ and $u(x,0) = \sin x$, $u_t(x,0) = 0$. We have

$$u_0 = k_1(x) + k_2(x)t + L_t^{-1}g$$

from the $L_t u$ equation and use of the two-fold definite integration L_t^{-1} and

$$u_0 = k_3(t) + k_4(t)x - L_x^{-1}g$$

from the $L_x u$ equation and application of the two-fold indefinite integration L_x^{-1} . Either solution, which we have called "a partial solution", is already correct: they are equal when the spatial boundary conditions depend on t and

the initial conditions depend on x . When conditions on one variable are independent of the other variable, the partial solutions are asymptotically equal. From the specified conditions $u(x,0) = \sin x$ and $u_t(x,0) = 0$

$$k_1(x) = \sin x$$

$$k_2(x) = 0$$

so that

$$u_0 = \sin x - (\sin^2 x)(t/2 - 1/4 \sin 2t)$$

The n -term approximant φ_n is

$$\varphi_n = \sum_{i=0}^{n-1} (L_t^{-1} L_x)^i u_0 - L_t^{-1} (\partial/\partial t) \sum_{m=0}^{n-1} A_m$$

where $A_0 = u_0 u_0$, $A_1 = u_0 u_1$, $A_2 = u_0 u_1 + u_1 u_1$, $\dots$

The contribution of the term $L^{-1}g$ to u_0 results in self-canceling terms, or "noise" terms. Hence, rather than calculating exactly, we observe that if we use only $u_0 = \sin x$, we get $u_1 = (-t^2/2!) \sin x$, $u_2 = (t^4/4!) \sin x$, etc. which appears to be $u = \cos t \sin x$. Thus the solution is $u = \cos t \sin x$ + other terms. We write $u = \cos t \sin x + N$ and substitute in the equation for u and find that $N = 0$, i.e., the neglected terms are self-canceling and $u = \cos t \sin x$ is the correct solution. It is often useful to look for patterns to minimize or avoid unnecessary work.

To summarize the procedure, we can write the two operator equations

$$L_t u = g + L_x u - (\partial/\partial t) f(u)$$

$$L_x u = -g + L_t u + (\partial/\partial t) f(u)$$

Applying the operators L_t^{-1} to the first equation or L_x^{-1} to the second,

$$u = k_1(x) + k_2(x)t + L_t^{-1}g + L_t^{-1}L_x u - L_t^{-1}(\partial/\partial t)f(u)$$

$$u = k_3(t) + k_4(t)x - L_x^{-1}g + L_x^{-1}L_t u + L_x^{-1}(\partial/\partial t)f(u)$$

Substituting $u = \sum_{n=0}^{\infty} u_n$ and $f(u) = \sum_{n=0}^{\infty} A_n$, where A_n are defined for $f(u)$, we have

$$u_0 = k_1(x) + k_2(x)t + L_t^{-1}g$$

$$u_{n+1} = L_t^{-1}L_x u_n - L_t^{-1}(\partial/\partial t)A_n$$

α

$$u_0 = k_3(t) + k_4(t)x - L_x^{-1}g$$

$$u_{n+1} = L_x^{-1}L_t u_n + L_x^{-1}(\partial/\partial t)A_n$$

where, if both make a contribution, either can be solved to get an n -term approximation φ_n satisfying the conditions to get the approximant to u . The partial solution as noted earlier is sufficient. The integration "constants" are evaluated from the specific conditions, i.e., each φ_n satisfies the appropriate conditions for any n .

The following example illustrates avoidance of often difficult integrations for solution to sufficient accuracy in a particular problem. The exact solution, which will then be used for a solution to two or three decimal places in physical problems, is often unnecessary. We can sometimes guess the sum of the decomposition series in a closed form and sometimes determine it by Euler, Padé, Shanks, or modified Shanks acceleration techniques. However, whether we see this final form or not, the series is the solution we need.

EXAMPLE: $u_{tt} - u_{xx} + (\partial/\partial t)f(u) = g(x, t)$

Let $g = 2e^{-t} \sin x - 2e^{-2t} \sin x \cos x$ and $f(u) = uu_x$. The initial/boundary conditions are:

$$u(x, 0) = \sin x$$

$$u_t(x, 0) = -\sin x$$

$$u(0, t) = u(\pi, t) = 0$$

Let $L_t = \partial^2/\partial t^2$ and write the equation as

$$L_t u = g - (\partial/\partial t)f(u) + (\partial^2/\partial x^2)u$$

(By the partial solutions technique, we need only the one operator equation the $\partial^2/\partial x^2$ is treated like the R operator in an ordinary differential equation.)

Operating with L_t^{-1} defined as the two-fold integration from 0 to t and writing $u = \sum_{n=0}^{\infty} u_n$ and $f(u) = \sum_{n=0}^{\infty} A_n$ where the A_n are generated for $f(u) = uu_x$, we obtain the decomposition components

$$u_0 = u(x, 0) + tu_1(x, 0) + L_t^{-1} g$$

$$u_{m+1} = -L_t^{-1}(\partial / \partial t) A_m + L_t^{-1}(\partial^2 / \partial x^2) u_m$$

for $m \geq 0$. Then, since $\sum_{n=0}^{\infty} u_n$ is a (rapidly) converging series, the partial sum $\phi_m = \sum_{i=0}^{m-1} u_i$ is our approximant to the solution.

We can calculate the above terms $u_0, u_1, \dots, u_m$ as given. However, since we calculate approximants, we can simplify the integrations by approximating g by a few terms of its double Maclaurin series representation. Thus we will drop terms involving t^3 and x^3 and higher terms. Then

$$e^{-t} = 1 - t + \frac{t^2}{2}$$

$$\sin x = x$$

$$\cos x = 1 - \frac{x^2}{2}$$

so that

$$g = 2\left(1 - t + \frac{t^2}{2}\right)(x) - 2(1 - 2t + 2t^2)x\left(1 - \frac{x^2}{2}\right)$$

Then $L^{-1} g \approx 0$ to the assumed approximation. Hence

$$u_0 = x - tx$$

$$u_1 = L_t^{-1}(\partial / \partial t) A_0 + L_t^{-1}(\partial^2 / \partial x^2) u_0 = xt^2/2$$

Thus the two-term approximation is

$$\phi_2 = u_0 + u_1 = x - tx + xt^2/2$$

$$= (1 - t^2 + t^2/2)x = e^{-t} \sin x$$

Although we can calculate more terms using u_{m+1} for $m \geq 0$, substitution verifies that $u = e^{-t} \sin x$ is already the correct solution.

If we need to recognize the exact solution, we can carry the series for g to a higher approximation to see the clear convergence to $e^{-t} \sin x$. Once we guess the solution may be $e^{-t} \sin x$, we can verify it by substitution, or substitute $e^{-t} \sin x + N$ and show that $N = 0$.

EQUALITY OF PARTIAL SOLUTIONS:

In solving linear or nonlinear partial differential equations by decomposition, we can solve separately for the term involving either of the highest-ordered linear operators* and apply the appropriate inverse operator to each equation. Each equation is solved for an n-term approximation. The solutions of the individual equations (e.g., for $L_x u$, $L_y u$, $L_x u$, or $L_y u$ in a four-dimensional problem) have been called "partial solutions" in earlier work. The reason was that they were to be combined into a general solution using all the conditions. However, it has now been shown [4] that in the general case, the partial solutions are equal and each is the solution. This permits a simpler solution which is not essentially different from solution of an ordinary differential equation. The other operators, if we solve for $L_x u$, for example, simply become the R operator in $Lu + Ru + Nu = g$. The procedure is now a single procedure for linear or nonlinear ordinary or partial differential equations. (When the u_0 term in one operator equation is zero, that equation does not contribute to the solution and the complete solution is obtained from the remaining equation or equations.) We will show that the partial solutions from each equation lead to the same solution (and explain why the one partial solution above does not contribute to the solution).

Consider the equation $L_x u + L_y u + Nu = 0$ with $L_x = \partial^2 / \partial x^2$ and $L_y = \partial^2 / \partial y^2$, although no limitation is implied and Nu is an analytic term accurately representable by the A_n polynomials. We choose conditions:

$$\begin{array}{ll} u(a_1, y) = \alpha_1(y) & u(x, b_1) = \beta_1(x) \\ u(a_2, y) = \alpha_2(y) & u(x, b_2) = \beta_2(x) \end{array}$$

Solving for the linear operator terms

$$\begin{array}{l} L_x u = -L_y u - Nu \\ L_y u = -L_x u - Nu \end{array}$$

Using the "x - solution", we have

$$L_x^{-1} L_x u = -L_x^{-1} L_y u - L_x^{-1} Nu$$

* Purely nonlinear equations or equations in which the highest-ordered operator is nonlinear require further consideration [3].

where L_x^{-1} is an indefinite two-fold integration and $L_x^{-1}(\cdot) = \iint (\cdot) dx dx + \Phi_x$ where $\Phi_x = \xi_0(y) + x\xi_1(y)$. The $\xi_0(y)$ and $\xi_1(y)$ are matching coefficients to the boundary conditions. Hence, $L_x^{-1}L_x u = u - \Phi_x$ and $\Phi_x = \xi_0(y) + x\xi_1(y)$, where ξ_0 and ξ_1 are the integration "constants". We now have

$$u = \Phi_x - L_x^{-1}L_y u - L_x^{-1}Nu$$

We let $u = \sum_{n=0}^{\infty} u_n$ and $Nu = \sum_{n=0}^{\infty} A_n$ where the A_n are defined specifically for Nu . The u_0 term is normally taken as Φ_x (or $\Phi_x + L_x^{-1}g$ when there is an inhomogeneous term).

We can take a somewhat different approach (double decomposition), discussed in Chapter 4, and decompose Φ_x as well. In that case, we write

$$\Phi_{x,m} = \xi_{0,m} + x\xi_{1,m}$$

and

$$u_0 = \Phi_{x,0}$$

Then instead of the following components being given by

$$u_{m+1} = -L_x^{-1}L_y u_m - L_x^{-1}A_m \text{ for } m \geq 1$$

we would have

$$\begin{aligned} u_1 &= \Phi_{x,1} - L_x^{-1}L_y u_0 - L_x^{-1}A_0 \\ u_2 &= \Phi_{x,2} - L_x^{-1}L_y u_1 - L_x^{-1}A_1 \\ u_3 &= \Phi_{x,3} - L_x^{-1}L_y u_2 - L_x^{-1}A_2 \\ &\vdots \\ u_m &= \Phi_{x,m} - L_x^{-1}L_y u_{m-1} - L_x^{-1}A_{m-1} \end{aligned}$$

The boundary conditions

$$\varphi_{m+1}(a_1, y) = \alpha_1(y)$$

$$\varphi_{m+1}(a_2, y) = \alpha_2(y)$$

determine $\xi_{0,m}$ and $\xi_{1,m}$. Then

$$u_0 = \Phi_{x,0}$$

$$u_1 = \Phi_{x,1} - L_x^{-1}L_y \Phi_{x,0} - L_x^{-1}A_0$$

$$\begin{aligned}
u_2 &= \Phi_{x,2} - L_x^{-1} L_y \Phi_{x,1} + (L_x^{-1} L_y)^2 \Phi_{x,0} + L_x^{-1} L_y L_x^{-1} A_0 - L_x^{-1} A_1 \\
u_3 &= \Phi_{x,3} - L_x^{-1} L_y \Phi_{x,2} + (L_x^{-1} L_y)^2 \Phi_{x,1} - (L_x^{-1} L_y)^3 \Phi_{x,0} \\
&\quad - (L_x^{-1} L_y)^2 L_x^{-1} A_0 + L_x^{-1} L_y L_x^{-1} A_1 - L_x^{-1} A_2 \\
&\quad \vdots \\
u_m &= \sum_{n=0}^m (-L_x^{-1} L_y)^n \Phi_{x,m-n} - \sum_{n=0}^{m-1} (-L_x^{-1} L_y)^{m-1-n} L_x^{-1} A_n
\end{aligned}$$

Now

$$u(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^m (-L_x^{-1} L_y)^n \Phi_{x,m-n} - \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} (-L_x^{-1} L_y)^{m-1-n} L_x^{-1} A_n$$

which is the solution to the equation in x . We can proceed in the same manner with the y equation; however, we return to the ordinary or regular decomposition for clarity. The additional decomposition is of no advantage for initial-value problems but speeds up convergence in boundary-value problems by giving us results for $u_0, u_1, \dots$ that are obtained by correcting constants of integration as we proceed, so that we can then use a corrected initial value without more matching to conditions. From the x partial solution.

$$\begin{aligned}
u_0 &= \Phi_x = \tilde{\zeta}_0(y) + x \tilde{\zeta}_1(y) \\
u_1 &= -L_x^{-1} L_y u_0 - L_x^{-1} A_0 \\
u_2 &= [L_x^{-1} L_y]^2 u_0 + [L_x^{-1} L_y] L_x^{-1} A_0 - L_x^{-1} A_1 \\
u_3 &= -[L_x^{-1} L_y]^3 u_0 - [L_x^{-1} L_y]^2 L_x^{-1} A_0 + [L_x^{-1} L_y] L_x^{-1} A_1 - L_x^{-1} A_2 \\
&\quad \vdots
\end{aligned}$$

From the y partial solution

$$\begin{aligned}
u_0 &= \Phi_y = \eta_0(x) + y \eta_1(x) \\
u_1 &= L_y^{-1} L_x u_0 - L_y^{-1} A_0 \\
u_2 &= [L_y^{-1} L_x]^2 u_0 + [L_y^{-1} L_x] L_y^{-1} A_0 - L_y^{-1} A_1 \\
&\quad \vdots
\end{aligned}$$

The m th approximant $\varphi_m = \sum_{n=0}^{m-1} u_n$ in each case above. The integration constants are determined by satisfying the given conditions by solution of the

matrix equations

$$\begin{bmatrix} 1 & a_1 \\ 1 & a_2 \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = \begin{bmatrix} \alpha_1(y) \\ \alpha_2(y) \end{bmatrix}$$

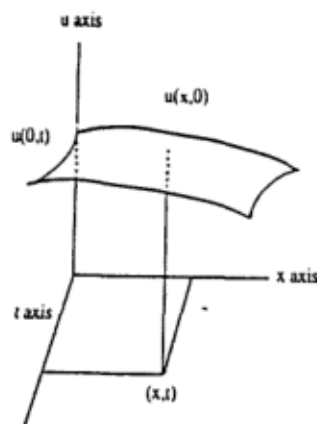
$$\begin{bmatrix} 1 & b_1 \\ 1 & b_2 \end{bmatrix} \begin{bmatrix} \eta_0 \\ \eta_1 \end{bmatrix} = \begin{bmatrix} \beta_1(x) \\ \beta_2(x) \end{bmatrix}$$

to determine $\xi_0, \xi_1, \eta_0, \eta_1$.

The limit as $m \rightarrow \infty$ of φ_m for the x equation and the y equation are respectively the x partial solution and y partial solution and are identically equal; either is the actual solution which satisfies the differential equation uniquely for the given initial/boundary conditions.

REMARKS: Suppose we consider a partial differential equation whose solution is the surface $u(x,t)$ in a Cartesian system. We write this in the form $L_t u + L_x u + Ru + Nu = g$. The intersections with the u, x plane is $u(x,0) = f(x)$. As t increases from this initial value, the surface u is generated. Similarly, the intersections with the u, t plane is $u(0,t) = g(t)$. As x increases, the surface is generated.

The partial solutions represent these two possibilities, i.e., we can determine u either by starting from $f(x)$ and using the t equations ($L_t u = g - L_x u - Ru - Nu$) or starting from $g(t)$ and using the x equation ($L_x u = g - L_t u - Ru - Nu$) and the appropriate inversions for each.



Consider, as an example, the simple heat flow equation $u_t = u_{xx}$, given that $u(x,0) = \sin(\pi x/\ell)$ and $u(0,t) = u(\ell,t) = 0$. The solution is

$$e^{-\pi^2 t/\ell^2} \sin(\pi x/\ell)$$

The equation in t is $L_t u = L_x u$. Applying the L_t^{-1} operator, we get $u = u(x, 0) + L_t^{-1} L_x \sum_{n=0}^{\infty} u_n$.

$$u_0 = \sin(\pi x/\ell)$$

$$u_1 = L_t^{-1} L_x u_0 = -(\pi^2 t/\ell^2) \sin(\pi x/\ell)$$

$$u_2 = (\pi^4 t^2/\ell^4) \sin(\pi x/\ell)$$

$$\vdots$$

$$u = \sum_{n=0}^{\infty} u_n = e^{-\pi^2 t/\ell^2} \sin(\pi x/\ell)$$

which is the complete solution usually obtained more easily than the textbook solutions of this problem. The x equation is $L_x u = L_t u$. Applying L_x^{-1} ,

$$u = k_1(t) + x k_2(t) + L_x^{-1} L_t \sum_{n=0}^{\infty} u_n$$

We see $u_0 = k_1(t) + x k_2(t) = 0$ which means all following components must be zero so this equation, as previously stated, makes no contribution. Here, the x conditions (boundary conditions) $u(0, t)$ and $u(\ell, t)$ do not depend on t . Hence the partial solutions are asymptotically equal—they both are zero at $t \rightarrow \infty$.

Use of the partial solutions technique as compared with the author's earlier treatments of partial differential equations [4] leads to substantially decreased computation and minimization of extraneous noise [5]. Also we note that the convergence region can be changed by the choice of the operator equation.

Since the partial solutions are equal, we need solve only one operator equation. (Exceptions occur when the u_0 term is zero in one of the equations or the initial/boundary conditions for one operator equation do not involve remaining variables.) The remaining highest-ordered linear differential operators can now be treated like the remainder operator R . Thus ordinary or partial differential equations are solved by a single method. The decision as to which operator equation to solve in a multidimensional problem is made

on the basis of the best known conditions and possibly also on the basis of the operator of lowest order to minimize integrations.

To make the procedure as clear as possible, we consider first the case where $Nu = 0$, i.e., a linear partial differential equation in R^2 ,

$$L_x u + L_y u + Ru = g$$

where $L_x = \partial^2 / \partial x^2$ and $L_y = \partial^2 / \partial y^2$ with the boundary conditions specified by boundary-operator equations

$$B_1 u|_{x=\xi_1} = \beta_1(y) \quad B_2 u|_{x=\xi_2} = \beta_2(y)$$

$$B_3 u|_{y=\eta_1} = \gamma_1(x) \quad B_4 u|_{y=\eta_2} = \gamma_2(x)$$

Solving for $L_x u$ we have $L_x u = g - L_y u - Ru$ and operating with L_x^{-1} we have

$$u = \Phi_x + L_x^{-1}g - L_x^{-1}L_y u - L_x^{-1}Ru$$

where Φ_x satisfies $L_x \Phi_x = 0$. The inverse operator L_x^{-1} is a two-fold (indefinite) integration operator since L_x is a second-order differential operator. The "constants of integration" are added for each indefinite integration. ** (This makes notation consistent with decomposition solution of initial-value problems where for $L_t = \partial / \partial t$ we define $L_t^{-1}[\cdot] = \int_0^t [\cdot] dt$, and for $L_t = \partial^2 / \partial t^2$, we have a two-fold definite integration from zero to t .) Now the decomposition $u = \sum_{n=0}^{\infty} u_n$ yields

$$u = \Phi_x + L_x^{-1}g - L_x^{-1}L_y \sum_{n=0}^{\infty} u_n - L_x^{-1}R \sum_{n=0}^{\infty} u_n$$

where we identify

$$u_0 = \Phi_x + L_x^{-1}g$$

**For linear ordinary differential equations, but not partial differential equations, we can view L_t^{-1} as a pure two-fold integration operator not involving constants and simply add the Φ_x for a general solution.

as the initial term of the decomposition. Since L_x^{-1} is a two-fold integration, $\Phi_x = c_0(y) + xc_1(y)$ and $u_0 = c_0(y) + xc_1(y) + L_x^{-1}g$. Hence

$$u = u_0 - L_x^{-1}L_y \sum_{n=0}^{\infty} u_n - L_x^{-1}R \sum_{n=0}^{\infty} u_n$$

Then for $m \geq 0$

$$u_{m+1} = -L_x^{-1}L_y u_m - L_x^{-1}R u_m$$

for components after u_0 . Consequently, all components of the decomposition are identified and calculable. We can now form successive approximants $\phi_n = \sum_{m=0}^{n-1} u_m$ as n increases which we match to the boundary conditions. Thus $\phi_1 = u_0$, $\phi_2 = \phi_1 + u_1$, $\phi_3 = \phi_2 + u_2$, serve as approximate solutions of increasing accuracy as $n \rightarrow \infty$ and must, of course, satisfy the boundary conditions. Beginning with $\phi_1 = u_0 = c_0(y) + xc_1(y) + L_x^{-1}g$, we evaluate c_0 and c_1 from the boundary conditions

$$B_1 \phi_1 \Big|_{x=b_1} = \beta_1(y) \quad B_2 \phi_1 \Big|_{x=b_2} = \beta_2(y)$$

Thus ϕ_1 is now determined. Since u_0 or ϕ_1 is now completely known, we form

$$u_1 = -L_x^{-1}L_y u_0 - L_x^{-1}R u_0$$

Then

$$\phi_2 = \phi_1 + u_1$$

which must satisfy the boundary conditions. Continuing to an acceptable approximation ϕ_n , we must match the conditions at each value of n for a satisfactory solution as decided either by substitution or by a stabilized numerical answer to a desired number of decimals.

For the special case of a linear ordinary differential equation, we have the simpler alternative of using the unevaluated u_0 to get u_1 , simply carrying along the constants in u_0 and continuing in this way to some ϕ_n . Thus, in this case, only one evaluation of the constants is necessary. (For nonlinear cases or for partial differential equations, the simpler procedure is not generally possible.) To make this clear, consider some examples:

$$d^2u/dx^2 - 40xu = 2$$

$$u(-1) = u(1) = 0$$

Write

$$Lu = 2 + 40xu$$

or

$$u = c_1 + c_2x + L^{-1}(2) + 40L^{-1}x \sum_{n=0}^{\infty} u_n$$

We can identify

$$u_0 = c_1 + c_2x + L^{-1}(2) = c_1 + c_2x + x^2$$

Now instead of evaluating the constants at this stage, we write

$$u_1 = 40L^{-1}xu_0 - (20/3)c_1x^3 + (10/3)c_2x^4 + 2x^5$$

$$u_2 = 40L^{-1}xu_1 - (80/9)c_1x^5 + (200/63)c_2x^7 + (10/7)x^8$$

⋮

If, for example, ϕ_3 is sufficient,

$$\begin{aligned}\phi_3 = & c_1 + c_2x + x^2 + (20/3)c_1x^3 \\ & + (10/3)c_2x^4 + 2x^5 + (80/9)c_1x^5 \\ & + (200/63)c_2x^7 + (10/7)x^8\end{aligned}$$

Imposing the boundary conditions at -1 and 1 on ϕ_3 we have $\phi_3(-1) = \phi_3(1) = 0$ or

$$\begin{pmatrix} 149/9 & 473/63 \\ 29/9 & -53/63 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -31/7 \\ -3/7 \end{pmatrix}$$

determining c_1 and c_2 and therefore ϕ_3 in a single evaluation. (By ϕ_{12} this yielded seven-digit accuracy.)

Another example is $d^2y/dx^2 + 2x dy/dx = 0$ with $y(0) = 0$ and $y(a) = 1$. The solution is $y(x) = (\operatorname{erf} x) / (\operatorname{erf} a)$ or

$$\frac{x - (x^3/3) + (x^5/10) - (x^7/42) + \dots}{a - (a^3/3) + (a^5/10) - (a^7/42) + \dots}$$

Write $Ly = -2x(dy/dx)y$ or

If we satisfy the conditions with $\varphi_1 = y_0$ we have $y_0 = x/a$ as our first approximant. If we continue to some φ_n and evaluate only then, we have

$$\begin{aligned}y_1 &= -2L^{-1}x(d/dx)y_0 = -2L^{-1}x(d/dx)(c_1 + c_2x) = -c_2x^3/3 \\y_2 &= -2L^{-1}x(d/dx)y_1 = -2L^{-1}x(d/dx)(-c_2x^3/3) = -c_2x^5/10 \\&\vdots\end{aligned}$$

If we stop at

$$\begin{aligned}\varphi_3 &= y_0 + y_1 + y_2 \\ \varphi_3 &= c_1 + c_2x - c_2x^3/3 + c_2x^5/10\end{aligned}$$

and now satisfy the conditions we have

$$y = \frac{x - (x^3/3) + (x^5/10)}{a - (a^3/3) + (a^5/10)}$$

which is $(\operatorname{erf} x)/(\operatorname{erf} a)$ to this approximation which can, of course, be carried as far as we like.

For linear ordinary differential equations, both procedures will work, i.e., we can use the evaluated u_0 to get u_1 , add it to the unevaluated u_0 to get φ_2 , then satisfy the conditions at the boundaries, or carry the constants along as in the examples. The last procedure is most convenient because of the single evaluation: the first is more general since it applies to nonlinear ordinary differential equations and linear or nonlinear partial differential equations as well [6.7].

EVALUATION OF COEFFICIENTS FOR A LINEAR PARTIAL DIFFERENTIAL EQUATION:

$$u_{xx} - u_{yy} = 0$$

for $0 \leq x \leq \pi/2$, $0 \leq y \leq \pi/2$ with the conditions given as

$$u(0, y) = u(x, 0) = 0$$

$$u(\pi/2, y) = \sin y$$

$$u(x, \pi/2) = \sin x$$

Let $L_x = \partial^2/\partial x^2$ and $L_y = \partial^2/\partial y^2$ to write $L_x u = L_y u$. If we apply inversion to the L_x operator, we have $u = k_1(y) + xk_2(y) + L_x^{-1}L_y u$.

Now $\Phi_x = k_1(y) + xk_2(y)$. Hence $u = \Phi_x + L_x^{-1}L_y u$. The one-term approximant is $\varphi_1 = u_0 = \Phi_x$. A two-term approximant is $\varphi_2 = \varphi_1 + u_1$ and $u_1 = L_x^{-1}L_y u_0$. The x conditions are $u(0, y) = 0$ and $u(\pi/2, y) = \sin y$. Applying these conditions to $k_1(y) + xk_2(y)$ we see that $k_1 = 0$ and $k_2 = (2/\pi)\sin y$. Thus, if the one-term approximant φ_1 were sufficient, the "solution" would be $\varphi_1 = (2/\pi)x \sin y$.

The next term is $u_1 = L_x^{-1}L_y u_0 = L_x^{-1}L_y [(2/\pi)x \sin y]$. Then $\varphi_2 = u_0 + u_1$ is given by $\varphi_2 = k_1 + xk_2 - (2/\pi)(x^3/3!)\sin y$. Because of the condition at $x = 0$, we have $k_1 = 0$. From the condition on x at $\pi/2$,

$$(\pi/2)k_2(y) - (2/\pi)((\pi/2)^3/3!)\sin y = \sin y$$

$$k_2(y) = (2/\pi - \pi/12)\sin y$$

hence

$$\varphi_2 = (2/\pi - \pi/12)x \sin y$$

The first coefficient was $(2/\pi) = 0.637$. The second (from φ_2) is 0.899. As $n \rightarrow \infty$, the coefficient approaches 1.0 so that $u = x \sin y$ is the solution.

Notice that if we try to carry along the constants of integration, k_1 and k_2 , to some φ_n and do a single evaluation for determination of the constants, we have $u_1 = L_x^{-1}L_y u_0 = L_x^{-1}L_y [k_1(y) + xk_2(y)]$ which we cannot carry out; we must use the evaluated preceding terms rather than a single evaluation at φ_n . We have used only the one operator equation; the same results are obtained from either.

Let us consider a more general form for the linear partial differential equation $L_x u + L_y u + Ru = g$ where $L_x = \partial^2 / \partial x^2$ and $L_y = \partial^2 / \partial y^2$ with conditions specified by $B_1 u(x)|_{x=b_1} = \beta_1(y)$ and $B_2 u(x)|_{x=b_2} = \beta_2(y)$. Solving for $L_x u$ and applying the inverse operator, we have

$$u = \Phi + L_x^{-1}g - L_x^{-1}L_y u - L_x^{-1}Ru$$

where $\Phi = c_1(y) + xc_2(y)$ is the solution of $L\Phi = 0$.

Let $u = \sum_{m=0}^{\infty} u_m$ and identify the initial term as $u_0 = c_1(y) + xc_2(y) + L_x^{-1}g$. Now for $m \geq 0$, the remaining components are identified

$$u_{m+1} = -L_x^{-1}L_y u_m - L_x^{-1}R u_m$$

We now form successive approximants $\varphi_n = \sum_{m=0}^{n-1} u_m$ which we match to the boundary conditions. Thus $\varphi_1 = u_0$, $\varphi_2 = \varphi_1 + u_1$, $\varphi_3 = \varphi_2 + u_2$, ... serve as approximate solutions of increasing accuracy as n approaches infinity and must satisfy the boundary conditions. Beginning with

$$\varphi_1 = u_0 = c_1(y) + x c_2(y) + L_x^{-1}g$$

we use $\beta_1 \varphi_1|_{b_1} = \beta_1(y)$ and $\beta_2 \varphi_1|_{b_2} = \beta_2(y)$ to determine $c_1(y)$ and $c_2(y)$ so that φ_1 is completely determined. Since u_0 is now known, we can form $u_1 = -L_x^{-1}L_y u_0 - L_x^{-1}R u_0$. Then $\varphi_2 = \varphi_1 + u_1$ which must also satisfy the boundary conditions. Continuing to some φ_n we match the conditions for a sufficient approximation. Thus carrying along constants to φ_n for a single evaluation doesn't work except for linear ordinary differential equations. For linear partial differential equations, we must use the already evaluated preceding terms and can do so also for nonlinear ordinary differential equations.

COEFFICIENT-GENERATING ALGORITHMS FOR SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS IN SERIES FORM:

Let's consider a model system in the form

$$\frac{\partial^2 u}{\partial t^2} + \alpha u + \beta \frac{\partial^2 u}{\partial x^2} = g(t, x)$$

assuming conditions given in the form $u(0, x) = \sigma(x)$ and $\partial u / \partial t(0, x) = \eta(x)$. Write

$$g(t, x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g_{m,n} t^m x^n$$

$$\sigma(x) = \sum_{n=0}^{\infty} \sigma_n x^n$$

$$\eta(x) = \sum_{n=0}^{\infty} \eta_n x^n$$

We note that

$$g(t, x) = \sum_{m=0}^{\infty} g_m(x) t^m$$

where

$$g_m(x) = \sum_{n=0}^{\infty} g_{m,n} x^n$$

Defining $L = \partial^2 / \partial t^2$ and L^{-1} as a two-fold definite integration from 0 to t , we can write

$$Lu + \alpha u + \beta(\partial^2 / \partial x^2)u = g(t, x)$$

or

$$Lu = g(t, x) - \alpha u - \beta(\partial^2 / \partial x^2)u$$

Operating with L^{-1} , we have

$$u = u_0 - L^{-1}\alpha u - L^{-1}\beta(\partial^2 / \partial x^2)u$$

where

$$u_0 = u(0, x) + t \frac{\partial u}{\partial t}(0, x) + L^{-1}g(t, x)$$

or

$$u_0 = \sigma(x) + t\eta(x) + \sum_{m=0}^{\infty} \frac{g_m(x)t^{m+2}}{(m+1)(m+2)}$$

which we will write as

$$u_0 = \sum_{m=0}^{\infty} a_m^{(0)}(x)t^m$$

where

$$a_m^{(0)}(x) = \sum_{n=0}^{\infty} a_{m,n}^{(0)} x^n$$

Thus the coefficients are

$$a_0^{(0)}(x) = \sigma(x)$$

$$a_1^{(0)}(x) = \eta(x)$$

$$a_{m+2}^{(0)}(x) = \frac{g_m(x)}{(m+1)(m+2)}$$

Using decomposition $u = \sum_{m=0}^{\infty} u_m$,

$$u = u_0 - L^{-1} \alpha u - L^{-1} \beta (\partial^2 / \partial x^2) u$$

$$u = u_0 - L^{-1} \alpha \sum_{m=0}^{\infty} u_m - L^{-1} \beta (\partial^2 / \partial x^2) \sum_{m=0}^{\infty} u_m$$

so that for $m = 0$,

$$u_0 = \sum_{m=0}^{\infty} a_m^{(0)}(x) t^m$$

and for $m > 0$,

$$u_m = -L^{-1} \alpha u_{m-1} - L^{-1} \beta (\partial^2 / \partial x^2) u_{m-1}$$

Since we can also write

$$u_0 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(0)} t^m x^n$$

we note

$$a_{0,n}^{(0)} = \sigma_n$$

$$a_{1,n}^{(0)} = \eta_n$$

$$a_{m+2,n}^{(0)} = \frac{\xi_{m,n}}{(m+1)(m+2)}$$

The next component u_1 is

$$u_1 = -L^{-1} \alpha u_0 - L^{-1} \beta (\partial^2 / \partial x^2) u_0$$

Since

$$\begin{aligned} (\partial^2 / \partial x^2) u_0 &= (\partial^2 / \partial x^2) \sum_{m=0}^{\infty} a_m^{(0)}(x) t^m \\ &= (\partial^2 / \partial x^2) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(0)} t^m x^n \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (n+1)(n+2) a_{m,n-2}^{(0)} t^m x^n \end{aligned}$$

Thus

$$\begin{aligned} u_0 &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(0)} t^m x^n \\ u_1 &= -\alpha \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_{m,n}^{(0)} t^{m+2} x^n}{(m+1)(m+2)} \\ &\quad -\beta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n-2}^{(0)} t^{m+2} x^n \frac{(n+1)(n+2)}{(m+1)(m+2)} \end{aligned}$$

which we write as

$$u_1 = t^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(1)} t^m x^n$$

where

$$a_{m,n}^{(1)} = \frac{-\alpha a_{m,n}^{(0)} - \beta(n+1)(n+2)a_{m,n+2}^{(0)}}{(m+1)(m+2)}$$

Proceeding in the same way, write

$$\begin{aligned} u_2 &= -L^{-1}\alpha u_1 - L^{-1}\beta(\partial^2/\partial x^2)u_1 \\ (\partial^2/\partial x^2)u_1 &= t^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n+2}^{(1)}(n+1)(n+2)t^m x^n \\ u_2 &= -\alpha \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_{m,n}^{(1)} t^{m+4} x^n}{(m+3)(m+4)} \\ &\quad -\beta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n+2}^{(1)} \frac{(n+1)(n+2)t^{m+4} x^n}{(m+3)(m+4)} \end{aligned}$$

which is now rewritten as

$$u_2 = t^4 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(2)} t^m x^n$$

where

$$a_{m,n}^{(2)} = \frac{-\alpha a_{m,n}^{(1)} - \beta(n+1)(n+2)a_{m,n+2}^{(1)}}{(m+3)(m+4)}$$

Continuing, we calculate $u_3, u_4, \dots$ and see that we can write for the μ th component of u

$$u_\mu = t^{2\mu} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\mu)} t^m x^n \quad (\mu > 0)$$

with

$$a_{m,n}^{(\mu)} = \frac{-\alpha a_{m,n}^{(\mu-1)} - \beta(n+1)(n+2)a_{m,n+2}^{(\mu-1)}}{(m+2\mu-1)(m+2\mu)}$$

as a coefficient-generating algorithm. Thus for $\mu = 0$

$$a_{0,n}^{(0)} = \sigma_n$$

$$a_{1,n}^{(0)} = \eta_n$$

$$a_{m+2,n}^{(0)} = \frac{g_{m,n}}{(m+1)(m+2)}$$

Then the solution is $u = \sum_{\mu=0}^{\infty} u_{\mu}$. Consequently

$$u = \sum_{\mu=0}^{\infty} t^{2\mu} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\mu)} t^m x^n$$

is the solution with

$$\phi_v = \sum_{\mu=0}^{v-1} t^{2\mu} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\mu)} t^m x^n$$

as the v th approximant to the solution which becomes an increasingly accurate representation of u as v increases.

MIXED DERIVATIVES:

Consider the equation $u_{xy} = -u$ given the conditions $u(x,0) = e^x$ and $u(0,y) = e^{-y}$. Let

$$L_x = \partial/\partial x$$

$$L_y = \partial/\partial y$$

Then $L_x^{-1}(\cdot) = \int_c^x (\cdot) dx$ and $L_y^{-1}(\cdot) = \int_0^y (\cdot) dy$. In operator form, we have $L_x L_y u = -u$. Operating with L_x^{-1} we have

$$L_x^{-1} L_x (L_y u) = -L_x^{-1} u$$

$$L_y u - L_y u(0, y) = -L_x^{-1} u$$

$$L_y u = L_y u(0, y) - L_x^{-1} u$$

Operating now with L_y^{-1}

$$L_y^{-1} L_y u = L_y^{-1} L_y u(0, y) - L_y^{-1} L_x^{-1} u$$

$$u - u(x, 0) = u(0, y) - u(0, 0) - L_y^{-1} L_x^{-1} u$$

$$u = u(x, 0) + u(0, y) - u(0, 0) - L_y^{-1} L_x^{-1} u$$

Let

$$u_0 = u(x, 0) + u(0, y) - u(0, 0)$$

$$u_0 = e^x + e^{-y} - 1$$

$$u_1 = -L_y^{-1} L_x^{-1} u_0$$

$$u_2 = -L_y^{-1} L_x^{-1} u_1$$

Hence $u = \sum_{n=0}^{\infty} u_n$ or

$$u = \sum_{n=0}^{\infty} (-L_y^{-1} L_x^{-1})^n u_0$$

Since

$$u_m = (-L_y^{-1} L_x^{-1})^m (e^x + e^{-y} - 1)$$

we have

$$\begin{aligned} u_m &= (-L_y^{-1} L_x^{-1})^m \sum_{\mu=0}^{\infty} \frac{x^{\mu}}{\mu!} + (-L_y^{-1} L_x^{-1})^m \sum_{\mu=1}^{\infty} \frac{(-y)^{\mu}}{\mu!} \\ &= \frac{(-y)^m}{m!} \sum_{\mu=0}^{\infty} \frac{x^{m+\mu}}{(m+\mu)!} + \frac{x^m}{m!} \sum_{\mu=1}^{\infty} \frac{(-y)^{m+\mu}}{(m+\mu)!} \end{aligned}$$

Because $u = \sum_{m=0}^{\infty} u_m$,

$$u = \sum_{m=0}^{\infty} \left\{ \frac{(-y)^m}{m!} \sum_{\mu=0}^{\infty} \frac{x^{m+\mu}}{(m+\mu)!} + \frac{x^m}{m!} \sum_{\mu=1}^{\infty} \frac{(-y)^{m+\mu}}{(m+\mu)!} \right\}$$

is the solution. However, we can rearrange the terms to get a simpler representation using staggered summation:

$$u = \sum_{m=0}^{\infty} \sum_{\mu=0}^m \frac{x^{m-\mu}}{(m-\mu)!} \frac{(-y)^{\mu}}{\mu!}$$

or

$$u = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^m \binom{m}{\mu} x^{m-\mu} (-y)^{\mu}$$

where

$$\binom{m}{\mu} = \frac{m!}{(m-\mu)! \mu!}$$

We recognize the binomial expansion of $(x-y)^m$ and write

$$u = \sum_{m=0}^{\infty} \frac{1}{m!} (x-y)^m$$

which is, of course, the exponential series of $(x-y)$ so that $u = e^{x-y}$ which is the same result in a convenient form.

REMARK: If we write $\varphi_2 = u_0 + u_1$, we can recognize the first six terms of $(1+x+x^2/2) \cdot (1-y+y^2/2) \approx e^x e^{-y}$. Write $u = e^x e^{-y} + N$, substitute into the original equation and see that N must vanish in the limit.

EXERCISE: $u_{xy} = u_x + u_y - u$ with $u(x, 0) = e^x$ and $u(0, y) = e^{-y}$. (The solution is $u = e^{x-y}$.)

EXERCISE: $u_{xy} = [4xy/(1+x^2y^2)]u$ with $u(x, 0) = u(0, y) = 1$. (The solution is $u = 1 + x^2y^2$.) A generalization to $u_{xy} + k(x, y)u = g(x, y)$ with $u(0, y) = \xi(y)$ and $u(x, 0) = \eta(x)$ can also be considered using power series expansions of the functions to several terms.

MODIFIED DECOMPOSITION SOLUTION:

$u_{xy} = u_x + u_y - u$ with $u(x, 0) = e^x$ and $u(0, y) = e^{-y}$. Let

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} x^m y^n$$

Then

$$u_x = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+1) a_{m+1,n} x^m y^n$$

$$u_y = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (n+1) a_{m,n+1} x^m y^n$$

$$u_{xy} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+1)(n+1) a_{m+1,n+1} x^m y^n$$

$$u_{x,0} = \sum_{m=0}^{\infty} a_{m,0} x^m$$

$$u(0,y) = \sum_{n=0}^{\infty} a_{0,n} y^n$$

We note that $u(x, 0) = \sum_{m=0}^{\infty} x^m / m!$ and $u(0, y) = \sum_{n=0}^{\infty} (-y)^n / n!$. Substituting in the equation,

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+1)(n+1) a_{m+1, n+1} x^m y^n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+1) a_{m+1, n} x^m y^n \\ + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (n+1) a_{m, n+1} x^m y^n - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m, n} x^m y^n$$

Equating like powers

$$a_{m+1, n+1} = \frac{(m+1)a_{m+1, n} + (n+1)a_{m, n+1} - a_{m, n}}{(m+1)(n+1)} \\ a_{m, 0} = 1/m! \\ a_{0, n} = (-1)^n/n!$$

We can now compute a table of coefficients in a convenient triangular form

$$\begin{array}{cccc} & & a_{0,0} & \\ & & a_{1,0} & a_{0,1} \\ & a_{2,0} & a_{1,1} & a_{0,2} \\ a_{3,0} & a_{2,1} & a_{1,2} & a_{0,3} \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

which is given as:

$$\begin{array}{cccccc} & & & & 1 & \\ & & & & 1 & -1 \\ & & & \frac{1}{2} & -1 & \frac{1}{2} \\ & & \frac{1}{6} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{6} \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{4} & -\frac{1}{6} & \frac{1}{24} & \end{array}$$

and by induction,

$$a_{m, n} = \frac{(-1)^n}{m!n!}$$

Therefore

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m, n} x^m y^n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{m!n!} x^m y^n \\ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{x^m (-y)^n}{m!n!} = \sum_{m=0}^{\infty} \frac{x^m}{m!} \sum_{n=0}^{\infty} \frac{(-y)^n}{n!}$$

Consequently, $u = e^{x-y}$. From the table of coefficients, we see that

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{x^{m-n}}{(m-n)!} \cdot \frac{(-y)^n}{n!} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^{\infty} \binom{m}{n} x^{m-n} (-y)^n$$

Since

$$(x-y)^m = \sum_{n=0}^m \binom{m}{n} x^{m-n} (-y)^n$$

$$u = \sum_{m=0}^{\infty} \frac{(x-y)^m}{m!} = e^{x-y}$$

ADDENDUM: From the table of coefficients in triangular form, we have

$$a_{m,n} = \frac{(-1)^n}{m!n!}$$

Therefore by substitution,

$$\begin{aligned} & (m+1)a_{m+1,n} + (n+1)a_{m,n+1} \\ &= \left\{ (m+1) \frac{(-1)^n}{(m+1)n!} + (n+1) \frac{(-1)^{n+1}}{m!(n+1)!} \right\} \\ &= \left\{ \frac{(-1)^n}{m!n!} - \frac{(-1)^n}{m!n!} \right\} = 0 \end{aligned}$$

Consequently we can derive the recurrence relation by substitution:

$$a_{m-1,n-1} = \frac{-a_{m,n}}{(m+1)(n+1)}$$

$$a_{m,0} = 1/m!$$

$$a_{0,n} = (-1)^n/n!$$

so that

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} x^m y^n$$

GENERALIZATION OF THE A_n POLYNOMIALS TO FUNCTIONS OF SEVERAL VARIABLES:

In applying the decomposition method to nonlinear differential equations arising in physical problems, we may encounter nonlinearities involving several variables [8]. We now generalize the algorithm for A_n for $f(u)$ to analytic functions of several variables such as $f(u,v)$, where $f(u,v)$ is not factorable into $f_1(u)f_2(v)$. (The latter case, of course, is solvable as a "product

nonlinearity" by developing the A_n for each factor and obtaining their product.) Examples appear in [2]. Our objective is to extend the class of solvable systems.

In the use of the method, the solution of a differential or partial differential equation is written $u = \sum_{n=0}^{\infty} u_n$ and $f(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n)$ where u_0 is a function involving initial/boundary conditions, the forcing function, and an integral operator. This amounts to the assumption that the solution and functions of the solution are expanded in the A_n polynomials since A_n reduces to u_n for $f(u) = u$. For development of an algorithm for the A_n , it is convenient to assume parametrized forms of the u and $f(u)$. The following expressions have been given by the author [2] as:

$$A_n = (1/n!)(d^n/d\lambda^n)f(u(\lambda))|_{\lambda=0} \quad (1)$$

or simply $A_n = (1/n!)D^n f|_{\lambda=0}$ where $D^n \equiv \partial^n/\partial \lambda^n$ and

$$(\partial^n/\partial \lambda^n)u(\lambda)|_{\lambda=0} = n!u_n$$

The $D^n f$ term for $n > 0$ can be written as a sum from $v = 1$ to n of terms $d^v f/du^v$ with coefficients which are polynomials in the $d^v u/d\lambda^v$. Thus,

$$D^1 f = (df/du)(du/d\lambda)$$

$$D^2 f = (d^2 f/du^2)(du/d\lambda)^2 + (df/du)(d^2 u/d\lambda^2)$$

$$D^3 f = (d^3 f/du^3)(du/d\lambda)^3 + 3(d^2 f/du^2)(du/d\lambda)(d^2 u/d\lambda^2) + (df/du)(d^3 u/d\lambda^3)$$

$$\vdots$$

The result for A_n can finally be given in a very convenient form which we have referred to as Rach's Rule,

$$A_n = \sum_{v=1}^n c(v,n) f^{(v)}(u_0) \quad (2)$$

Here $f^{(v)}(u_0)$ means the v th derivative of $f(u)$ at $u = u_0$ and the $1/n!$ is absorbed in the $c(v,n)$. The first index of $c(v,n)$ progresses from 1 to n along with the order of the derivative. The second index is the order of the polynomial.

The A_n is a function of $u_0, u_1, \dots, u_n$, i.e., of the components of the solution u in the decomposition. The $c(v, n)$ are products (or sums of products) of v components of u whose subscripts sum up to n with the result divided by the factorial of the number of repeated subscripts. For example $c(1, 3)$ can only be u_3 (a single subscript which must be 3). $c(2, 3)$ can only be $u_1 u_2$ (two subscripts adding to 3). $c(3, 3) = (1/3!)u_1^3$. $c(2, 6)$ has two subscripts adding to 6 for which we have three possibilities for $u_i u_j$ using (2, 4), (1, 5) and (3, 3). Hence $c(2, 6) = u_2 u_4 + u_1 u_5 + (1/2!)u_3^2$. The result is

$$\begin{aligned} A_0 &= f(u_0) \\ A_1 &= u_1 (d/du_0) f(u_0) \\ A_2 &= u_2 (d/du_0) f(u_0) + (u_1^2/2!) (d^2/du_0^2) f(u_0) \\ A_3 &= u_3 (d/du_0) f(u_0) + u_1 u_2 (d^2/du_0^2) f(u_0) \\ &\quad + (u_1^3/3!) (d^3/du_0^3) f(u_0) \\ &\quad \vdots \end{aligned}$$

ANALYTIC FUNCTION OF TWO VARIABLES $f(u, v)$:

Proceeding analogously to the case of one variable,

$$\begin{aligned} D^1 f &= (df/du)(du/d\lambda) + (df/dv)(dv/d\lambda) \\ D^2 f &= (d^2 u/d\lambda^2)(df/du) + (d^2 v/d\lambda^2)(df/dv) + (du/d\lambda)^2 (d^2 f/du^2) \\ &\quad + 2(du/d\lambda)(dv/d\lambda)(d^2 f/dudv) + (dv/d\lambda)^2 (d^2 f/dv^2) \\ D^3 f &= (d^3 u/d\lambda^3)(df/du) + (d^3 v/d\lambda^3)(df/dv) \\ &\quad + 3(du/d\lambda)(d^2 u/d\lambda^2)(d^2 f/du^2) + 3(dv/d\lambda)(d^2 v/d\lambda^2)(d^2 f/dv^2) \\ &\quad + 3(du/d\lambda)(d^2 v/d\lambda^2)(d^2 f/dudv) + 3(dv/d\lambda)(d^2 u/d\lambda^2)(d^2 f/dvdu) \\ &\quad + 3(dv/d\lambda)(du/d\lambda)^2 (d^3 f/dvdu^2) + 3(du/d\lambda)(dv/d\lambda)^2 (d^3 f/du^2 dv) \\ &\quad + (du/d\lambda)^3 (d^3 f/du^3) + (dv/d\lambda)^3 (d^3 f/dv^3) \\ D^4 f &= (d^4 u/d\lambda^4)(df/du) + (d^4 v/d\lambda^4)(df/dv) \\ &\quad + 6(du/d\lambda)^2 (d^2 u/d\lambda^2)(d^3 f/du^3) + 6(du/d\lambda)^2 (d^2 v/d\lambda^2)(d^3 f/du^2 dv) \\ &\quad + 12(du/d\lambda)(dv/d\lambda)(d^2 u/d\lambda^2)(d^3 f/du^2 dv) \\ &\quad + 12(du/d\lambda)(dv/d\lambda)(d^2 v/d\lambda^2)(d^3 f/dudv^2) \\ &\quad + 6(dv/d\lambda)^2 (d^2 u/d\lambda^2)(d^3 f/dudv^2) + 6(dv/d\lambda)^2 (d^2 v/d\lambda^2)(d^3 f/dv^3) \\ &\quad + 3(d^2 u/d\lambda^2)^2 (d^2 f/du^2) + 4(du/d\lambda)(d^3 u/d\lambda^3)(d^2 f/du^2) \end{aligned}$$

$$\begin{aligned}
& + 4(dv/d\lambda)(d^3u/d\lambda^3)(d^2f/dvdu) + 6(d^2v/d\lambda^2)(d^2u/d\lambda^2)(d^2f/dudv) \\
& + 4(du/d\lambda)(d^3v/d\lambda^3)(d^2f/dudv) + 4(dv/d\lambda)(d^3v/d\lambda^3)(d^2f/dv^2) \\
& + 3(d^2v/d\lambda^2)^2(d^2f/dv^2) + 4(dv/d\lambda)(du/d\lambda)^3(d^4f/dvdu^3) \\
& + 6(dv/d\lambda)^2(du/d\lambda)^2(d^4f/dv^2du^2) + 4(du/d\lambda)(dv/d\lambda)^3(d^4f/dudv^3) \\
& + (du/d\lambda)^4(d^4f/du^4) + (dv/d\lambda)^4(d^4f/dv^4)
\end{aligned}$$

The A_n for $f(u)$ are written $A_n\{f(u)\}$. Generalizing to $A_n\{f(u,v)\}$ or $A_n\{f(u(\lambda), v(\lambda))\}$ we introduce the notation

$$\frac{\partial^{\mu+\nu}}{\partial u^\mu \partial v^\nu} f(u(\lambda), v(\lambda)) \Big|_{\lambda=0} = f_{\mu,\nu}(u_0, v_0)$$

Proceeding analogously to the $c(\nu, n)$ and $f^{(0)}(u_0)$ for $f(u)$, we can now write $c(\mu, \nu, n)$ and $f^{\mu,\nu}$, or $f_{\mu,\nu}(u_0, v_0)$ for a function $f(u, v)$.

$$\begin{aligned}
A_1 &= c(1, 0, 1)f_{1,0}(u_0, v_0) + c(0, 1, 1)f_{0,1}(u_0, v_0) \\
&= c(1, 0, 1)df/du_0 + c(0, 1, 1)df/dv_0
\end{aligned}$$

Comparing with D^1f we see that $c(1, 0, 1) = du/d\lambda$ which must be evaluated at $\lambda = 0$. Since $u = \sum_{n=0}^{\infty} \lambda^n u_n$, $du/d\lambda|_{\lambda=0} = u_1$. Hence

$$\begin{aligned}
A_1 &= u_1 df/du_0 + v_1 df/dv_0 \\
A_1 &= u_1 f_{1,0} + v_1 f_{0,1}
\end{aligned}$$

Proceeding in the same way we can list the $A_n\{f(u, v)\}$

$$\begin{aligned}
A_0 &= f_{0,0} \\
A_1 &= u_1 f_{1,0} + v_1 f_{0,1} \\
A_2 &= u_2 f_{1,0} + v_2 f_{0,1} + (u_1^2/2!)f_{2,0} + u_1 v_1 f_{1,1} + (v_1^2/2!)f_{0,2} \\
A_3 &= u_3 f_{1,0} + v_3 f_{0,1} + u_1 u_2 f_{2,0} \\
&\quad + [u_1 v_2 + u_2 v_1]f_{1,1} + v_1 v_2 f_{0,2} + (u_1^3/3!)f_{3,0} + (u_1^2/2!)v_1 f_{2,1} \\
&\quad + (u_1 v_1^2/2!)f_{1,2} + (v_1^3/3!)f_{0,3} \\
A_4 &= u_4 f_{1,0} + v_4 f_{0,1} + [(u_2^2/2!) + u_1 u_3]f_{2,0} + [u_1 v_3 + u_2 v_2 + u_3 v_1]f_{1,1} \\
&\quad + [(v_2^2/2!) + v_1 v_3]f_{0,2} + (u_1^2/2!)u_2 f_{3,0} + [(u_1^2/2!)v_2 + u_1 u_2 v_1]f_{2,1} \\
&\quad + [u_1 v_1 v_2 + u_2 v_1^2/2!]f_{1,2} + (v_1^2/2!)v_2 f_{0,3} + (u_1^4/4!)f_{4,0} + (u_1^3/3!)v_1 f_{3,1} \\
&\quad + (u_1^2/2!)(v_1^2/2!)f_{2,2} + (u_1 v_1^3/3!)f_{1,3} + (v_1^4/4!)f_{0,4}
\end{aligned}$$

Perhaps more conveniently we will write in symmetric form [1,2] where the indices of f_{nm} start from $n,0$, subtracting 1 from n and adding 1 to 0 for the next set, ..., and finally reaching $0,n$. Thus

$$\begin{aligned} A_0 &= f_{0,0} \\ A_1 &= u_1 f_{1,0} + v_1 f_{0,1} \\ A_2 &= (u_1^2/2!) f_{2,0} + u_1 v_1 f_{1,1} + (v_1^2/2!) f_{0,2} + u_2 f_{1,0} + v_2 f_{0,1} \\ A_3 &= (u_1^3/3!) f_{3,0} + (u_1^2/2!) v_1 f_{2,1} + (v_1^2/2!) u_1 f_{1,2} + (v_1^3/3!) f_{0,3} + u_1 u_2 f_{2,0} \\ &\quad + [v_1 u_2 + u_1 v_2] f_{1,1} + v_1 v_2 f_{0,2} + u_3 f_{1,0} + v_3 f_{0,1} \\ A_4 &= u_4 f_{1,0} + v_4 f_{0,1} + (u_1^2/2!) u_2 f_{3,0} + (u_1^2/2!) v_2 f_{2,1} + u_1 v_1 u_2 f_{2,1} \\ &\quad + u_1 v_1 v_2 f_{1,2} + (v_1^2/2!) u_2 f_{1,2} + (v_1^2/2!) v_2 f_{0,3} + u_2^2/2! f_{2,0} + u_1 u_3 f_{2,0} \\ &\quad + v_1 u_3 f_{1,1} + v_2 u_3 f_{1,1} + u_1 v_3 f_{1,1} + v_1 v_3 f_{0,2} + (v_2^2/2!) f_{0,2} \\ &\quad + (v_1 u_1^3/3!) f_{3,1} + (v_1^2/2!)(u_1^2/2!) f_{2,2} + (u_1 v_1^3/3!) f_{1,3} + (u_1^4/4!) f_{4,0} \\ &\quad + (v_1^4/4!) f_{0,4} \end{aligned}$$

For A_0 , we have $c(0,0;0) = 1$. We can list the A_n as follows

$$\begin{aligned} A_1 &= c(1,0;1) f_{1,0} + c(0,1;1) f_{0,1} \\ A_2 &= c(2,0;2) f_{2,0} + c(1,1;2) f_{1,1} + c(0,2;2) f_{0,2} + c(1,0;2) f_{1,0} + c(0,1;2) f_{0,1} \\ A_3 &= c(3,0;3) f_{3,0} + c(0,1;3) f_{0,1} + c(2,0;3) f_{2,0} + c(1,1;3) f_{1,1} + c(0,2;3) f_{0,2} \\ &\quad + c(3,0;3) f_{3,0} + c(2,1;3) f_{2,1} + c(1,2;3) f_{1,2} + c(0,3;3) f_{0,3} \\ A_4 &= c(1,0;4) f_{1,0} + c(0,1;4) f_{0,1} + c(2,0;4) f_{2,0} + c(1,1;4) f_{1,1} \\ &\quad + c(0,2;4) f_{0,2} + c(3,0;4) f_{3,0} + c(2,1;4) f_{2,1} + c(1,2;4) f_{1,2} \\ &\quad + c(0,3;4) f_{0,3} + c(4,0;4) f_{4,0} + c(3,1;4) f_{3,1} + c(2,2;4) f_{2,2} \\ &\quad + c(1,3;4) f_{1,3} + c(0,4;4) f_{0,4} \\ A_5 &= c(1,0;5) f_{1,0} + c(0,1;5) f_{0,1} + c(2,0;5) f_{2,0} + c(1,1;5) f_{1,1} + c(0,2;5) f_{0,2} \\ &\quad + c(3,0;5) f_{3,0} + c(2,1;5) f_{2,1} + c(1,2;5) f_{1,2} + c(0,3;5) f_{0,3} \\ &\quad + c(4,0;5) f_{4,0} + c(3,1;5) f_{3,1} + c(2,2;5) f_{2,2} + c(1,3;5) f_{1,3} + c(0,4;5) f_{0,4} \\ &\quad + c(5,0;5) f_{5,0} + c(4,1;5) f_{4,1} + c(3,2;5) f_{3,2} + c(2,3;5) f_{2,3} \\ &\quad + c(1,4;5) f_{1,4} + c(0,5;5) f_{0,5} \\ A_6 &= c(1,0;6) f_{1,0} + c(0,1;6) f_{0,1} + c(2,0;6) f_{2,0} + c(1,1;6) f_{1,1} + c(0,2;6) f_{0,2} \\ &\quad + c(3,0;6) f_{3,0} + c(2,1;6) f_{2,1} + c(1,2;6) f_{1,2} + c(0,3;6) f_{0,3} + c(4,0;6) f_{4,0} \\ &\quad + c(3,1;6) f_{3,1} + c(2,2;6) f_{2,2} + c(1,3;6) f_{1,3} + c(0,4;6) f_{0,4} + c(5,0;6) f_{5,0} \\ &\quad + c(4,1;6) f_{4,1} + c(3,2;6) f_{3,2} + c(2,3;6) f_{2,3} + c(1,4;6) f_{1,4} + c(0,5;6) f_{0,5} \\ &\quad + c(6,0;6) f_{6,0} + c(5,1;6) f_{5,1} + c(4,2;6) f_{4,2} + c(3,3;6) f_{3,3} + c(2,4;6) f_{2,4} \\ &\quad + c(1,5;6) f_{1,5} + c(0,6;6) f_{0,6} \end{aligned}$$

$$\begin{aligned}
A_7 = & c(1,0;7)f_{1,0} + c(0,1;7)f_{1,0} + c(2,0;7)f_{2,0} + c(1,1;7)f_{1,1} + c(0,2;7)f_{0,2} \\
& + c(3,0;7)f_{3,0} + c(2,1;7)f_{2,1} + c(1,2;7)f_{1,2} + c(0,3;7)f_{0,3} + c(4,0;7)f_{4,0} \\
& + c(3,1;7)f_{3,1} + c(2,2;7)f_{2,2} + c(1,3;7)f_{1,3} + c(0,4;7)f_{0,4} + c(5,0;7)f_{5,0} \\
& + c(4,1;7)f_{4,1} + c(3,2;7)f_{3,2} + c(2,3;7)f_{2,3} + c(1,4;7)f_{1,4} + c(0,5;7)f_{0,5} \\
& + c(4,3;7)f_{4,3} + c(3,4;7)f_{3,4} + c(2,5;7)f_{2,5} + c(1,6;7)f_{1,6} + c(0,7;7)f_{0,7} \\
& + c(6,0;7)f_{6,0} + c(5,1;7)f_{5,1} + c(4,2;7)f_{4,2} + c(3,3;7)f_{3,3} + c(2,4;7)f_{2,4} \\
& + c(1,5;7)f_{1,5} + c(0,6;7)f_{0,6} + c(7,0;7)f_{7,0} + c(6,1;7)f_{6,1} + c(5,2;7)f_{5,2}
\end{aligned}$$

$$\begin{aligned}
A_8 = & c(1,0;8)f_{1,0} + c(0,1;8)f_{0,1} + c(2,0;8)f_{2,0} + c(1,1;8)f_{1,1} \\
& + c(0,2;8)f_{0,2} + c(3,0;8)f_{3,0} + c(2,1;8)f_{2,1} + c(1,2;8)f_{1,2} + c(0,3;8)f_{0,3} \\
& + c(4,0;8)f_{4,0} + c(3,1;8)f_{3,1} + c(2,2;8)f_{2,2} + c(1,3;8)f_{1,3} + c(0,4;8)f_{0,4} \\
& + c(5,0;8)f_{5,0} + c(4,1;8)f_{4,1} + c(3,2;8)f_{3,2} + c(2,3;8)f_{2,3} + c(1,4;8)f_{1,4} \\
& + c(0,5;8)f_{0,5} + c(6,0;8)f_{6,0} + c(5,1;8)f_{5,1} + c(4,2;8)f_{4,2} + c(3,3;8)f_{3,3} \\
& + c(2,4;8)f_{2,4} + c(1,5;8)f_{1,5} + c(0,6;8)f_{0,6} + c(7,0;8)f_{7,0} + c(6,1;8)f_{6,1} \\
& + c(5,2;8)f_{5,2} + c(4,3;8)f_{4,3} + c(3,4;8)f_{3,4} + c(2,5;8)f_{2,5} + c(1,6;8)f_{1,6} \\
& + c(0,7;8)f_{0,7} + c(8,0;8)f_{8,0} + c(7,1;8)f_{7,1} + c(6,2;8)f_{6,2} + c(5,3;8)f_{5,3} \\
& + c(4,4;8)f_{4,4} + c(3,5;8)f_{3,5} + c(2,6;8)f_{2,6} + c(1,7;8)f_{1,7} + c(0,8;8)f_{0,8}
\end{aligned}$$

$$\begin{aligned}
A_9 = & c(1,0;9)f_{1,0} + c(0,1;9)f_{0,1} + c(2,0;9)f_{2,0} + c(1,1;9)f_{1,1} + c(0,2;9)f_{0,2} \\
& + c(3,0;9)f_{3,0} + c(2,1;9)f_{2,1} + c(1,2;9)f_{1,2} + c(0,3;9)f_{0,3} + c(4,0;9)f_{4,0} \\
& + c(3,1;9)f_{3,1} + c(2,2;9)f_{2,2} + c(1,3;9)f_{1,3} + c(0,4;9)f_{0,4} + c(5,0;9)f_{5,0} \\
& + c(4,1;9)f_{4,1} + c(3,2;9)f_{3,2} + c(2,3;9)f_{2,3} + c(1,4;9)f_{1,4} + c(0,5;9)f_{0,5} \\
& + c(6,0;9)f_{6,0} + c(5,1;9)f_{5,1} + c(4,2;9)f_{4,2} + c(3,3;9)f_{3,3} + c(2,4;9)f_{2,4} \\
& + c(1,5;9)f_{1,5} + c(0,6;9)f_{0,6} + c(7,0;9)f_{7,0} + c(6,1;9)f_{6,1} + c(5,2;9)f_{5,2} \\
& + c(4,3;9)f_{4,3} + c(3,4;9)f_{3,4} + c(2,5;9)f_{2,5} + c(1,6;9)f_{1,6} + c(0,7;9)f_{0,7} \\
& + c(8,0;9)f_{8,0} + c(7,1;9)f_{7,1} + c(6,2;9)f_{6,2} + c(5,3;9)f_{5,3} + c(4,4;9)f_{4,4} \\
& + c(3,5;9)f_{3,5} + c(2,6;9)f_{2,6} + c(1,7;9)f_{1,7} + c(0,8;9)f_{0,8} + c(9,0;9)f_{9,0} \\
& + c(8,1;9)f_{8,1} + c(7,2;9)f_{7,2} + c(6,3;9)f_{6,3} + c(5,4;9)f_{5,4} + c(4,5;9)f_{4,5} \\
& + c(3,6;9)f_{3,6} + c(2,7;9)f_{2,7} + c(1,8;9)f_{1,8} + c(0,9;9)f_{0,9}
\end{aligned}$$

$$\begin{aligned}
A_{10} = & c(1,0;10)f_{1,0} + c(0,1;10)f_{0,1} + c(2,0;10)f_{2,0} + c(1,1;10)f_{1,1} \\
& + c(0,2;10)f_{0,2} + c(3,0;10)f_{3,0} + c(2,1;10)f_{2,1} + c(1,2;10)f_{1,2} \\
& + c(0,3;10)f_{0,3} + c(4,0;10)f_{4,0} + c(3,1;10)f_{3,1} + c(2,2;10)f_{2,2} \\
& + c(1,3;10)f_{1,3} + c(0,4;10)f_{0,4} + c(5,0;10)f_{5,0} + c(4,1;10)f_{4,1} \\
& + c(3,2;10)f_{3,2} + c(2,3;10)f_{2,3} + c(1,4;10)f_{1,4} + c(0,5;10)f_{0,5} \\
& + c(6,0;10)f_{6,0} + c(5,1;10)f_{5,1} + c(4,2;10)f_{4,2} + c(3,3;10)f_{3,3}
\end{aligned}$$

$$\begin{aligned}
& + c(2,4;10)f_{2,4} + c(1,5;10)f_{1,5} + c(5,2;10)f_{5,2} + c(4,3;10)f_{4,3} \\
& + c(3,4;10)f_{3,4} + c(2,5;10)f_{2,5} + c(1,6;10)f_{1,6} + c(0,7;10)f_{0,7} \\
& + c(8,0;10)f_{8,0} + c(7,1;10)f_{7,1} + c(6,2;10)f_{6,2} + c(5,3;10)f_{5,3} \\
& + c(4,4;10)f_{4,4} + c(3,5;10)f_{3,5} + c(2,6;10)f_{2,6} + c(1,7;10)f_{1,7} \\
& + c(0,8;10)f_{0,8} + c(9,0;10)f_{9,0} + c(8,1;10)f_{8,1} + c(7,2;10)f_{7,2} \\
& + c(6,3;10)f_{6,3} + c(5,4;10)f_{5,4} + c(4,5;10)f_{4,5} + c(3,6;10)f_{3,6} \\
& + c(2,7;10)f_{2,7} + c(1,8;10)f_{1,8} + c(0,9;10)f_{0,9} + c(10,0;10)f_{10,0} \\
& + c(9,1;10)f_{9,1} + c(8,2;10)f_{8,2} + c(7,3;10)f_{7,3} + c(6,4;10)f_{6,4} \\
& + c(5,5;10)f_{5,5} + c(4,6;10)f_{4,6} + c(3,7;10)f_{3,7} + c(2,8;10)f_{2,8} \\
& + c(1,9;10)f_{1,9} + c(0,10;10)f_{0,10}
\end{aligned}$$

For $A_1 = \sum_{\mu+v=1}^1 c(\mu, v, 1)f_{\mu, v}$ we need only $c(1, 0; 1) = u_1$ and $c(0, 1; 1) = v_1$

For $A_2 = \sum_{\mu+v=1}^2 c(\mu, v, 2)f_{\mu, v}$, we need

$$\begin{aligned}
c(1, 0; 2) &= u_2 \\
c(0, 1; 2) &= v_2 \\
c(2, 0; 2) &= u_1^2/2! \\
c(1, 1; 2) &= u_1 v_1 \\
c(0, 2; 2) &= v_1^2/2! \\
A_2 &= \sum_{\mu+v=1}^3 c(\mu, v, 2)f_{\mu, v}
\end{aligned}$$

$$\begin{aligned}
c(1, 0; 3) &= u_3 \\
c(0, 1; 3) &= v_3 \\
c(2, 0; 3) &= u_1 u_2 \\
c(1, 1; 3) &= u_1 v_2 + u_2 v_1 \\
c(0, 2; 3) &= v_1 v_2 \\
c(3, 0; 3) &= u_1^3/3! \\
c(2, 1; 3) &= u_1^2 v_1/2! \\
c(1, 2; 3) &= u_1 v_1^2/2 \\
c(0, 3; 3) &= v_1^3/3!
\end{aligned}$$

$$\begin{aligned}
A_3 &= \sum_{\mu+v=1}^4 c(\mu, v, 4)f_{\mu, v} \\
c(1, 0; 4) &= u_4 \\
c(0, 1; 4) &= v_4 \\
c(2, 0; 4) &= u_1 v_3 + u_2^2/2!
\end{aligned}$$

$$c(1,1;4) = u_1 v_3 + u_2 v_2 + u_3 v_1$$

$$c(0,2;4) = v_1 v_3 + v_2^2/2!$$

$$c(3,0;4) = u_1^2 u_2/2!$$

$$c(2,1;4) = u_1 u_2 v_1 + u_1^2 v_2/2!$$

$$c(1,2;4) = u_1 v_1 v_2 + u_2 v_1^2/2!$$

$$c(0,3;4) = v_1^2 v_2/2!$$

$$c(4,0;4) = u_1^4/4!$$

$$c(3,1;4) = u_1^3 v_1/3!$$

$$c(2,2;4) = u_1^2 v_1^2/2!2!$$

$$c(1,3;4) = u_1 v_1^3/3!$$

$$c(0,4;4) = v_1^4/4!$$

$$A_5 = \sum_{\mu+v=1}^5 c(\mu, v, 5) f_{\mu, v}$$

$$c(1,0;5) = u_5$$

$$c(0,1;5) = v_5$$

$$c(2,0;5) = u_1 u_4 + u_2 u_3$$

$$c(1,1;5) = u_1 v_4 + u_2 v_3 + u_3 v_2 + u_4 v_1$$

$$c(0,2;5) = v_1 v_4 + v_2 v_3$$

$$c(3,0;5) = u_1 u_2^2/2! + u_1^2 u_3/2!$$

$$c(2,1;5) = u_1 u_2 v_2 + u_1 u_3 v_1 + u_1^2 v_3/2! + u_2^2 v_1/2!$$

$$c(1,2;5) = u_1 v_1 v_3 + u_2 v_1 v_2 + u_1 v_2^2/2! + u_3 v_1^2/2!$$

$$c(0,3;5) = v_1 v_2^2/2! + v_1^2 v_3/2!$$

$$c(4,0;5) = u_1^3 u_2/3!$$

$$c(3,1;5) = u_1^3 v_2/3! + u_1^2 u_2 v_1/2!$$

$$c(2,2;5) = u_1^2 v_1 v_2/2! + u_1 u_2 v_1^2/2!$$

$$c(1,3;5) = u_2 v_1^3/3! + u_1 v_1^2 v_2/2!$$

$$c(0,4;5) = v_1^3 v_2/3!$$

$$c(5,0;5) = u_1^5/5!$$

$$c(4,1;5) = u_1^4 v_1/4!$$

$$c(3,2;5) = u_1^3 v_1^2/2!3!$$

$$c(2,3;5) = u_1^2 v_1^3/2!3!$$

$$c(1,4;5) = u_1 v_1^4/4!$$

$$c(0,5;5) = v_1^5/5!$$

$$A_6 = \sum_{\mu+\nu=1}^6 c(\mu, \nu, 6) f_{\mu, \nu}$$

$$c(1, 0; 6) = u_6$$

$$c(0, 1; 6) = v_6$$

$$c(2, 0; 6) = u_1 u_5 + u_2 u_4 + u_3^2 / 2!$$

$$c(1, 1; 6) = u_1 v_5 + u_2 v_4 + u_3 v_3 + u_4 v_2 + u_5 v_1$$

$$c(0, 2; 6) = v_1 v_5 + v_2 v_4 + v_3^2 / 2!$$

$$c(3, 0; 6) = u_1 u_2 u_3 + u_1^2 u_4 / 2! + u_2^3 / 3!$$

$$c(2, 1; 6) = u_1 u_2 v_3 + u_1 u_3 v_2 + u_1 u_4 v_1 + u_2 u_3 v_1 + u_1^2 v_4 / 2! + u_2^2 v_2 / 2!$$

$$c(1, 2; 6) = u_1 v_1 v_4 + u_1 v_2 v_3 + u_2 v_1 v_3 + u_3 v_1 v_2 + u_2 v_2^2 / 2! + u_4 v_1^2 / 2!$$

$$c(0, 3; 6) = v_1 v_2 v_3 + v_1^2 v_4 / 2! + v_2^3 / 3!$$

$$c(4, 0; 6) = u_1^2 u_2^2 / 2! 2! + u_1^3 u_3 / 3!$$

$$c(3, 1; 6) = v_1 u_1 u_2^2 / 2! + v_1 u_1^2 u_3 / 2! + v_2 u_3 u_1^2 / 2! + v_3 u_1^3 / 3!$$

$$c(2, 2; 6) = u_1^2 v_1^2 / 2! 2! + v_1^2 u_2^2 / 2! 2! + u_1^2 v_1 v_3 / 2! + u_1 u_3 v_1^2 / 2!$$

$$c(1, 3; 6) = u_1 v_1 v_2^2 / 2! + u_1 v_1^2 v_3 / 2! + u_2 v_2 v_1^2 / 2! + u_3 v_1^3 / 3!$$

$$c(0, 4; 6) = v_1^2 v_2^2 / 2! 2! + v_1^3 v_3 / 3!$$

$$c(5, 0; 6) = u_1^5 u_2 / 4!$$

$$c(4, 1; 6) = u_1^4 v_2 / 4! + u_1^3 u_2 v_1 / 3!$$

$$c(3, 2; 6) = u_1^3 v_1 v_2 / 3! + u_1^2 u_2 v_1^2 / 2! 2!$$

$$c(2, 3; 6) = v_1^3 u_1 u_2 / 3! + v_1^2 v_2 u_1^2 / 2! 2!$$

$$c(1, 4; 6) = v_1^4 u_2 / 4! + v_1^3 v_2 u_1^2 / 3!$$

$$c(0, 5; 6) = v_1^5 v_2 / 4!$$

$$c(6, 0; 6) = u_1^6 / 6!$$

$$c(5, 1; 6) = u_1^5 v_1 / 5!$$

$$c(4, 2; 6) = u_1^4 v_1^2 / 2! 4!$$

$$c(3, 3; 6) = u_1^3 v_1^3 / 3! 3!$$

$$c(2, 4; 6) = u_1^2 v_1^4 / 2! 4!$$

$$c(1, 5; 6) = v_1^5 u_1 / 5!$$

$$c(0, 6; 6) = v_1^6 / 6!$$

$$A_7 = \sum_{\mu+\nu=1}^7 c(\mu, \nu, 7) f_{\mu, \nu}$$

$$c(1, 0; 7) = u_7$$

$$c(0, 1; 7) = v_7$$

$$c(2, 0; 7) = u_1 u_6 + u_2 u_5 + u_3 u_4$$

$$\begin{aligned}
c(1,1;7) &= u_1 v_6 + u_2 v_5 + u_3 v_4 + u_4 v_3 + u_5 v_2 + u_6 v_1 \\
c(0,2;7) &= v_1 v_6 + v_2 v_5 + v_3 v_4 \\
c(3,0;7) &= u_1 u_2 u_4 + u_1 u_3^2/2! + u_1^2 u_5/2! + u_2^2 u_3/2! \\
c(2,1;7) &= v_1 u_2 u_4 + v_2 u_1 u_4 + v_4 u_1 u_2 + v_1 u_3^2/2! + v_3 u_1 u_3 \\
&\quad + u_1^2 v_5/2! + v_1 u_1 u_5 + u_2^2 v_3/2! + v_2 u_2 u_3 \\
c(1,2;7) &= u_1 v_2 v_4 + u_2 v_1 v_4 + u_4 v_1 v_2 + u_1 v_3^2/2! + u_3 v_1 v_3 \\
&\quad + v_1^2 u_5/2! + u_1 v_1 v_5 + v_2^2 u_3/2! + u_2 v_2 v_3 \\
c(0,3;7) &= v_1 v_2 v_4 + v_1 v_3^2/2! + v_1^2 v_5/2! + v_2^2 v_3/2! \\
c(4,0;7) &= u_3^2 u_1/3! + u_1^2 u_2 u_3/2! + u_4 u_1^3/3! \\
c(3,1;7) &= v_1 u_2^3/3! + v_2 u_2^2 u_1/2! + v_1 u_1 u_2 u_3 + v_2 u_1^2 u_3/2! \\
&\quad + v_3 u_1^2 u_2/2! + v_4 u_1^3/3! + v_1 u_4 u_1^2/2! \\
c(2,2;7) &= u_2^2 v_2 v_1/2! + v_2^2 u_2 u_1/2! + u_1^2 v_2 v_3/2! + v_1^2 u_2 u_3/2! \\
&\quad + u_1 u_2 v_1 v_3 + v_1 v_2 u_1 u_3 + u_1 u_4 v_1^2/2! + v_1 v_4 u_1^2/2! \\
c(1,3;7) &= u_1 v_2^3/3! + u_2 v_2^2 v_1/2! + u_1 v_1 v_2 v_3 \\
&\quad + u_2 v_1^2 v_3/2! + u_3 v_1^2 v_2/2! + u_4 v_1^3/3! + u_1 v_4 v_1^2/2! \\
c(0,4;7) &= v_2^3 v_1/3! + v_1^2 v_2 v_3/2! + v_4 v_1^3/3! \\
c(5,0;7) &= u_1^3 u_2^2/2!3! + u_1^4 u_3/4! \\
c(4,1;7) &= u_1^3 u_2 v_2/3! + u_1^2 u_2^2 v_1/2!2! + u_1^4 v_3/4! + u_1^3 u_3 v_1/3! \\
c(3,2;7) &= u_1^3 v_2^2/2!3! + u_1 u_2^2 v_1^2/2!2! + u_1^2 u_2 v_1 v_2/2! \\
&\quad + u_1^3 v_1 v_3/3! + u_1^2 u_3 v_1^2/2!2! - \\
c(2,3;7) &= v_1^3 u_2^2/2!3! + v_1 v_2^2 u_1^2/2!2! + v_1^2 v_2 u_1 u_2/2! \\
&\quad + v_1^3 u_1 u_3/3! + v_1^2 v_3 u_1^2/2!2! \\
c(1,4;7) &= v_1^3 v_2 u_2/3! + v_1^2 v_2^2 u_1/2!2! + v_1^4 u_3/4! + v_1^3 v_3 u_1/3! \\
c(0,5;7) &= v_1^3 v_2^2/2!3! + v_1^4 v_3/4! \\
c(6,0;7) &= u_1^5 u_2/5! \\
c(5,1;7) &= u_1^5 v_2/5! + u_1^4 u_2 v_1/4! \\
c(4,2;7) &= u_1^4 v_1 v_2/4! + u_1^3 u_2 v_1^2/2!3! \\
c(3,3;7) &= u_1^3 v_1^2 v_2/2!3! + v_1^3 u_1^2 u_2/2!3! \\
c(2,4;7) &= v_1^4 u_1 u_2/4! + v_1^3 v_2 u_1^2/2!3! \\
c(1,5;7) &= v_1^5 u_2/5! + v_1^4 v_2 u_1/4! \\
c(0,6;7) &= v_1^5 v_2/5! \\
c(7,0;7) &= u_1^7/7! \\
c(6,1;7) &= u_1^6 v_1/6!
\end{aligned}$$

$$c(5,2;7) = u_1^5 v_1^2 / 2!5!$$

$$c(4,3;7) = u_1^4 v_1^3 / 3!4!$$

$$c(3,4;7) = v_1^4 u_1^3 / 3!4!$$

$$c(2,5;7) = u_1^5 u_1^2 / 2!5!$$

$$c(1,6;7) = v_1^6 u_1 / 6!$$

$$c(0,7;7) = v_1^7 / 7!$$

$$A_8 = \sum_{\mu+v=1}^8 c(\mu, v, 8) f_{\mu, v}$$

$$c(1,0;8) = u_8$$

$$c(0,1;8) = v_8$$

$$c(2,0;8) = u_1 u_7 + u_2 u_6 + u_3 u_5 + u_4^2 / 2!$$

$$c(1,1;8) = u_1 v_7 + u_2 v_6 + u_3 v_5 + u_4 v_4 + u_5 v_3 + u_6 v_2 + u_7 v_1$$

$$c(0,2;8) = v_1 v_7 + v_2 v_6 + v_3 v_5 + v_4^2 / 2!$$

$$c(3,0;8) = u_1 u_2 u_5 + u_1 u_3 u_4 + u_1^2 u_6 / 2! + u_2^2 u_4 / 2! + u_3^2 u_2 / 2!$$

$$\begin{aligned} c(2,1;8) = & v_1 u_2 u_5 + u_1 v_2 u_5 + u_1 u_2 v_5 + v_1 u_3 u_4 \\ & + u_1 v_3 u_4 + u_1 u_3 v_4 + v_1 u_1 u_6 + u_1^2 v_6 / 2! \\ & + u_2 v_2 u_4 + u_2^2 v_4 / 2! + v_3 u_3 u_2 + u_3^2 v_2 / 2! \end{aligned}$$

$$\begin{aligned} c(1,2;8) = & u_1 v_2 v_5 + v_1 u_2 v_5 + v_1 v_2 u_5 + u_1 v_3 v_4 \\ & + v_1 u_3 v_4 + v_1 v_3 u_4 + u_1 v_1 v_6 + v_1^2 u_6 / 2! \\ & + v_2 u_2 v_4 + v_2^2 u_4 / 2! + u_3 v_3 v_2 + v_3^2 u_2 / 2! \end{aligned}$$

$$c(0,3;8) = v_1 v_2 v_5 + v_1 v_3 v_4 + v_1^2 v_6 / 2! + v_2^2 v_4 / 2! + v_3^2 v_2 / 2!$$

$$c(4,0;8) = u_1^2 u_3^2 / 2!2! + u_1 u_2^2 u_3 / 2! + u_2^4 / 4! + u_1^2 u_2 u_4 / 2! + u_1^3 u_5 / 3!$$

$$\begin{aligned} c(3,1;8) = & v_1 u_1 u_3^2 / 2! + u_1^2 v_3 u_3 / 2! + v_1 u_1^2 u_3 / 2! \\ & + u_1 v_3 u_2^2 / 2! + u_1 u_3 u_2 v_2 + v_2 u_2^3 / 3! \\ & + u_1 v_1 u_2 u_4 + u_1^2 v_2 u_4 / 2! + u_1^2 u_2 v_4 / 2! \\ & + v_1 u_1^2 u_5 / 2! + u_1^3 v_5 / 3! \end{aligned}$$

$$\begin{aligned} c(2,2;8) = & u_1^2 v_3^2 / 2!2! + v_1^2 u_3^2 / 2!2! + u_1 v_1 u_3 v_3 \\ & + u_1 u_3 v_2^2 / 2! + v_1 v_3 u_2^2 / 2! + u_1 u_2 v_2 v_3 + v_1 v_2 u_2 u_3 \\ & + u_2^2 v_2^2 / 2!2! + u_1^2 v_2 v_4 / 2! + v_1^2 u_2 u_4 / 2! \\ & + u_1 u_2 v_1 v_4 + v_1 v_2 u_1 u_4 + u_1^2 v_1 v_5 / 2! + v_1^2 u_1 u_5 / 2! \end{aligned}$$

$$\begin{aligned} c(1,3;8) = & u_1 v_1 v_3^2 / 2! + v_1^2 u_3 v_3 / 2! + u_1 v_2^2 v_3 / 2! \\ & + v_1 u_3 v_2^2 / 2! + v_1 v_3 v_2 u_2 + u_2 v_2^3 / 3! + v_1 u_1 v_2 v_4 \\ & + v_1^2 u_2 v_4 / 2! + v_1^2 v_2 u_4 / 2! + u_1 v_1 v_5 / 2! + v_1^3 u_5 / 3! \end{aligned}$$

$$\begin{aligned}
c(0,4;8) &= v_1^2 v_2^3 / 2!2! + v_1 v_2^2 v_3 / 2! + v_2^4 / 4! \\
&\quad + v_1^2 v_2 v_4 / 2! + v_1^3 v_3 / 3! \\
c(5,0;8) &= u_1^2 u_2^3 / 2!3! + u_1^3 u_2 u_3 / 3! + u_1^4 u_4 / 4! \\
c(4,1;8) &= v_1 u_1 u_2^3 / 3! + u_1^2 u_2^2 v_2 / 2!2! + v_1 u_1^2 u_2 u_3 / 2! \\
&\quad + u_1^3 v_2 u_3 / 3! + u_1^3 u_2 v_3 / 3! + u_1^3 v_1 u_4 / 3! + u_1^4 v_4 / 4! \\
c(3,2;8) &= u_2^3 v_1^2 / 2!3! + u_1^2 u_2 v_2^2 / 2!2! + v_1 v_2 u_1 u_2^2 / 2! \\
&\quad + v_1^2 u_1 u_2 u_3 / 2! + u_1^2 v_1 v_2 u_3 / 2! \\
&\quad + u_1^2 v_1 u_2 v_3 / 2! + u_1^3 v_2 v_3 / 3! + u_1^3 v_1 v_4 / 3! \\
&\quad + u_1^2 u_4 v_1^2 / 2!2! \\
c(2,3;8) &= v_2^3 u_1^2 / 2!3! + v_1^2 v_2 u_2^2 / 2!2! + u_1 u_2 v_1 v_2^2 / 2! \\
&\quad + u_1^2 v_1 v_2 v_3 / 2! + v_1^2 u_1 u_2 v_3 / 2! \\
&\quad + v_1^2 u_1 v_2 u_3 / 2! + v_1^3 u_2 u_3 / 3! + v_1^3 u_1 u_4 / 3! + v_1^2 v_4 u_1^2 / 2!2! \\
c(1,4;8) &= u_1 v_1 v_2^3 / 3! + v_1^2 v_2^2 u_2 / 2!2! + u_1 v_1^2 v_2 v_3 / 2! \\
&\quad + v_1^3 u_2 v_3 / 3! + v_1^3 v_2 u_3 / 3! + v_1^3 u_1 v_4 / 3! + v_1^4 u_4 / 4! \\
c(0,5;8) &= v_1^2 v_2^3 / 2!3! + v_1^3 v_2 v_3 / 3! + v_1^4 v_4 / 4! \\
c(6,0;8) &= u_1^4 u_2^2 / 2!4! + u_1^5 u_3 / 5! \\
c(5,1;8) &= v_1 u_1^3 u_2^2 / 2!3! + v_2 u_1^4 u_2 / 4! + v_1 u_1^4 u_3 / 4! + u_1^5 v_3 / 5! \\
c(4,2;8) &= u_1^4 v_2^2 / 2!4! + v_1^2 u_1^2 u_2^2 / 2!2!2! \\
&\quad + v_1 v_2 u_1^3 u_2 / 3! + u_1^4 v_1 v_3 / 4! + u_1^3 v_1^2 u_3 / 2!3! \\
c(3,3;8) &= u_1^3 v_1 v_2^2 / 2!3! + v_1^3 u_1 u_2^2 / 2!3! \\
&\quad + v_1^2 v_2 u_2 u_1^2 / 2!2! + u_1^3 v_1^2 v_3 / 2!3! + v_1^3 u_1^2 u_3 / 2!3! \\
c(2,4;8) &= v_1^4 u_2^2 / 2!4! + u_1^2 v_1^2 v_2^2 / 2!2!2! \\
&\quad + u_1 u_2 v_1^3 v_2 / 3! + v_1^4 u_1 u_3 / 4! + v_1^3 u_1^2 v_3 / 2!3! \\
c(1,5;8) &= u_1 v_1^3 v_2^2 / 2!3! + u_2 v_1^4 v_2 / 4! + u_1 v_1^4 v_3 / 4! + v_1^5 u_3 / 5! \\
c(0,6;8) &= v_1^4 v_2^2 / 2!4! + v_1^5 v_3 / 5! \\
c(7,0;8) &= u_1^6 u_2 / 6! \\
c(6,1;8) &= u_1^6 v_2 / 6! + v_1 u_1^5 u_2 / 5! \\
c(5,2;8) &= u_1^5 v_1 v_2 / 5! + v_1^2 u_2 u_1^4 / 2!4! \\
c(4,3;8) &= u_1^4 v_1^2 v_2 / 2!4! + u_1^3 v_1^3 u_2 / 3!3! \\
c(3,4;8) &= v_1^4 u_1^2 u_2 / 2!4! + v_1^3 u_1^3 v_2 / 3!3! \\
c(2,5;8) &= v_1^5 u_1 u_2 / 5! + u_1^2 v_2 v_1^4 / 2!4! \\
c(1,6;8) &= v_1^6 u_2 / 6! + u_1 v_1^5 v_2^2 / 5! \\
c(0,7;8) &= v_1^6 v_2 / 6! \\
c(8,0;8) &= u_1^8 / 8!
\end{aligned}$$

$$c(7,1;8) = u_1^7 v_1 / 7!$$

$$c(6,2;8) = u_1^6 v_1^2 / 2! 6!$$

$$c(5,3;8) = u_1^5 v_1^3 / 3! 5!$$

$$c(4,4;8) = u_1^4 v_1^4 / 4! 4!$$

$$c(3,5;8) = v_1^5 u_1^3 / 3! 5!$$

$$c(2,6;8) = v_1^6 u_1^2 / 2! 6!$$

$$c(1,7;8) = v_1^7 u_1 / 7!$$

$$c(0,8;8) = v_1^8 / 8!$$

$$A_9 = \sum_{\mu+v=1}^9 c(\mu, v, 9) f_{\mu, v}$$

$$c(1,0;9) = u_9$$

$$c(0,1;9) = v_9$$

$$c(2,0;9) = u_1 u_8 + u_2 u_7 + u_3 u_6 + u_4 u_5$$

$$c(1,1;9) = u_1 v_8 + u_2 v_7 + u_3 v_6 + u_4 v_5 + u_5 v_4 + u_6 v_3 + u_7 v_2 + u_8 v_1$$

$$c(0,2;9) = v_1 v_8 + v_2 v_7 + v_3 v_6 + v_4 v_5$$

$$c(3,0;9) = u_1 u_2^2 / 2! + u_2^2 u_5 / 2! + u_2 u_3 u_4 + u_3^3 / 3! + u_1^2 u_7 / 2! + u_1 u_2 u_6 + u_1 u_3 u_5$$

$$c(2,1;9) = v_1 u_2^2 / 2! + u_1 u_4 u_5 + u_2^2 v_5 / 2! + u_2 v_2 u_5 + v_2 u_3 u_4 + u_2 v_3 u_4 + u_2 u_3 v_4 + v_3 u_3^2 / 2! + v_1 u_1 u_7 + u_1^2 v_7 / 2! + v_1 u_2 u_6$$

$$+ u_1 v_2 u_6 + u_1 u_2 v_6 + v_1 u_3 u_5 + u_1 v_3 u_5 + u_1 u_3 v_5$$

$$c(1,2;9) = u_1 v_4^2 / 2! + v_1 v_4 u_5 + v_2^2 u_5 / 2! + v_2 u_2 v_5 + u_2 v_3 v_4 + v_2 u_3 v_4 + v_2 v_3 u_4 + u_2 v_3^2 / 2! + u_1 v_1 v_7 + v_1^2 u_7 / 2! + u_1 v_2 v_6$$

$$+ v_1 u_2 v_6 + v_1 v_2 u_6 + u_1 v_3 v_5 + v_1 u_3 v_5 + v_1 v_3 u_5$$

$$c(0,3;9) = v_1 v_4^2 / 2! + v_2^2 v_5 / 2! + v_2 v_3 v_4 + v_3^3 / 3! + v_1^2 v_7 / 2! + v_1 v_2 v_6 + v_1 v_3 v_5$$

$$c(4,0;9) = u_2^3 u_3 / 3! + u_1 u_2 u_3^2 / 2! + u_1 - u_2^2 u_4 / 2! + u_1^2 u_3 u_4 / 2! + u_1^2 u_2 u_5 / 2! + u_1^3 u_6 / 3!$$

$$c(3,1;9) = u_3^3 v_3 / 3! + v_2 u_2^2 u_3 / 2! + v_1 u_2 u_3^2 / 2! + u_1 v_2 u_3^2 / 2! + u_1 u_2 u_3 v_3 + v_1 u_2^2 u_4 / 2! + u_1 u_2 v_2 u_4 + u_1 u_2^2 v_4 / 2! + v_1 u_1 u_3 u_4 + u_1^2 v_3 u_4 / 2! + u_1^2 u_3 v_4 / 2! + v_1 u_1 u_2 u_5 + u_1^2 v_2 u_5 / 2! + u_1^2 u_2 v_5 / 2! + u_1^3 v_6 / 3! + v_1 u_1^2 u_6 / 2!$$

$$c(2,2;9) = u_2^2 v_2 v_3 / 2! + v_2^2 u_2 u_3 / 2! + v_1 v_2 u_3^2 / 2! + u_1 u_2 v_3^2 / 2! + v_1 v_3 u_2 u_3 + u_1 u_3 v_2 v_3 + v_1 v_4 u_2^2 / 2! + u_1 u_4 v_2^2 / 2!$$

$$\begin{aligned}
& + u_1 u_2 v_2 v_4 + v_1 v_2 u_2 u_4 + u_1^2 v_3 v_4 / 2! + v_1^2 u_3 u_4 / 2! \\
& + v_1 v_3 u_1 u_4 + u_1 u_3 v_1 v_4 + v_1^2 u_2 u_5 / 2! + u_1^2 v_2 v_5 / 2! \\
& + u_1 u_2 v_1 v_5 + v_1 v_2 u_1 u_5 + u_1^2 v_1 v_6 / 2! + v_1^2 u_1 u_6 / 2! \\
c(1,3;9) = & v_2^3 u_3 / 3! + u_2 v_2^2 v_3 / 2! + u_1 v_2 v_3^2 / 2! + v_1 u_2 v_3^2 / 2! \\
& + v_1 v_2 v_3 u_3 + u_1 v_2^2 v_4 / 2! + v_1 v_2 u_2 v_4 + v_1 v_2^2 u_4 / 2! \\
& + u_1 v_1 v_3 v_4 + v_1^2 u_3 v_4 / 2! + v_1^2 v_3 u_4 / 2! + u_1 v_1 v_2 v_5 \\
& + v_1^2 u_2 v_5 / 2! + v_1^2 v_2 u_5 / 2! + v_1^3 u_6 / 3! + u_1 v_1^2 v_6 / 2! \\
c(0,4;9) = & v_2^3 v_3 / 3! + v_1 v_2 v_3^2 / 2! + v_1 v_2^2 v_4 / 2! + v_1^2 v_3 v_4 / 2! \\
& + v_1^2 v_2 v_5 / 2! + v_1^3 v_6 / 3! \\
c(5,0;9) = & u_1 u_2^4 / 4! + u_1^2 u_2^2 u_3 / 2! 2! + u_1^3 u_3^2 / 2! 3! + u_1^3 u_2 u_4 / 3! + u_1^4 u_5 / 4! \\
c(4,1;9) = & v_1 u_2^4 / 4! + u_1 v_2 u_2^3 / 3! + v_1 u_1 u_2^2 u_3 / 2! + u_1^2 u_2 v_2 u_3 / 2! \\
& + u_1^2 u_2^2 v_3 / 2! 2! + v_1 u_1^2 u_3^2 / 2! 2! + u_1^3 v_3 u_3 / 3! \\
& + v_1 u_1^2 u_2 u_4 / 2! + u_1^3 v_2 u_4 / 3! + u_1^3 u_2 v_4 / 3! \\
& + v_1 u_1^3 u_5 / 3! + u_1^4 v_5 / 4! \\
c(3,2;9) = & u_2^3 v_1 v_2 / 3! + u_1 u_2^2 v_2^2 / 2! 2! + v_1^2 u_2^2 u_3 / 2! 2! \\
& + u_1^2 v_2^2 u_3 / 2! 2! + v_1 v_3 u_1 u_2^2 / 2! + v_1 v_2 u_1 u_2 u_3 \\
& + v_2 v_3 u_1^2 u_2 / 2! + u_1^3 v_3^2 / 2! 3! + v_1^2 u_1 u_3^2 / 2! 2! \\
& + v_1 v_3 u_1^2 u_3 / 2! + v_1^2 u_1 u_2 u_4 / 2! + v_2 v_4 u_1^3 / 3! \\
& + v_1 v_2 u_4 u_1^2 / 2! + v_1 v_4 u_4 u_1^2 u_2 / 2! + v_1^2 u_1^2 u_5 / 2! 2! + v_1 v_5 u_1^3 / 3! \\
c(2,3;9) = & v_2^3 u_1 u_2 / 3! + v_1 v_2^2 u_2^2 / 2! 2! + u_1^2 v_2^2 v_3 / 2! 2! \\
& + v_1^2 u_2^2 v_3 / 2! 2! + u_1 u_3 v_1 v_2^2 / 2! + u_1 u_2 v_1 v_2 v_3 \\
& + u_2 u_3 v_1^2 v_2 / 2! + v_1^3 u_3^2 / 2! 3! + u_1^2 v_1 v_3^2 / 2! 2! \\
& + u_1 u_3 v_1^2 v_3 / 2! + u_1^2 v_1 v_2 v_4 / 2! + u_2 u_4 v_1^3 / 3! \\
& + u_1 u_2 v_4 v_1^2 / 2! + u_1 u_4 v_1^2 v_2 / 2! + u_1^2 v_1^2 v_5 / 2! 2! + u_1 u_5 v_1^3 / 3! \\
c(1,4;9) = & u_1 v_2^4 / 4! + v_1 u_2 v_2^3 / 3! + u_1 v_1 v_2^2 v_3 / 2! + v_1^2 v_2 v_3 / 2! \\
& + v_1^2 v_2^2 u_3 / 2! 2! + u_1 v_1^2 v_3^2 / 2! 2! + v_1^3 u_3 v_3 / 3! \\
& + u_1 v_1^2 v_2 v_4 / 2! + v_1^3 u_2 v_4 / 3! + v_1^3 v_2 u_4 / 3! \\
& + u_1 v_1^3 v_5 / 3! + v_1^4 u_5 / 4! \\
c(0,5;9) = & v_1 v_2^4 / 4! + v_1^2 v_2^2 v_3 / 2! 2! + v_1^3 v_3^2 / 2! 3! \\
& + v_1^3 v_2 v_4 / 3! + v_1^4 v_5 / 4! \\
c(6,0;9) = & u_1^3 u_2^3 / 3! 3! + u_1^4 u_2 u_3 / 4! + u_1^5 u_4 / 5! \\
c(5,1;9) = & v_1 u_1^2 u_2^3 / 2! 3! + v_2 u_1^3 u_2^2 / 2! 3! + u_1^4 v_2 u_3 / 4! \\
& + u_1^4 u_2 v_3 / 4! + v_1 u_1^3 u_2 u_3 / 3! + v_4 u_1^5 / 5! + v_1 u_1^4 u_4 / 4!
\end{aligned}$$

$$c(4,2;9) = v_1^2 u_1 u_2^3 / 2!3! + v_2^2 u_2 u_1^3 / 2!3! + v_1 v_2 u_1^2 u_2^2 / 2!2! \\ + v_2 v_3 u_1^4 / 4! + v_1^2 u_1^2 u_2 u_3 / 2!2! + v_1 v_2 u_1^3 u_3 / 3! \\ + v_1 v_3 u_1^3 u_2 / 3! + v_1^2 u_1^3 u_4 / 2!3! + v_1 v_4 u_1^4 / 4!$$

$$c(3,3;9) = u_1^3 v_2^3 / 3!3! + v_1^3 u_2^3 / 3!3! + u_1^2 u_2 v_2^2 v_1 / 2!2! + u_2^2 u_1 v_1^2 v_2 / 2!2! \\ + u_1^3 v_1 v_2 v_3 / 3! + v_1^3 u_1 u_2 u_3 / 3! + u_1^2 v_1^2 u_2 v_3 / 2!2! \\ + u_1^2 v_1^2 v_2 u_3 / 2!2! + u_1^3 v_1^2 v_4 / 2!3! + v_1^3 u_1^2 u_4 / 2!3!$$

$$c(2,4;9) = u_1^2 v_1 v_2^3 / 2!3! + u_2^2 v_2 v_1^3 / 2!3! + u_1 u_2 v_1^2 v_2^2 / 2!2! \\ + u_2 u_3 v_1^4 / 4! + u_1^2 v_1^2 v_2 v_3 / 2!2! + u_1 u_2 v_1^3 v_3 / 3! \\ + u_1 u_3 v_1^3 v_2 / 3! + u_1^2 v_1^3 v_4 / 2!3! + u_1 u_4 v_1^4 / 4!$$

$$c(1,5;9) = u_1 v_1^2 v_2^3 / 2!3! + u_2 v_1^3 v_2^2 / 2!3! + v_1^4 u_2 v_3 / 4! \\ + v_1^4 v_2 u_3 / 4! + u_1 v_1^3 v_2 v_3 / 3! + u_4 v_1^5 / 5! + u_1 v_1^4 v_4 / 4!$$

$$c(0,6;9) = v_1^3 u_2^3 / 3!3! + v_1^4 v_2 v_3 / 4! + v_1^5 v_4 / 5!$$

$$c(7,0;9) = u_1^5 v_2^2 / 2!5! + u_1^6 u_3 / 6!$$

$$c(6,1;9) = u_1^5 u_2 v_2 / 5! + u_1^4 u_2^2 v_1 / 2!4! + u_1^6 v_3 / 6! + u_1^5 u_3 v_1 / 5!$$

$$c(5,2;9) = u_1^5 v_2^2 / 2!5! + u_1^3 u_2^2 v_1^2 / 2!2!3! + u_1^4 u_2 v_1 v_2 / 4! \\ + u_1^5 v_1 v_3 / 5! + u_1^4 u_3 v_1^2 / 2!4!$$

$$c(4,3;9) = u_1^4 v_1 v_2^2 / 2!4! + u_1^2 u_2^2 v_1^3 / 2!2!3! + u_1^3 u_2 v_1^2 v_2 / 2!3! \\ + u_1^4 v_1^2 v_3 / 2!4! + u_1^3 u_3 v_1^3 / 3!3!$$

$$c(3,4;9) = v_1^4 u_1 u_2^2 / 2!4! + v_1^2 v_2^2 u_1^3 / 2!2!3! + v_1^3 v_2 u_1^2 u_2 / 2!3! \\ + v_1^4 u_1^2 u_3 / 2!4! + v_1^3 v_3 u_1^3 / 3!3!$$

$$c(2,5;9) = v_1^5 v_2^2 / 2!5! + v_1^3 v_2^2 u_1^2 / 2!2!3! + v_1^4 v_2 u_1 u_2 / 4! \\ + v_1^5 u_1 u_3 / 5! + v_1^4 v_3 u_1^2 / 2!4!$$

$$c(1,6;9) = v_1^5 v_2 u_2 / 5! + v_1^4 v_2^2 u_1 / 2!4! + v_1^6 u_3 / 6! + v_1^5 v_3 u_1 / 5!$$

$$c(0,7;9) = v_1^5 v_2^2 / 2!5! + v_1^6 v_3 / 6!$$

$$c(8,0;9) = u_1^7 u_2 / 7!$$

$$c(7,1;9) = u_1^7 v_2 / 7! + u_1^6 u_2 v_1 / 6!$$

$$c(6,2;9) = u_1^6 v_1 v_2 / 6! + u_1^5 u_2 v_1^2 / 2!5!$$

$$c(5,3;9) = u_1^5 v_1^2 v_2 / 2!5! + u_1^4 u_2 v_1^3 / 3!4!$$

$$c(4,4;9) = u_1^3 u_2 v_1^4 / 3!4! + v_1^3 v_2 u_1^4 / 3!4!$$

$$c(3,5;9) = v_1^5 u_1^2 u_2 / 2!5! + v_1^4 v_2 u_1^3 / 3!4!$$

$$c(2,6;9) = v_1^6 u_1 u_2 / 6! + v_1^5 v_2 u_1^2 / 2!5!$$

$$c(1,7;9) = v_1^7 u_2 / 7! + v_1^6 v_2 u_1 / 6!$$

$$\begin{aligned}
c(0,8;9) &= v_1^7 v_2 / 7! \\
c(9,0;9) &= u_1^9 / 9! \\
c(8,1;9) &= u_1^8 v_1 / 8! \\
c(7,2;9) &= u_1^7 v_1^2 / 2! 7! \\
c(6,3;9) &= u_1^6 v_1^3 / 3! 6! \\
c(5,4;9) &= u_1^5 v_1^4 / 4! 5! \\
c(4,5;9) &= u_1^4 v_1^5 / 4! 5! \\
c(3,6;9) &= u_1^3 v_1^6 / 3! 6! \\
c(2,7;9) &= u_1^2 v_1^7 / 2! 7! \\
c(1,8;9) &= u_1 v_1^8 / 8! \\
c(0,9;9) &= v_1^9 / 9!
\end{aligned}$$

CONVENIENT RULES FOR USE:

The A_n have been written in detail as a convenient reference and an aid in calculations. However, they can now be written by simply remembering the algorithm. The $c(\mu, \nu; n)$ are written by considering all possibilities for μ and ν with $\mu + \nu = n$. Inspection of the listed $c(\mu, \nu; n)$ will make it clear that μ tells us how many times u appears and ν tells us how many times v appears. Further, we see that the sum of all the subscripts is n and as with functions of a single variable, repeated indices require division by the factorial of the number of repetitions.

ANALYTIC FUNCTION OF SEVERAL VARIABLES:

Let's consider $f(u, v, w) \neq f_1(u)f_2(v)f_3(w)$. Thus, $N[u, v, w]$, with N a non-linear operator, acting on u is an analytic function $f(u, v, w)$ which we set equal to $\sum_{n=0}^{\infty} A_n$. Now we define

$$f_{\mu, \nu, \omega} = (\partial^\mu / \partial u_0) (\partial^\nu / \partial v_0) (\partial^\omega / \partial w_0) f(u_0, v_0, w_0)$$

Now

$$A_0 = f_{0,0,0}$$

$$A_1 = c(1,0,0;1)f_{1,0,0} + c(0,1,0;1)f_{0,1,0} + c(0,0,1;1)f_{0,0,1}$$

$$\begin{aligned}
A_2 &= c(1,0,0;2)f_{1,0,0} + c(0,1,0;2)f_{0,1,0} + c(0,0,1;2)f_{0,0,1} + c(2,0,0;2)f_{2,0,0} \\
&\quad + c(1,1,0;2)f_{1,1,0} + c(1,0,1;2)f_{1,0,1} \\
&\quad + c(0,1,1;2)f_{0,1,1} + c(0,2,0;2)f_{0,2,0} + c(0,0,2;2)f_{0,0,2}
\end{aligned}$$

The values of the $c(\mu, \nu, \omega)$ above are

$$\begin{array}{ll} c(1,0,0;1) = u_1 & c(1,1,0;2) = u_1 v_1 \\ c(0,1,0;1) = v_1 & c(1,0,1;2) = u_1 w_1 \\ c(0,0,1;1) = w_1 & c(0,1,1;2) = v_1 w_1 \\ c(1,0,0;2) = u_2 & c(2,0,0;2) = u_1^2/2! \\ c(0,1,0;2) = v_2 & c(0,2,0;2) = v_1^2/2! \\ c(0,0,1;2) = w_2 & c(0,0,2;2) = w_1^2/2! \end{array}$$

Thus

$$\begin{aligned} A_0 &= f(u_0, v_0, w_0) \\ A_1 &= u_1(\partial f / \partial u_0) + v_1(\partial f / \partial v_0) + w_1(\partial f / \partial w_0) \\ A_2 &= u_2(\partial f / \partial u_0) + v_2(\partial f / \partial v_0) + w_2(\partial f / \partial w_0) \\ &\quad + u_1 v_1(\partial^2 f / \partial u_0 \partial v_0) + u_1 w_1(\partial^2 f / \partial u_0 \partial w_0) \\ &\quad + v_1 w_1(\partial^2 f / \partial v_0 \partial w_0) + (u_1^2/2!)(\partial^2 f / \partial u_0^2) \\ &\quad + (v_1^2/2!)(\partial^2 f / \partial v_0^2) + (w_1^2/2!)(\partial^2 f / \partial w_0^2) \end{aligned}$$

etc. for A_n . We can proceed analogously for determination of A_n for functions $f(u_1, u_2, \dots, u_m)$.

APPLICATIONS:

The A_n for $f(u, v, w)$ is needed to solve three coupled nonlinear differential equations. In the author's form [2] for coupled equations, using decomposition we have

$$\begin{aligned} u &= L_1^{-1} g_1 - L_1^{-1} R_1(u, v, w) - L_1^{-1} N_1(u, v, w) \\ v &= L_2^{-1} g_2 - L_2^{-1} R_2(u, v, w) - L_2^{-1} N_2(u, v, w) \\ w &= L_3^{-1} g_3 - L_3^{-1} R_3(u, v, w) - L_3^{-1} N_3(u, v, w) \end{aligned}$$

We let $u = \sum_{n=0}^{\infty} u_n$, $v = \sum_{n=0}^{\infty} v_n$, $w = \sum_{n=0}^{\infty} w_n$ and we write $N_i(u, v, w) = f_i(u, v, w) = \sum_{n=0}^{\infty} A_n \{f(u, v, w)\}$ for $i = 1, 2, 3$. Then

$$\begin{aligned} u_0 &= \Phi_1 + L_1^{-1} g_1 \quad \text{where } L_1 \Phi_1 = 0 \\ v_0 &= \Phi_2 + L_2^{-1} g_2 \quad \text{where } L_2 \Phi_2 = 0 \\ w_0 &= \Phi_3 + L_3^{-1} g_3 \quad \text{where } L_3 \Phi_3 = 0 \end{aligned}$$

Similarly we require $A_n \{f(u_1, \dots, u_m)\}$ for m coupled operator equations. An example for a non-factorable nonlinearity $f(u, v)$ is the pair of coupled

equations

$$du/dx + a_1 u + b_1 v + f_1(u, v) = g_1$$

$$dv/dx + a_2 u + b_2 v + f_2(u, v) = g_2$$

Finally, we consider $N(u, v) = f(u, v) = e^{u+v}$. This is an interesting case for comparison purposes since it is a factorable nonlinearity: $e^{u+v} = e^u \cdot e^v$, so we can solve it as a product nonlinearity using $A_n\{f(u)\}$ or with the present results for $A_n\{f(u, v)\}$. We can now consider a set of two coupled equations in the general form:

$$L_1 u + R_1(u, v) + N(u, v) = g_1$$

$$L_2 v + R_2(u, v) + N(u, v) = g_2$$

where $N(u, v) = e^{u+v}$.

SOME FINAL REMARKS:

The definition of the L operator avoids difficult integrations involving Green's functions. The use of a finite approximation in series form for the excitation term, and calculation only to necessary accuracy simplifies integrations still further. (With Maclaurin expansion, for example, of trigonometric terms, one needs only integrals of t^n .) The avoidance of the necessity for perturbation and linearization means physically more correct solutions in many cases. The avoidance of discretized or grid methods avoids the computationally intensive procedures inherent in such methods. The decomposition method is continuous and requires significantly less processing time for the computation of results. It has been demonstrated that very few terms of the decomposition series are necessary for an accurate solution, and also that the integrations can be made simple by the suggested methods, or by symbolic methods, and use quite simple computer codes in comparison with methods such as finite differences or finite elements.

As we have shown, partial differential equations can be solved by choosing one operator for the inversion and considering all other derivatives to be included in the R operator. Hence we solve exactly as with an ordinary differential equation. We have the additional advantage of a single global method (for ordinary or partial differential equations as well as many other types of equations). The convergence is always sufficiently rapid to be valuable for numerical work. The initial term must be bounded (a reasonable assumption for a physical system) and L must be the highest-ordered differential.

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SUGGESTED READING

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DOUBLE DECOMPOSITION

In solving boundary-value problems by the decomposition method, we have seen that we can either retain the "constants" of integration in the u_0 term for the case of linear ordinary differential equations, re-evaluating the constants as more terms of the approximate solution φ_n are computed, or, we can use the u_0 evaluated to satisfy the boundary conditions and add constants of integration for each successive term u_n .

For a linear ordinary differential equation, it is more efficient to calculate an n -term approximation φ_n , carrying along the constants u_0 , and finally force φ_n to satisfy the boundary conditions, thus evaluating the constants of integration.

We now introduce an effective procedure which allows us decreased computation, especially in partial differential equations. This is done by a further decomposition, i.e., we now decompose the initial term $u_0 = \Phi$ into

$$\sum_{m=0}^{\infty} \Phi_m, \text{ i.e., } u_0 = \sum_{m=0}^{\infty} u_{0,m} \quad [1]$$

At first thought, this seems like an ill-advised procedure which can only slow convergence, since the new initial term Φ_0 or $u_{0,0}$ will be farther from the optimum value for u_0 . However, we will see that, as a result, we can use Φ_0 to determine a Φ which can then be used for further terms of φ_n without further evaluations. The boundary-value problem becomes an equivalent initial-value formulation in terms of Φ . This eliminates further matching to boundary conditions.

Let us again consider the equation $u_{xx} - u_{yy} = 0$ with $u(0,y) = 0$, $u(x,0) = 0$, $u(\pi/2,y) = \sin y$, and $u(x,\pi/2) = \sin x$ whose solution by decomposition is $u(x,y) = \sin x \sin y$. We will again use decomposition and also the concept of equality of the partial solutions of the operator equations, so only one operator equation needs to be considered. Also, we will decompose the u_0 term of the decomposition series, which means a double decomposition of the solution u . (This is a much preferable method to that of eigenvalue expansion in m dimensions.)

We have $L_x u = L_y u$ and can apply the inverse operator L_x^{-1} on both sides. Thus $L_x^{-1} L_x u = u - \Phi_x$ or $u = \Phi_x + L_x^{-1} L_y u$ with $u(0,y) = 0$ and $u(\pi/2,y) = \sin y$. Equivalently, we can start with $L_y u = L_x u$ and apply L_y^{-1} to write $u = \Phi_y + L_y^{-1} L_x u$ with $u(x,0) = 0$ and $u(x,\pi/2) = \sin x$.

As usual, we assume $u = \sum_{m=0}^{\infty} u_m$ but now we also decompose u_0 into $\sum_{m=0}^{\infty} u_{0,m}$. For the x conditions, we have

$$\sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} \Phi_{x,m} + L_x^{-1} L_y \sum_{m=0}^{\infty} u_m$$

with $u_0 = \Phi_{x,0}$ and $u_{m \geq 1} = \Phi_{x,m} + L_x^{-1} L_y u_{m-1}$. We can also write, using the y conditions, the equation

$$\sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} \Phi_{y,m} + L_y^{-1} L_x \sum_{m=0}^{\infty} u_m$$

with $u_0 = \Phi_{y,0}$ and $u_{m \geq 1} = \Phi_{y,m} + L_y^{-1} L_x u_{m-1}$. Since $L_x \Phi_x = 0$ and $L_y \Phi_y = 0$, we have

$$\Phi_{x,0} = \xi_0(y) + x\xi_1(y)$$

$$\Phi_{x,m} = \xi_{0,m}(y) + x\xi_{1,m}(y)$$

$$\Phi_{y,0} = \eta_0(x) + y\eta_1(x)$$

$$\Phi_{y,m} = \eta_{0,m}(x) + y\eta_{1,m}(x)$$

where the ξ 's and η 's arise from the indefinite integrations. The conditions given determine these integration "constants" for the approximate solution $\varphi_{m+1} = \sum_{n=0}^m u_n$. Thus $\varphi_{m+1}(0, y) = 0$ and $\varphi_{m+1}(\pi/2, y) = \sin y$ determine $\xi_{0,m}(y)$ and $\xi_{1,m}(y)$. Similarly, $\varphi_{m+1}(x, 0) = 0$ and $\varphi_{m+1}(x, \pi/2) = \sin x$ determine $\eta_{0,m}(x)$ and $\eta_{1,m}(x)$.

Let us consider improving approximations to the x -dimensional solution as we calculate increasing terms of the decomposition series. Of course, the approximation is the solution in the limit $m \rightarrow \infty$.

$$u = \Phi_x + L_x^{-1} L_y u = \xi_0 + x\xi_1 + L_x^{-1} L_y u$$

$$\varphi_1 = u_0 = \xi_{0,0} + x\xi_{1,0}$$

Since $\varphi_1(0, y) = 0$, $\xi_{0,0} = 0$. Since $\varphi_1(\pi/2, y) = \sin y$, $\xi_{1,0} = (2/\pi) \sin y$. Therefore, $\varphi_1 = u_0 = (2/\pi)x \sin y$. To calculate u_1 we have

$$\begin{aligned}
u_1 &= \xi_{0,1} + x\xi_{1,4} + L_x^{-1}L_y u_0 \\
L_y u_0 &= -(2/\pi)x \sin y \\
L_x^{-1}L_y u_0 &= -(2/\pi)(x^3/3!) \sin y \\
u_1 &= \xi_{0,1} + x\xi_{1,4} - (2/\pi)(x^3/3!) \sin y
\end{aligned}$$

A two-term approximation is given by $\varphi_2 = \varphi_1 + u_1$ (or $u_0 + u_1$); hence

$$\varphi_2 = (2/\pi)x \sin y - (2/\pi)(x^3/3!) \sin y + \xi_{0,1} + x\xi_{1,4}$$

Since $\varphi_2(0, y) = 0$, we have $\xi_{0,1} = 0$, and since $\varphi_2(\pi/2, y) = \sin y$, we have $\xi_{1,4} = (\pi/2)(\sin y)/3!$

$$\begin{aligned}
u_1 &= (\pi/2)(x \sin y)/3! - (2/\pi)(x^3 \sin y)/3! \\
u_2 &= \xi_{0,2} + x\xi_{1,2} + L_x^{-1}L_y u_1 \\
L_y u_1 &= (2/\pi)(x^3 \sin y)/3! - (\pi/2)(x \sin y)/3! \\
L_x^{-1}L_y u_1 &= (2/\pi)(x^5 \sin y)/5! - (\pi/2)(x^3 \sin y)/(3!)^2 \\
u_2 &= \xi_{0,2} + x\xi_{1,2} + (2/\pi)(x^5 \sin y)/5! - (\pi/2)(x^3 \sin y)/(3!)^2 \\
\varphi_3 &= \varphi_2 + u_2 \text{ or } u_0 + u_1 + u_2
\end{aligned}$$

etc. Summarizing, the components of u are

$$\begin{aligned}
u_0 &= (2/\pi)x \sin y \\
u_1 &= (\pi/2)(x \sin y)/3! - (2/\pi)(x^3 \sin y)/3! \\
u_2 &= \left\{ -(\pi/2)^3/5! + (\pi/2)^3/(3!)^2 \right\} x \sin y \\
&\quad - (\pi/2)(x^3 \sin y)/(3!)^2 + (2/\pi)(x^5 \sin y)/5!
\end{aligned}$$

etc. The approximate solutions $\varphi_1, \varphi_2, \varphi_3, \dots$ are:

$$\begin{aligned}
\varphi_1 &= (2/\pi)x \sin y \\
\varphi_2 &= (2/\pi + (\pi/2)/3!)x \sin y + (2/\pi)(-x^3/3!) \sin y \\
\varphi_3 &= 1/(\pi/2) + (\pi/2)/3! - (\pi/2)^3/5! + (\pi/2)^3/(3!)^2 x \sin y \\
&\quad + 1/(\pi/2) + (\pi/2)(3!)(-x^3/3!) \sin y \\
&\quad + 1/(\pi/2)(x^5/5!) \sin y
\end{aligned}$$

etc., or

$$\varphi_1 = (.6366198)x \sin y$$

$$\varphi_2 = (.8984192)x \sin y + (.6366198)(-x^3/3!) \sin y$$

$$\begin{aligned} \varphi_3 = & (.9737817)x \sin y + (.8984192)(-x^3/3!) \sin y \\ & + (.6366198)(x^5/5!) \sin y \end{aligned}$$

which converges very rapidly to the given solution. It is interesting to write the result as

$$\varphi_m = a_{m,0}x \sin y + a_{m,1}(-x^3/3!) \sin y + a_{m,2}(x^5/5!) \sin y + \dots$$

or

$$\varphi_m = \sum_{n=0}^{m-1} a_{m,n}(-1)^n(x^{2n+1})/(2n+1)! \sin y$$

where the $a_{m,n}$ are numerical series whose sum is 1; each term is delayed behind the preceding term. Now,

$$\lim_{m \rightarrow \infty} \varphi_m = \lim_{m \rightarrow \infty} \sum_{n=0}^{m-1} a_{m,n}(-1)^n(x^{2n+1})/(2n+1)! \sin y$$

where $\lim_{m \rightarrow \infty} a_{m,n} = 1$ for all n . Then

$$u = \lim_{m \rightarrow \infty} \varphi_m = \sum_{n=0}^{\infty} \{(-1)^n(x^{2n+1})/(2n+1)!\} \sin y = \sin x \cdot \sin y$$

The y -dimensional solution is $u = \sin y \sin x$ since, by symmetry, y is interchanged with x ; i.e., the partial solutions are identical.

Consider the example $u_{xx} + u_{yy} = g(x,y) = x^2 + y^2$ with $u(0,y) = 0$, $u(x,0) = 0$, $u(1,y) = y^2/2$, $u(x,1) = x^2/2$. We have shown previously, using decomposition, that the solution $u = x^2y^2/2$ can be obtained in only two terms. It is also clear that either the operator equation for $L_x u$ or for $L_y u$ can be used with appropriate inversions. Thus

$$L_x u = x^2 + y^2 - L_y u$$

$$L_x^{-1} L_x u = L_x^{-1}(x^2 + y^2) - L_x^{-1} L_y u$$

and since $L_x^{-1} L_x u = u - \Phi_x$

$$u = \Phi_x + L_x^{-1}(x^2 + y^2) - L_x^{-1}L_y u \quad (1)$$

Similarly,

$$u = \Phi_y + L_y^{-1}(x^2 + y^2) - L_y^{-1}L_x u \quad (2)$$

Using (1),

$$\begin{aligned} u_0 &= \Phi_x + L_x^{-1}(x^2 + y^2) \\ \sum_{m=0}^{\infty} u_m &= u_0 - L_x^{-1}L_y \sum_{m=0}^{\infty} u_m \\ u_{m+1} &= -L_x^{-1}L_y u_m \end{aligned}$$

for $m \geq 0$. Now, if we decompose the u_0 term as well, we write

$$\sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} \Phi_{x,m} + L_x^{-1}(x^2 + y^2) - L_x^{-1}L_y \sum_{m=0}^{\infty} u_m$$

Identifying $u_0 = \Phi_{x,0} + L_x^{-1}(x^2 + y^2)$, all other components are determined by

$$u_{m+1} = \Phi_{x,m+1} - L_x^{-1}L_y u_m \quad (3)$$

Proceeding analogously using (2)

$$\begin{aligned} u_0 &= \Phi_{y,0} + L_y^{-1}(x^2 + y^2) \\ u_{m+1} &= \Phi_{y,m+1} - L_y^{-1}L_x u_m \end{aligned} \quad (4)$$

Continuing with the x equation, i.e., (1) and (3),

$$\begin{aligned} \Phi_x &= \xi_0(y) + x\xi_1(y) \\ \Phi_{x,m} &= \xi_{0,m}(y) + x\xi_{1,m}(y) \end{aligned} \quad (5)$$

or from (2) and (4)

$$\begin{aligned} \Phi_y &= \eta_0(x) + y\eta_1(x) \\ \Phi_{y,m} &= \eta_{0,m}(x) + y\eta_{1,m}(x) \end{aligned} \quad (6)$$

The "constants" of (indefinite) integration are now matched with the approximate solutions φ_n for $n = 1, 2, \dots$ where $\varphi_{m+1} = \sum_{n=0}^m u_n$. Thus $\varphi_{m+1}(0, y) = 0$, $\varphi_{m+1}(1, y) = y^2/2$ determines $\xi_{0,m}(y)$ and $\xi_{1,m}(y)$ in (5). Similarly, $\varphi_{m+1}(x, 0) = 0$ and $\varphi_{m+1}(x, 1) = x^2/2$ determines $\eta_{0,m}(x)$ and $\eta_{1,m}(x)$.

Proceeding with the x -dimensional solution, $\Phi_x = \xi_0(y) + x\xi_1(y)$ and $u_0 = \xi_0 + x\xi_1 + L_x^{-1}(x^2 + y^2)$; after decomposition of u_0 ,

$$u_0 = \xi_{0,0} + x\xi_{1,0} + L_x^{-1}(x^2 + y^2)$$

$$u_{m+1} = \xi_{0,m+1} + x\xi_{1,m+1} - L_x^{-1}L_y u_m$$

Our first approximation is $\varphi_1 = u_0$, or

$$\varphi_1 = \xi_{0,0} + x\xi_{1,0} + x^4/12 + x^2y^2/2$$

where $\varphi_1(0, y) = 0$, $\varphi_1(1, y) = y^2/2$. Since $\varphi_1 = 0$, $\xi_{0,0} = 0$. Since $\varphi_1(1, y) = y^2/2$,

$$\xi_{1,0} + 1/12 + y^2/2 = y^2/2$$

or $\xi_{1,0} = -1/12$. Hence

$$u_0 = -x/12 + x^4/12 + x^2y^2/2$$

Then

$$u_1 = \xi_{0,1} + x\xi_{1,1} - L_x^{-1}L_y u_0$$

Since $L_y u_0 = x^2$ and $L_x^{-1}L_y u_0 = L_x^{-1}x^2 = x^4/12$

$$u_1 = \xi_{0,1} + x\xi_{1,1} - x^4/12$$

Then

$$\begin{aligned} \varphi_2 &= u_0 + u_1 = \varphi_1 + u_1 \\ &= \left\{ -\frac{x}{12} + \frac{x^4}{12} + \frac{x^2y^2}{2} \right\} + \left\{ \xi_{0,1} - x\xi_{1,1} - \frac{x^4}{12} \right\} \\ &= -\frac{x}{12} + \frac{x^2y^2}{2} + \xi_{0,1} + x\xi_{1,1} \end{aligned}$$

Since

$$\begin{aligned} \varphi_2(0, y) &= 0 & \xi_{0,1} &= 0 \\ \varphi_2(1, y) &= y^2/2 & \xi_{1,1} &= 1/12 \\ u_1 &= x/12 - x^4/12 \end{aligned}$$

We now have $\varphi_2 = x^2 y^2 / 2$, i.e., the exact solution in two terms. If we proceed further

$$u_2 = \xi_{0,2} + x\xi_{1,2} - L_x^{-1} L_y u_1$$

We have $L_y u_1 = 0$, $L_x^{-1} L_y u_1 = 0$

$$\varphi_3 = \varphi_2 + u_2 = x^2 y^2 / 2 + \xi_{0,2} + x\xi_{1,2}$$

and since $\varphi_3(0, y) = 0$, $\xi_{0,2} = 0$. Since $\varphi_3(1, y) = y^2 / 2$, $\xi_{1,2} = 0$; hence $u_2 = 0$ so $\varphi_3 = x^2 y^2 / 2$. We can continue to see $\lim_{m \rightarrow \infty} \varphi_{m+1} = u = x^2 y^2 / 2$. The same result is obtained from the y -dimensional solution.

We now apply the double decomposition to a linear ordinary differential equation represented by $Lu + Ru = g$ where L is the highest-ordered linear differential operator—in this example we choose $L = d^2/dx^2$ and R is a linear operator (the "remainder" operator) which can contain for this L no derivatives higher than the first (the order of R is always less than the order of L).

DIRICHLET CONDITIONS:

$$u(b_1) = \beta_1 \text{ and } u(b_2) = \beta_2$$

Solving for Lu and operating with L^{-1} , we have $u = \Phi + L^{-1}g - L^{-1}Ru$ where $L\Phi = 0$. Now let $u = \sum_{m=0}^{\infty} u_m$ and $\Phi = \sum_{m=0}^{\infty} \Phi_m$; then

$$\sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} \Phi_m + L^{-1}g - L^{-1}R \sum_{m=0}^{\infty} u_m$$

(where L^{-1} is a pure integration not involving constants). Let $\Phi_m = c_{0,m} + xc_{1,m}$ and define $u_0 = \Phi_0 + L^{-1}g$. Now $\Phi_0 = c_{0,0} + xc_{1,0}$. Matching φ_1 to the boundary conditions $c_{0,0}$ and $c_{1,0}$ are determined by two linear equations. Suppose $g = 0$ for simplicity. Then

$$c_{0,0} + b_1 c_{1,0} = \beta_1$$

$$c_{0,0} + b_2 c_{1,0} = \beta_2$$

or in matrix form

$$\begin{bmatrix} 1 & b_1 \\ 1 & b_2 \end{bmatrix} \cdot \begin{bmatrix} c_{0,0} \\ c_{1,0} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

if the determinant of the first matrix is non-zero. We now go to the next approximation φ_2 by first determining

$$u_1 = \Phi_1 - L^{-1}Ru_0$$

to get $\varphi_2 = \varphi_1 + u_1$. Matching φ_2 to the boundary conditions to evaluate the constants, φ_2 is determined completely. Continuing in this manner, we determine $u_2, u_3, \dots$ until we arrive at a satisfactory φ_m verifiable by substitution or stabilized numerically to sufficient accuracy. We have

$$u_m = \Phi_m - L^{-1}Ru_{m-1}$$

where $\Phi_m = c_{0,m} + b_1 c_{1,m}$ and $\varphi_{m+1} = \varphi_m + u_m$. Matching φ_{m+1} to the boundary conditions, we require

$$\varphi_{m+1}(b_1) = \varphi_m(b_1) + u_m(b_1)$$

$$u_m(b_1) = \Phi_m(b_1) - L^{-1}Ru_{m-1}(b_1)$$

where $\Phi_m(b_1) = c_{0,m} + b_1 c_{1,m}$. Substituting and matching the conditions,

$$c_{0,m} + b_1 c_{1,m} - L^{-1}Ru_{m-1}(b_1) + \varphi_m(b_1) = \beta_1$$

$$c_{0,m} + b_2 c_{1,m} - L^{-1}Ru_{m-1}(b_2) + \varphi_m(b_2) = \beta_2$$

Rearranging,

$$c_{0,m} + b_1 c_{1,m} = \beta_1 - \varphi_m(b_1) + L^{-1}Ru_{m-1}(b_1) \equiv \beta_{1,m}$$

$$c_{0,m} + b_2 c_{1,m} = \beta_2 - \varphi_m(b_2) + L^{-1}Ru_{m-1}(b_2) \equiv \beta_{2,m}$$

which we write simply as

$$c_{0,m} + b_1 c_{1,m} = \beta_{1,m}$$

$$c_{0,m} + b_2 c_{1,m} = \beta_{2,m}$$

or

$$\begin{bmatrix} 1 & b_1 \\ 1 & b_2 \end{bmatrix} \begin{bmatrix} c_{0,m} \\ c_{1,m} \end{bmatrix} = \begin{bmatrix} \beta_{1,m} \\ \beta_{2,m} \end{bmatrix}$$

where

$$\begin{bmatrix} \beta_{1,m} \\ \beta_{2,m} \end{bmatrix} = \begin{bmatrix} \beta_1 - \varphi_m(b_1) + L^{-1}Ru_{m-1}(b_1) \\ \beta_2 - \varphi_m(b_2) + L^{-1}Ru_{m-1}(b_2) \end{bmatrix}$$

Thus,

$$\begin{bmatrix} c_{0,m} \\ c_{1,m} \end{bmatrix} = \begin{bmatrix} 1 & b_1 \\ 1 & b_2 \end{bmatrix}^{-1} \begin{bmatrix} \beta_{1,m} \\ \beta_{2,m} \end{bmatrix}$$

Now Φ_m or $c_{0,m}$ and $c_{1,m}$ are determined and we remark that

$$\lim_{m \rightarrow \infty} \begin{bmatrix} c_{0,m} \\ c_{1,m} \end{bmatrix} = \lim_{m \rightarrow \infty} \begin{bmatrix} \beta_{1,m} \\ \beta_{2,m} \end{bmatrix} = 0$$

The decomposition of the initial term can be used for nonlinear boundary-value problems (for ordinary or partial differential equations) and also for linear partial differential equations. It is not necessary in linear ordinary differential equations where we can carry along the unevaluated u_0 and evaluate all at once in the φ_m , a simpler procedure. The objective of the decomposition of u_0 is to allow a convenient matching of the boundary conditions to any approximant φ_m , i.e., for any value of m . Each integration involves constants which are added to get a better u_0 . This gives us a useful procedure.

EXAMPLE: $u_{xx} + u_{yy} = 0$ with the conditions

$$u(a_1, y) = \alpha_1(y)$$

$$u(a_2, y) = \alpha_2(y)$$

$$u(x, b_1) = \beta_1(x)$$

$$u(x, b_2) = \beta_2(x)$$

Write $L_x u + L_y u = 0$. If we solve for $L_x u$, we have $u = \Phi_x - L_x^{-1} L_y u$ where $\Phi_x = \xi_0(y) + x\xi_1(y)$. Now decompose Φ_x also; thus $\Phi_x = \sum_{m=0}^{\infty} \Phi_{x,m}$. Then

$$u_0 = \xi_{0,0} + x\xi_{1,0}$$

$$u_1 = \xi_{0,1} + x\xi_{1,1} - L_x^{-1} L_y (\xi_{0,0} + x\xi_{1,0})$$

$$u_2 = \xi_{0,2} + x\xi_{1,2} - L_x^{-1} L_y (\xi_{0,1} + x\xi_{1,1}) + (L_x^{-1} L_y)^2 (\xi_{0,0} + x\xi_{1,0})$$

$\vdots$

$$u_m = \sum_{n=0}^m (-L_x^{-1} L_y)^n (\xi_{0,m-n} + x\xi_{1,m-n})$$

where $\xi_{0,m}$ and $\xi_{1,m}$ are determined by satisfying the boundary conditions with the approximate solution $\varphi_{m+1} = \sum_{i=0}^{\infty} u_i$; thus,

$$\varphi_{m+1}(a_1, y) = \alpha_1(y)$$

$$\varphi_{m+1}(a_2, y) = \alpha_2(y)$$

The solution is $\sum_{m=0}^{\infty} u_m$ or

$$\begin{aligned} u &= \sum_{m=0}^{\infty} \sum_{n=0}^m (-L_x^{-1} L_y)^n (\xi_{0,m-n} + x \xi_{1,m-n}) \\ u &= \sum_{m=0}^{\infty} \sum_{n=0}^m (-1)^n (x^{2n}/(2n)!)(\partial^{2n}/\partial y^{2n}) \xi_{0,m-n}(y) \\ &\quad + \sum_{m=0}^{\infty} \sum_{n=0}^m (-1)^n (x^{2n+1}/(2n+1)!)(\partial^{2n}/\partial y^{2n}) \xi_{1,m-n}(y) \\ u &= \sum_{n=0}^{\infty} (-1)^n (x^{2n}/(2n)!)(\partial^{2n}/\partial y^{2n}) \sum_{v=0}^{\infty} \xi_{0,v}(y) \\ &\quad + \sum_{n=0}^{\infty} (-1)^n (x^{2n+1}/(2n+1)!)(\partial^{2n}/\partial y^{2n}) \sum_{v=0}^{\infty} \xi_{1,v}(y) \end{aligned}$$

Since we decomposed u_0 ,

$$\xi_0(y) = \sum_{v=0}^{\infty} \xi_{0,v}(y)$$

$$\xi_1(y) = \sum_{v=0}^{\infty} \xi_{1,v}(y)$$

so that we obtain the solution

$$\begin{aligned} u &= \sum_{n=0}^{\infty} (-1)^n (x^{2n}/(2n)!)(\partial^{2n}/\partial y^{2n}) \xi_0(y) \\ &\quad + \sum_{n=0}^{\infty} (-1)^n (x^{2n+1}/(2n+1)!)(\partial^{2n}/\partial y^{2n}) \xi_1(y) \end{aligned}$$

We have seen that the solution can also be obtained from the equation for $L_y u$. Thus, if we write $L_y u = -L_x u$ and apply the inverse L_y^{-1} we have $u = \Phi_y - L_y^{-1} L_x u$ where

$$\Phi_y = \eta_0(x) + y\eta_1(x)$$

Now $u = \eta_0(x) + y\eta_1(x) - L_y^{-1}L_x u$ where $L_x = \partial^2/\partial x^2$ and L_y^{-1} is a two-fold indefinite integration with respect to y . We let $u = \sum_{n=0}^{\infty} u_n$ where u_0 is normally given by $\eta_0(x) + y\eta_1(x)$. We now decompose the u_n also, i.e.,

$$\begin{aligned} u_0 &= \eta_{0,0} + y\eta_{1,0} \\ u_1 &= \eta_{0,1} + y\eta_{1,1} - L_y^{-1}L_x[\eta_{0,0} + y\eta_{1,0}] \\ u_2 &= \eta_{0,2} + y\eta_{1,2} - L_y^{-1}L_x[\eta_{0,1} + y\eta_{1,1}] \\ &\quad + (L_y^{-1}L_x)^2[\eta_{0,0} + y\eta_{1,0}] \\ &\quad \vdots \\ u_m &= \sum_{n=0}^m (-L_y^{-1}L_x)^n [\eta_{0,m-n} + y\eta_{1,m-n}] \end{aligned}$$

where

$$\begin{aligned} \phi_{m+1}(x, b_1) &= \beta_1(x) \\ \phi_{m+1}(x, b_2) &= \beta_2(x) \end{aligned}$$

Now $u = \lim_{m \rightarrow \infty} \phi_{m+1}$. In the inhomogeneous case $g \neq 0$,

$$\begin{aligned} u_0 &= \Phi_0 + L^{-1}g \\ u_1 &= \Phi_1 - (L^{-1}R)L^{-1}g - (L^{-1}R)\Phi_0 \\ &\quad \vdots \\ u_m &= \sum_{n=0}^m (-L^{-1}R)^n \Phi_{m-n} + (-L^{-1}R)^m L^{-1}g \end{aligned}$$

Finally, summing to get the solution

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^m (-L^{-1}R)^n \Phi_{m-n} + (-L^{-1}R)^m L^{-1}g$$

Rearranging terms,

$$u = \sum_{m=0}^{\infty} (-L^{-1}R)^m \sum_{n=0}^{\infty} \Phi_n + (-L^{-1}R)^m L^{-1}g$$

$$u = \sum_{m=0}^{\infty} (-L^{-1}R)^m \{\Phi + L^{-1}g\}$$

The value of decomposition of the initial term is the matching of the boundary conditions for each ϕ_m for any m . Every integration has new constants to evaluate. Finally, we can add all the $c_{0,m}$ and $c_{1,m}$ separately to form a new c_0 and c_1 or equivalently, a new u_0 which now is close to a final value which would be reached as $n \rightarrow \infty$ in ϕ_n .

NONLINEAR CASE:

Consider the ordinary differential equation $Lu + Ru + Nu = g$. Solving for Lu and applying L^{-1} :

$$u = c_0(y) + xc_1(y) + L^{-1}g - L^{-1}Ru - L^{-1}Nu$$

(Again L^{-1} is an indefinite integration—in this case, two-fold.) Since $\Phi_x = c_0(y) + xc_1(y)$ is decomposed,

$$\begin{aligned} u_0 &= \Phi_{x,0} + L^{-1}g \\ u_1 &= \Phi_{x,1} - L^{-1}Ru_0 + L^{-1}A_0 \\ &\vdots \\ u_m &= \Phi_{x,m} - L^{-1}Ru_{m-1} - L^{-1}A_{m-1} \end{aligned}$$

The solution $u = \sum_{m=0}^{\infty} u_m$ and $\Phi_x = \sum_{m=0}^{\infty} \Phi_{x,m}$ where

$$\Phi_{x,m} = c_{0,m}(y) + xc_{1,m}(y)$$

NONLINEAR CASE—PARTIAL DIFFERENTIAL EQUATION:

Consider $L_x u + L_y u + Ru + Nu = g$. We assume that $L_x = \partial^2 / \partial x^2$ and consider the equation for $L_x u$ with L_y treated as another R term. If there are also operators L_z and L_t , they are treated exactly like L_y ; then,

$$L_x u = g - L_y u - Ru - Nu$$

$$u = \Phi_x + L_x^{-1}g - L_x^{-1}L_y u - L_x^{-1}Ru - L_x^{-1}Nu$$

where $\Phi_x = c_0(y) + xc_1(y)$. Decomposing Φ_x where $\Phi_x = c_0(y) + xc_1(y)$,

$$u_0 = c_{0,0} + xc_{1,0} + L_x^{-1}g$$

$$u_1 = c_{0,1} + xc_{1,1} + L_x^{-1}L_y u_0 - L_x^{-1}Ru_0 - L_x^{-1}A_0$$

⋮

To evaluate, we have $\phi_1 = u_0$ which is matched to the boundary conditions. Suppose the x conditions are $u(b_1, y) = \beta_1$ and $u(b_2, y) = \beta_2$. Then

$$c_{0,0} + b_1 c_{1,0} + L_x^{-1}g = \beta_1$$

$$c_{0,0} + b_2 c_{1,0} + L_x^{-1}g = \beta_2$$

so that

$$c_{0,0} + b_1 c_{1,0} = \beta_1 - L_x^{-1}g$$

$$c_{0,0} + b_2 c_{1,0} = \beta_2 - L_x^{-1}g$$

or

$$\begin{bmatrix} 1 & b_1 \\ 1 & b_2 \end{bmatrix} \cdot \begin{bmatrix} c_{0,0} \\ c_{1,0} \end{bmatrix} = \begin{bmatrix} \beta_1 - L_x^{-1}g \\ \beta_2 - L_x^{-1}g \end{bmatrix}$$

from which we determine $c_{0,0}$ and $c_{1,0}$, so that $\phi_1 = u_0$ is determined completely. Now u_1 is calculated from

$$u_1 = c_{0,1} + xc_{1,1} - L_x^{-1}L_y u_0 - L_x^{-1}Ru_0 - L_x^{-1}A_0$$

Since u_0 has been determined, u_1 is calculable, so we obtain $\phi_2 = \phi_1 + u_1$ which is matched to the boundary conditions using

$$c_{0,1} + b_1 c_{1,1} - L_x^{-1}L_y u_0 - L_x^{-1}Ru_0 - L_x^{-1}A_0 + \phi_1(b_1) = \beta_1$$

$$c_{0,1} + b_2 c_{1,1} - L_x^{-1}L_y u_0 - L_x^{-1}Ru_0 - L_x^{-1}A_0 + \phi_1(b_2) = \beta_2$$

so that ϕ_2 is known. The process is continued to a satisfactory ϕ_n .

Consider the equation $Lu + Ru = 0$ where $L = d^n/dt^n$ and R is a linear operator possibly involving differentials of lower order. The integral representation is:

$$u = \Phi - L^{-1}Ru$$

where L^{-1} is an integral operator defined as an n -fold integration and $u = \sum_{m=0}^{\infty} u_m$ yields the solution in series form. It is interesting to consider a double series representation

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{m,n}$$

$$u_m = \sum_{n=0}^{\infty} u_{m,n}$$

If L is of n th order in the single series, we have $u_0 = \Phi$ where $L\Phi = 0$ and Φ has n terms. Suppose then we decompose $\Phi = \sum_{m=0}^{\infty} \Phi_m$ and let $u_0 = \Phi_0$ only. Now u_1 , which was previously identified as $u_1 = -L^{-1}R\Phi$, becomes

$$u_1 = \Phi_1 - L^{-1}R\Phi_0$$

$$u_2 = \Phi_2 - L^{-1}R\Phi_1 + (L^{-1}R)^2\Phi_0$$

$$\vdots$$

$$u_m = \Phi_m - L^{-1}R\Phi_{m-1} + \dots + (-L^{-1}R)^{m-1}\Phi_1 + (-L^{-1}R)^m\Phi_0$$

Now

$$\sum_{n=0}^m u_n = \sum_{n=0}^m \Phi_n - L^{-1}R \sum_{n=0}^{m-1} \Phi_n + \dots + (-L^{-1}R)^m \Phi_0$$

The approximate (m -term) solution is given by:

$$\phi_{m+1} = \sum_{n=0}^m \Phi_n - L^{-1}R \sum_{n=0}^{m-1} \Phi_n + \dots + (-L^{-1}R)^{m-1} \sum_{n=0}^1 \Phi_n + (-L^{-1}R)^m \Phi_0$$

which we can write as a double summation

$$\begin{aligned}\phi_{m+1} &= \sum_{r=1}^m \sum_{n=0}^{m-r+1} (-L^{-1}R)^{r-1} \Phi_n \\ \lim_{m \rightarrow \infty} \phi_{m+1} &= \lim_{m \rightarrow \infty} \sum_{r=1}^m \sum_{n=0}^{m-r+1} (-L^{-1}R)^{r-1} \Phi_n = \sum_{r=1}^{\infty} \sum_{n=0}^{\infty} (-L^{-1}R)^{r-1} \Phi_n \\ &= \sum_{r=1}^{\infty} (-L^{-1}R)^{r-1} \sum_{n=0}^{\infty} \Phi_n = \sum_{m=0}^{\infty} (-L^{-1}R)^m \Phi = u\end{aligned}$$

so we see that our m -term approximation becomes the *exact* solution in the limit as expected.

We can now view initial-value and boundary-value problems in the same way, offering clear advantages over finite difference or shooting methods. Thus in *initial-value* problems,

$$\begin{array}{lll} u_0 = \Phi & u_{0,n} = \Phi_n & \sum_{n=0}^{\infty} u_{0,n} = \sum_{n=0}^{\infty} \Phi_n \\ u_1 = -L^{-1}R\Phi & u_{1,n} = -L^{-1}R\Phi_n & \sum_{n=0}^{\infty} u_{1,n} = -L^{-1}R \sum_{n=0}^{\infty} \Phi_n \\ u_2 = (-L^{-1}R)^2 \Phi & u_{2,n} = (-L^{-1}R)^2 \Phi_n & \sum_{n=0}^{\infty} u_{2,n} = (-L^{-1}R)^2 \sum_{n=0}^{\infty} \Phi_n \\ \vdots & & \\ u_m = (-L^{-1}R)^m \Phi & u_{m,n} = (-L^{-1}R)^m \Phi_n & \sum_{n=0}^{\infty} u_{m,n} = (-L^{-1}R)^m \sum_{n=0}^{\infty} \Phi_n \end{array}$$

The approximation ϕ_r is given by

$$\phi_r = \sum_{m=0}^{r-1} (-L^{-1}R)^m \sum_{n=0}^{\infty} \Phi_n$$

In the limit $r \rightarrow \infty$, this becomes $\sum_{m=0}^{\infty} (-L^{-1}R)^m \Phi = u$.

In the *boundary-value* representation of the decomposition components of the solution u ,

$$\begin{aligned}u_0 &= u_{0,0} = \Phi_0 \\ u_1 &= u_{0,1} + u_{1,0} = \Phi_1 - L^{-1}R\Phi_0\end{aligned}$$

$$\begin{aligned}
u_2 &= u_{0,2} + u_{1,1} + u_{2,0} = \Phi_2 - L^{-1}R\Phi_1 + (L^{-1}R)^2\Phi_0 \\
&\vdots \\
u_m &= u_{0,m} + u_{1,m-1} + \dots + u_{m-1,1} + u_{m,0} \\
&= \Phi_m - L^{-1}R\Phi_{m-1} + \dots + (-L^{-1}R)^{m-1}\Phi_1 + (-L^{-1}R)^m\Phi_0
\end{aligned}$$

and the approximant to the solution ϕ_{m+1} is given by the staggered summations. Thus,

$$\phi_r = \sum_{m=0}^{r-1} \sum_{n=0}^{r-1-m} (-L^{-1}R)^m \Phi_n$$

Again in the limit of the approximations we get u ; thus

$$\begin{aligned}
\phi_r &= \sum_{m=0}^{r-1} \sum_{n=0}^{r-1-m} (-L^{-1}R)^m \Phi_n \\
\lim_{r \rightarrow \infty} \phi_r &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-L^{-1}R)^m \Phi_n = \sum_{m=0}^{\infty} (-L^{-1}R)^m \sum_{n=0}^{\infty} \Phi_n = \sum_{m=0}^{\infty} (-L^{-1}R)^m \Phi = u
\end{aligned}$$

We now have an initial-value format for boundary-value problems. We can determine Φ_m by evaluating an approximation ϕ_{m+1} at the boundary conditions, then use it in the initial-value format for a better solution, i.e., one even closer to the final u . The two limiting forms of our approximation are equal.

To summarize, in a boundary-value problem we compute $\phi_{m+1} = \phi_m + u_m$ and evaluate at the boundary conditions. Now we can approximate a good value for u_0 from

$$u_0 = \Phi \approx \sum_{n=0}^{r-1} \Phi_n$$

Then, using the approximation for Φ , calculate

$$\phi_k = \sum_{m=0}^{k-1} (-L^{-1}R)^m \Phi$$

for k as large as we wish without further evaluations at the boundary conditions of ϕ_m for higher values of the index m .

HOMOGENEOUS NONLINEAR ORDINARY DIFFERENTIAL EQUATION WITH GIVEN BOUNDARY CONDITIONS:

Starting from the usual decomposition form $Lu + Nu = 0$ where Nu is an analytic function $f(u)$, the usual integral representation of the solution $u = \Phi - L^{-1}Nu$ with the (single) decomposition $u = \sum_{m=0}^{\infty} u_m$ and $Nu = \sum_{m=0}^{\infty} A_m$ is now written using double decomposition,

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{m,n}$$

$$u_m = \sum_{n=0}^{\infty} u_{m,n}$$

Now our usual polynomials A_m must also be doubly decomposed; thus:

$$f(u) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n}$$

$$A_m = \sum_{n=0}^{\infty} A_{m,n}$$

Now

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{m,n} = \sum_{m=0}^{\infty} \Phi_m - L^{-1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n}$$

where we have decomposed the initial, or u_0 , term taking only $u_{0,0} = \Phi_0$ as the first term. It is not essential, but we can also utilize the analytic grouping parameter λ . In such a case,

$$\sum_{m=0}^{\infty} \lambda^m u_m = \sum_{m=0}^{\infty} \lambda^m \Phi_m - L^{-1} \sum_{m=0}^{\infty} \lambda^{m+1} A_m$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda^{m+n} u_{m,n} = \sum_{m=0}^{\infty} \lambda^m \Phi_m - L^{-1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda^{m+n+1} A_{m,n}$$

Without λ

$$\sum_{m=0}^{\infty} u_m = \Phi - L^{-1} \sum_{m=0}^{\infty} A_m$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{m,n} = \sum_{m=0}^{\infty} \Phi_m - L^{-1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n}$$

An initial-value solution to $Lu + Nu = 0$ is provided by:

$$\begin{aligned}u_0 &= \Phi \\u_1 &= -L^{-1}A_0 \\u_2 &= -L^{-1}A_1 \\\vdots \\u_m &= -L^{-1}A_{m-1}\end{aligned}$$

A boundary-value solution using decomposition of the initial term gives us

$$\begin{aligned}u_0 &= \Phi_0 \\u_1 &= \Phi_1 - L^{-1}A_0 \\u_2 &= \Phi_2 - L^{-1}A_1 \\\vdots \\u_m &= \Phi_m - L^{-1}A_{m-1}\end{aligned}$$

(There are no integration constants implied by L^{-1} , i.e., it represents a pure two-fold integration with no constants.) The A_m in the boundary-value problem are decomposed into $A_{m,n}$; thus:

$$\begin{aligned}A_0 &= A_{0,0} \\A_1 &= A_{0,1} + A_{1,0} \\A_2 &= A_{0,2} + A_{1,1} + A_{2,0} \\\vdots \\A_n &= \sum_{m=0}^n A_{m,n-m} \\u_0 &= \Phi_0 \\u_1 &= \Phi_1 - L^{-1}(A_{0,0}) \\u_2 &= \Phi_2 - L^{-1}(A_{0,1} + A_{1,0}) \\\vdots \\u_n &= \Phi_n - L^{-1} \sum_{m=0}^{n-1} A_{m,n-1-m}\end{aligned}$$

$$\begin{aligned}
 u_{n-1} &= \Phi_{n-1} - L^{-1} \sum_{m=0}^n A_{m,n-m} \\
 \phi_r &= \sum_{m=0}^{r-1} \Phi_m - L^{-1} \sum_{m=0}^{r-2} \sum_{n=0}^{r-2-m} A_{m,n} \\
 \lim_{r \rightarrow \infty} \phi_r &= \sum_{m=0}^{\infty} \Phi_m - L^{-1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n} = \Phi - L^{-1} \sum_{m=0}^{\infty} A_m = u
 \end{aligned}$$

demonstrating convergence. The value of Φ_m is determined by evaluating $\phi_{m-1} = \phi_m + u_m$ at the boundary conditions.

In the initial-value problem, $\phi_r = \sum_{m=0}^{r-1} u_m$ and $\lim_{r \rightarrow \infty} \phi_r = \sum_{m=0}^{\infty} u_m = u$ or again,

$$\begin{aligned}
 \phi_r &= \Phi - L^{-1} \sum_{m=0}^{r-2} A_m \\
 \lim_{r \rightarrow \infty} \phi_r &= \Phi - L^{-1} \sum_{m=0}^{\infty} A_m = u
 \end{aligned}$$

What we accomplish, after computing several terms of ϕ_n , is to minimize our computation for boundary-value problems by avoiding the further matching to the boundary conditions. Because of the rapid convergence in decomposition, computation was already minimal in comparison to discretized methods, and the above procedure offers further decrease.

EXAMPLES OF BOUNDARY-VALUE PROBLEMS:

We now calculate two boundary-value problems to demonstrate how one can change the boundary-value format to an equivalent initial-value format, decreasing computation and accelerating convergence by using the concept of double decomposition.

The first example is an ordinary differential equation. The second is a partial differential equation which is considered both in the temporal format (t-coordinate partial solution) and the spatial format (x-coordinate partial solution); convergence of the spatial solution is accelerated by transforming it into the initial-value format. The procedure can also be used for nonlinear equations.

Consider the ordinary differential equation

$$d^2u / dx^2 + \alpha u = \beta(x)$$

with boundary conditions

$$\begin{aligned}
 u(x_1) &= \xi_1 \\
 u(x_r) &= \xi_{r-2} \\
 \phi_r &= \Phi - L^{-1} \sum_{m=0}^r A_m \\
 \lim_{r \rightarrow \infty} \phi_r &= \Phi - L^{-1} \sum_{m=0}^{\infty} A_m = u \\
 \beta(x) &= \sum_{m=0}^{\infty} \beta_m x^m
 \end{aligned}$$

In operator format with $L = d^2 / dx^2$, we have

$$Lu + \alpha u = \beta(x)$$

Then solving for Lu and operating with L^{-1} , which is always an indefinite integration for boundary-value problems, we have

$$u = u_0 - L^{-1} \alpha u$$

where

$$u_0 = A_0 + B_0 x + \iint \beta(x) dx dx$$

By writing $u = \sum_{n=0}^{\infty} u_n$, the solution is decomposed into a sum of components to be determined, and $\phi_m = \sum_{n=0}^{m-1} u_n$ is the "approximant" to the solution, i.e., a λ -term approximation converging to u in the limit. The one-term approximant is $\phi_1 = u_0$ and we must have

$$\phi_1(x_1) = \xi_1$$

$$\phi_1(x_2) = \xi_2$$

Following components are given by

$$u_\ell = -L^{-1} \alpha u_{\ell-1}$$

and $\phi_{\lambda+1} = \phi_\lambda + u_\lambda$. The increasingly accurate approximations must still satisfy the boundary conditions, hence

$$\phi_{\lambda}(x_1) = \xi_1$$

$$\phi_{\lambda}(x_2) = \xi_2$$

$$\phi_{\lambda+1}(x_1) = \xi_1$$

$$\phi_{\lambda+1}(x_2) = \xi_2$$

Then for $\lambda > 0$,

$$u_{\lambda}(x_1) = 0$$

$$u_{\lambda}(x_2) = 0$$

while for $\lambda = 0$,

$$u_0(x_1) = \xi_1$$

$$u_0(x_2) = \xi_2$$

Since

$$u_{\gamma} = A_0 + B_0 x + L^{-1} \beta(x) = A_0 + B_0 x + \sum_{m=0}^{\infty} \frac{\beta_m x^{m+2}}{(m+1)(m+2)} = \phi_1$$

and we know that

$$\phi_1(x_1) = \xi_1$$

$$\phi_1(x_2) = \xi_2$$

we have

$$A_0 + B_0 x_1 + \sum_{m=0}^{\infty} \frac{\beta_m x_1^{m+2}}{(m+1)(m+2)} = \xi_1$$

$$A_0 + B_0 x_2 + \sum_{m=0}^{\infty} \frac{\beta_m x_2^{m+2}}{(m+1)(m+2)} = \xi_2$$

Let's write

$$A_0 + B_0 x_1 = \xi_1^{(0)}$$

$$A_0 + B_0 x_2 = \xi_2^{(0)}$$

where

$$\xi_1^{(0)} = \xi_1 - \sum_{m=0}^{\infty} \frac{\beta_m x_1^{m+2}}{(m+1)(m+2)}$$

$$\xi_2^{(0)} = \xi_2 - \sum_{m=0}^{\infty} \frac{\beta_m x_2^{m+2}}{(m+1)(m+2)}$$

Then,

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \cdot \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} \xi_1^{(0)} \\ \xi_2^{(0)} \end{pmatrix}$$

which we can symbolize as a simple matrix equation $xA = \xi$ or $A = x^{-1}\xi$ if $x_1 \neq x_2$, a trivial condition, since the points x_1, x_2 must be distinct. We now have

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \frac{1}{x_2 - x_1} \begin{pmatrix} x_2 & -x_1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \xi_1^{(0)} \\ \xi_2^{(0)} \end{pmatrix}$$

Thus

$$\begin{aligned} A_0 &= \frac{x_2 \xi_1^{(0)} - x_1 \xi_2^{(0)}}{x_2 - x_1} \\ B_0 &= \frac{\xi_2^{(0)} - \xi_1^{(0)}}{x_2 - x_1} \\ u_0 &= A_0 + B_0 x + \sum_{m=0}^{\infty} \frac{\beta_m x^{m+2}}{(m+1)(m+2)} \\ &= \sum_{m=0}^{\infty} a_m^{(0)} x^m = \phi_1 \end{aligned}$$

with

$$a_0^{(0)} = A_0$$

$$a_1^{(0)} = B_0$$

and

$$a_{m+2}^{(0)} = \frac{\beta_m}{(m+1)(m+2)}$$

Now we calculate the u_1 component to get the ϕ_2 approximant, recalling that L^{-1} represents indefinite integration.

$$\begin{aligned} u_1 &= -L^{-1} \alpha u_0 \\ &= A_1 + B_1 x - \iint \alpha \sum_{m=0}^{\infty} a_m^{(0)} x^m dx dx \\ &= A_1 + B_1 x - \sum_{m=0}^{\infty} \frac{\alpha a_m^{(0)} x^{m+2}}{(m+1)(m+2)} \end{aligned}$$

We have $u_1(x_1) = 0$ and $u_1(x_2) = 0$ and we let

$$A_1 + B_1 x_1 = \zeta_1^{(1)}$$

$$A_1 + B_1 x_2 = \zeta_2^{(1)}$$

and let

$$\zeta_1^{(1)} = x_1^2 \sum_{m=0}^{\infty} \frac{\alpha a_m^{(0)} x_1^m}{(m+1)(m+2)}$$

$$\zeta_2^{(1)} = x_2^2 \sum_{m=0}^{\infty} \frac{\alpha a_m^{(0)} x_2^m}{(m+1)(m+2)}$$

Proceeding as before to solve for the "constants of integration" which we prefer to call matching coefficients,

$$A_1 = \frac{x_2 \zeta_1^{(1)} - x_1 \zeta_2^{(1)}}{x_2 - x_1}$$

$$B_1 = \frac{\zeta_2^{(1)} - \zeta_1^{(1)}}{x_2 - x_1}$$

$$u_1 = A_1 + B_1 x - \alpha \sum_{m=0}^{\infty} \frac{a_m^{(0)} x^{m+2}}{(m+1)(m+2)}$$

$$u_1 = \sum_{m=0}^{\infty} a_m^{(1)} x^m$$

with $a_0^{(1)} = A_1$ and $a_1^{(1)} = B_1$ and

$$a_{m+2}^{(1)} = \frac{-\alpha a_m^{(0)}}{(m+1)(m+2)}$$

We now have $\phi_2 = \phi_1 + u_1$ and can proceed in the same manner to a general term u_ℓ .

$$u_\ell = -L^{-1} \alpha u_{\ell-1}$$

$$u_{\ell-1} = \sum_{m=0}^{\infty} a_m^{(\ell-1)} x^m$$

$$u_\ell = A_\ell + B_\ell x - x^2 \sum_{m=0}^{\infty} \frac{\alpha a_m^{(\ell-1)} x^m}{(m+1)(m+2)}$$

$$u_\ell(x_1) = u_\ell(x_2) = 0$$

$$A_\ell + B_\ell x_1 = \zeta_1^{(\ell)}$$

$$A_\ell + B_\ell x_2 = \zeta_2^{(\ell)}$$

where

$$\xi_1^{(\ell)} = x_1^2 \sum_{m=0}^{\infty} \frac{\alpha a_m^{(\ell-1)} x_1^m}{(m+1)(m+2)}$$

$$\xi_2^{(\ell)} = x_2^2 \sum_{m=0}^{\infty} \frac{\alpha a_m^{(\ell-1)} x_2^m}{(m+1)(m+2)}$$

or

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \cdot \begin{pmatrix} A_\ell \\ B_\ell \end{pmatrix} = \begin{pmatrix} \xi_1^{(\ell)} \\ \xi_2^{(\ell)} \end{pmatrix}$$

so that

$$\begin{pmatrix} A_\ell \\ B_\ell \end{pmatrix} = \frac{1}{x_2 - x_1} \begin{pmatrix} x_2 & -x_1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \xi_1^{(\ell)} \\ \xi_2^{(\ell)} \end{pmatrix}$$

We now have

$$A_\ell = \frac{x_2 \xi_1^{(\ell)} - x_1 \xi_2^{(\ell)}}{x_2 - x_1}$$

$$B_\ell = \frac{\xi_2^{(\ell)} - \xi_1^{(\ell)}}{x_2 - x_1}$$

(where, of course x_1, x_2 are distinct points in a boundary-value problem)
Therefore,

$$u_\ell = A_\ell + B_\ell x - x^2 \sum_{m=0}^{\infty} \frac{\alpha a_m^{(\ell-1)} x^m}{(m+1)(m+2)}$$

$$u_\ell = \sum_{m=0}^{\infty} a_m^{(\ell)} x^m$$

where

$$a_0^{(\ell)} = A_\ell$$

$$a_1^{(\ell)} = B_\ell$$

$$a_{m+2}^{(\ell)} = \frac{-\alpha a_m^{(\ell-1)}}{(m+1)(m+2)}$$

and finally $\phi_{\ell-1} = \phi_\ell + u_\ell$.

SUMMARY:

$$u_0 = \sum_{m=0}^{\infty} a_m^{(0)} x^m$$

$$u_1 = \sum_{m=0}^{\infty} a_m^{(1)} x^m$$

$$\vdots$$

$$u_l = \sum_{m=0}^{\infty} a_m^{(l)} x^m$$

and

$$\phi_{m+l} = \sum_{\ell=0}^m u_{\ell}$$

is the $(m+1)$ -term approximant to the solution u , which we can also write as

$$u = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_m^{(n)} x^m = \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{\infty} a_m^{(n)} \right\} x^m = \sum_{m=0}^{\infty} a_m x^m$$

We observe that

$$\lim_{m \rightarrow \infty} \phi_{m+1}\{u\} = u$$

$$\lim_{m \rightarrow \infty} \phi_{m+1}\{a_m\} = a_m$$

since

$$u = \sum_{n=0}^{\infty} u_n$$

$$u_n = \sum_{m=0}^{\infty} a_m^{(n)} x^m$$

Upon substitution,

$$u = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_m^{(n)} x^m = \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{\infty} a_m^{(n)} \right\} x^m = \sum_{m=0}^{\infty} a_m x^m$$

where

$$a_m = \sum_{n=0}^{\infty} a_m^{(n)}$$

Finally we note that no difficulty exists in extension to nonlinear cases since it only requires use of our A_n polynomials for the nonlinear term.

We again consider the same ordinary differential equation

$$d^2u/dx^2 + \alpha u = \beta(x) = \sum_{m=0}^{\infty} \beta_m x^m$$

but with the initial conditions

$$u(0) = A \approx \phi_{m+1}\{A\} = \sum_{n=0}^m A_n$$

$$\frac{du(0)}{dx} = B \approx \phi_{m+1}\{B\} = \sum_{n=0}^m B_n$$

The $(m+1)$ -term approximant ϕ_{m+1} of u_0 is

$$\phi_{m+1}\{u_0\} = \phi_{m+1}\{A\} + x \phi_{m+1}\{B\}$$

We can now avoid further evaluations of boundary conditions, as we did earlier, by recasting the problem into the initial-value problem format. We can then continue with less work, i.e., without further matching to the boundary conditions. This acceleration of convergence by recasting boundary-value problems into initial-value format becomes more helpful as the problem complexity grows and matching boundary conditions becomes more painful, because we then have a more accurate initial term to work with. Thus,

$$\begin{aligned} u(0) &= \phi_{m+1}\{A\} \\ \frac{du(0)}{dx} &= \phi_{m+1}\{B\} \end{aligned}$$

will yield identical analytical and numerical solutions to the boundary-value format solutions. Thus beginning with $Lu + \alpha u = B(x)$,

$$\begin{aligned} Lu &= \beta(x) - \alpha u \\ u &= u_0 - L^{-1} \alpha u \end{aligned}$$

where L^{-1} is the definite integration operator $L^{-1} = \int_0^x \int_0^x (\cdot) dx dx$ and

$$u_0 = u(0) + x \frac{du(0)}{dx} + L^{-1} \beta(x)$$

or

$$u_0 = A + Bx + \sum_{m=0}^{\infty} \frac{\beta_m x^{m+2}}{(m+1)(m+2)}$$

We can write u_0 as

$$u_0 = \sum_{m=0}^{\infty} a_m^{(0)} x^m$$

where

$$a_0^{(0)} = A$$

$$a_1^{(0)} = B$$

$$a_{m+2}^{(0)} = \frac{\beta_m}{(m+1)(m+2)}$$

We note now that no further boundary condition evaluations are necessary for computation of the u_m for any m . Continuing,

$$u_m = -L^{-1} \alpha u_{m-1}$$

$$u_1 = -L^{-1} \alpha u_0$$

$$u_0 = \sum_{m=0}^{\infty} a_m^{(0)} x^m$$

$$u_1 = L^{-1} \alpha \sum_{m=0}^{\infty} a_m^{(0)} x^m = -\alpha \sum_{m=0}^{\infty} \frac{a_m^{(0)} x^{m+2}}{(m+1)(m+2)}$$

$$u_1 = x^2 \sum_{m=0}^{\infty} a_m^{(1)} x^m$$

where

$$a_m^{(1)} = \frac{-\alpha a_m^{(0)}}{(m+1)(m+2)}$$

$$a_m^{(2)} = \frac{-\alpha a_m^{(1)}}{(m+3)(m+4)}$$

$$u_{n>0} = x^{2n} \sum_{m=0}^{\infty} a_m^{(n)} x^m$$

where

$$a_m^{(n)} = \frac{-\alpha a_m^{(n-1)}}{(2n+m-1)(2n+m)}$$

Since $\phi_{m-1} = \sum_{n=0}^m u_n$ and $u = \sum_{n=0}^{\infty} u_n$, $u = \sum_{n=0}^{\infty} x^{2n} \sum_{m=0}^{\infty} a_m^{(n)} x^m$. Staggered summation is applicable. (See Appendix II.)

SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS IN SPATIAL AND TEMPORAL FORMATS:

$$\frac{\partial^2 u}{\partial x^2} + \alpha(x, t)u + \frac{\partial^2 u}{\partial t^2} = \beta(x, t)$$

with the initial conditions

$$u(0, x) = \tau_1(x)$$

$$\frac{\partial u(0, x)}{\partial t} = \tau_2(x)$$

and boundary conditions

$$u(x_1, t) = \xi_1(t)$$

$$u(x_2, t) = \xi_2(t)$$

We suppose α and β are given in the form:

$$\alpha(x, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} x^m t^n$$

$$\beta(x, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta_{m,n} x^m t^n$$

and the conditions are also in series form:

$$u(0, x) = \sum_{m=0}^{\infty} \tau_m^{(1)} x^m$$

$$\frac{\partial u(0, x)}{\partial t} = \sum_{m=0}^{\infty} \tau_m^{(2)} x^m$$

$$u(x_1, t) = \sum_{n=0}^{\infty} \xi_n^{(1)} t^n$$

$$u(x_2, t) = \sum_{n=0}^{\infty} \xi_n^{(2)} t^n$$

Let $L = \partial^2 / \partial x^2$ and write

$$\begin{aligned}Lu &= \beta(x, t) - \alpha(x, t)u - \partial^2 u / \partial t^2 \\ u &= u_0 - L^{-1}\alpha(x, t)u - L^{-1}(\partial^2 / \partial t^2)u\end{aligned}$$

where

$$u_0 = A_0(t) + x B_0(t) + L^{-1} \beta(x, t)$$

The solution u is the decomposition $u = \sum_{n=0}^{\infty} u_n$ and the approximant is

$\phi_m = \sum_{n=0}^{m-1} u_n$. Now

$$\begin{aligned}\phi_1 &= u_0 \\ \phi_1(x_1, t) &= \xi_1(t) \\ \phi_1(x_2, t) &= \xi_2(t)\end{aligned}$$

The general component is

$$\begin{aligned}u_n &= -L^{-1} \alpha(x, t)u_{n-1} - L^{-1}(\partial^2 / \partial t^2)u_{n-1} \\ L^{-1}(\cdot) &= A_n(t) + x B_n(t) + \iint (\cdot) dx dx\end{aligned}$$

Clearly

$$\begin{aligned}\phi_{m+1} &= \phi_m + u_m \\ \phi_m(x_1, t) &= \xi_1(t) \\ \phi_m(x_2, t) &= \xi_2(t) \\ \phi_{m+1}(x_1, t) &= \xi_1(t) \\ \phi_{m+1}(x_2, t) &= \xi_2(t)\end{aligned}$$

Thus for $m > 0$,

$$\begin{aligned}u_m(x_1, t) &= 0 \\ u_m(x_2, t) &= 0\end{aligned}$$

and for $m = 0$,

$$\begin{aligned}u_0(x_1, t) &= \xi_1(t) \\ u_0(x_2, t) &= \xi_2(t)\end{aligned}$$

We have

$$u_0 = A_0(t) + x B_0(t) + \iint \beta(x, t) dx dx$$

$$\beta(x, t) = \sum_{m=0}^{\infty} \beta_m(t) x^m$$

$$\beta_m(t) = \sum_{n=0}^{\infty} \beta_{m,n} t^n$$

We can also write

$$A_0(t) = \sum_{n=0}^{\infty} A_n^{(0)} t^n$$

$$B_0(t) = \sum_{n=0}^{\infty} B_n^{(0)} t^n$$

Now

$$u_0 = A_0(t) + x B_0(t) + \sum_{m=0}^{\infty} \frac{\beta_m(t) x^{m+2}}{(m+1)(m+2)} = \phi_1$$

We now have the first approximant; it must satisfy the boundary conditions hence,

$$A_0(t) + x_1 B_0(t) + \sum_{m=0}^{\infty} \frac{\beta_m(t) x_1^{m+2}}{(m+1)(m+2)} = \xi_1(t)$$

$$A_0(t) + x_2 B_0(t) + \sum_{m=0}^{\infty} \frac{\beta_m(t) x_2^{m+2}}{(m+1)(m+2)} = \xi_2(t)$$

Let's write this as

$$A_0(t) + x_1 B_0(t) = \xi_1^{(0)}(t)$$

$$A_0(t) + x_2 B_0(t) = \xi_2^{(0)}(t)$$

where

$$\xi_1^{(0)}(t) = \xi_1(t) - \sum_{m=0}^{\infty} \frac{\beta_m(t) x_1^{m+2}}{(m+1)(m+2)}$$

$$\xi_2^{(0)}(t) = \xi_2(t) - \sum_{m=0}^{\infty} \frac{\beta_m(t) x_2^{m+2}}{(m+1)(m+2)}$$

or

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \cdot \begin{pmatrix} A_0(t) \\ B_0(t) \end{pmatrix} = \begin{pmatrix} \xi_1^{(0)}(t) \\ \xi_2^{(0)}(t) \end{pmatrix}$$

from which we find that

$$A_0(t) = \frac{x_2 \xi_1^{(0)}(t) - x_1 \xi_2^{(0)}(t)}{x_2 - x_1}$$

$$B_0(t) = \frac{\xi_2^{(0)}(t) - \xi_1^{(0)}(t)}{x_2 - x_1}$$

so that

$$u_0 = A_0(t) + x B_0(t) + \sum_{m=0}^{\infty} \frac{\beta_m(t) x^{m+1}}{(m+1)(m+2)}$$

becomes

$$u_0 = \sum_{m=0}^{\infty} a_m^{(0)}(t) x^m$$

where

$$a_0^{(0)}(t) = A_0(t)$$

$$a_1^{(0)}(t) = B_0(t)$$

and

$$a_{m+2}^{(0)}(t) = \frac{\beta_m(t)}{(m+1)(m+2)}$$

Since

$$A_0(t) = \sum_{n=0}^{\infty} A_n^{(0)} t^n$$

$$B_0(t) = \sum_{n=0}^{\infty} B_n^{(0)} t^n$$

$$\beta_m(t) = \sum_{n=0}^{\infty} \beta_{m,n}^{(0)} t^n$$

we can write

$$a_0^{(0)}(t) = \sum_{n=0}^{\infty} a_{0,n}^{(0)} t^n = \sum_{n=0}^{\infty} A_n^{(0)} t^n$$

$$a_1^{(0)}(t) = \sum_{n=0}^{\infty} a_{1,n}^{(0)} t^n = \sum_{n=0}^{\infty} B_n^{(0)} t^n$$

$$a_{m+2}^{(0)}(t) = \sum_{n=0}^{\infty} a_{m+2,n}^{(0)} t^n = \sum_{n=0}^{\infty} \frac{\beta_{m,n}^{(0)} t^n}{(m+1)(m+2)}$$

Thus

$$u_0 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(0)} x^m t^n = \phi_1$$

or

$$u_0 = \sum_{m=0}^{\infty} a_m^{(0)}(t) x^m$$

where

$$a_m^{(0)}(t) = \sum_{n=0}^{\infty} a_{m,n}^{(0)} t^n$$

We can now calculate the u_1 component and the $\phi_2 = \phi_1 + u_1$ approximant. (We point out that although we explain in considerable detail, the procedure is simple and straightforward and is easily programmed or even calculated by hand.)

For the u_1 component, we have

$$\begin{aligned} u_1 &= -L^{-1} \alpha(x, t) u_0 - L^{-1} (\partial^2 / \partial t^2) u_0 \\ u_0 &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(0)} x^m t^n \\ \alpha(x, t) u_0(x, t) &= \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} x^m t^n \right\} \cdot \left\{ \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} a_{\mu,\nu}^{(0)} x^{\mu} t^{\nu} \right\} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} \bar{a}_{\mu,\nu}^{(0)} \end{aligned}$$

which is the Cauchy product $\alpha(x, t) u_0(x, t)$. (See Appendix III.)

$$L^{-1}(\cdot) = A_1(t) + x B_1(t) + \iint (\cdot) dx dx$$

$$A_1(t) = \sum_{n=0}^{\infty} A_n^{(1)} t^n$$

$$B_1(t) = \sum_{n=0}^{\infty} B_n^{(1)} t^n$$

$$\begin{aligned} u_1 &= A_1(t) + x B_1(t) - \iint \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} \bar{a}_{\mu,\nu}^{(0)} dx dx \\ &\quad - \iint \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (n+1)(n+2) a_{m,n+2}^{(0)} x^m t^n dx dx \end{aligned}$$

$$u_1 = A_1(t) + xB_1(t)$$

$$-x^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{x^m t^n \left[(n+1)(n+2)a_{m,n+2}^{(0)} + \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} a_{\mu, \nu}^{(0)} \right]}{(m+1)(m+2)}$$

Since $u_1(x_1, t)$ and $u_1(x_2, t)$ must be zero

$$A_1(t) + x_1 B_1(t) = \xi_1^{(1)}(t)$$

$$A_1(t) + x_2 B_1(t) = \xi_2^{(1)}(t)$$

where

$$\xi_1^{(1)}(t) = x_1^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x_1^m t^n \frac{\left[(n+1)(n+2)a_{m,n+2}^{(0)} + \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} a_{\mu, \nu}^{(0)} \right]}{(m+1)(m+2)}$$

$$\xi_2^{(1)}(t) = x_2^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x_2^m t^n \frac{\left[(n+1)(n+2)a_{m,n+2}^{(0)} + \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} a_{\mu, \nu}^{(0)} \right]}{(m+1)(m+2)}$$

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \cdot \begin{bmatrix} A_1(t) \\ B_1(t) \end{bmatrix} = \begin{bmatrix} \xi_1^{(1)}(t) \\ \xi_2^{(1)}(t) \end{bmatrix}$$

so that we obtain

$$A_1(t) = \frac{x_2 \xi_1^{(1)}(t) - x_1 \xi_2^{(1)}(t)}{x_2 - x_1}$$

$$B_1(t) = \frac{\xi_2^{(1)}(t) - \xi_1^{(1)}(t)}{x_2 - x_1}$$

(Of course, $x_2 \neq x_1$.) We can now write

$$u_1 = \sum_{m=0}^{\infty} a_m^{(1)}(t) x^m$$

where

$$a_0^{(1)}(t) = A_1(t)$$

$$a_1^{(1)}(t) = B_1(t)$$

and

$$a_{m+2}^{(1)}(t) = - \sum_{n=0}^{\infty} t^n \frac{\left[(n+1)(n+2)a_{m,n+2}^{(0)} + \sum_{\mu=0}^m \sum_{v=0}^n \alpha_{m-\mu,n-v} \alpha_{\mu,v}^{(0)} \right]}{(m+1)(m+2)}$$

Note that

$$A_1(t) = \sum_{n=0}^{\infty} A_n^{(1)} t^n$$

$$B_1(t) = \sum_{n=0}^{\infty} B_n^{(1)} t^n$$

Finally

$$u_1 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(1)} x^m t^n$$

$$\phi_2 = \phi_1 + u_1$$

where

$$a_{0,n}^{(1)} = A_n^{(1)}$$

$$a_{1,n}^{(1)} = B_n^{(1)}$$

$$a_{m+2,n}^{(1)} = - \frac{\left[(n+1)(n+2)a_{m,n+2}^{(0)} + \sum_{\mu=0}^m \sum_{v=0}^n \alpha_{m-\mu,n-v} a_{\mu,v}^{(0)} \right]}{(m+1)(m+2)}$$

or

$$u_1 = \sum_{m=0}^{\infty} a_m^{(1)}(t) x^m$$

where

$$a_m^{(1)}(t) = \sum_{n=0}^{\infty} a_{m,n}^{(1)} t^n$$

We can continue in this manner to calculate $u_2, u_3, \dots$. To compute the general term u_ℓ and the $\phi_{\ell+1}$ approximant, we write

$$u_\ell = -L^{-1} \alpha(x, t) u_{\ell-1} - L^{-1} (\partial^2 / \partial t^2) u_{\ell-1}$$

$$u_{\ell-1} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\ell-1)} x^m t^n$$

$$\frac{\partial^2}{\partial t^2} u_{\ell-1} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (n+1)(n+2) a_{m,n-2}^{(\ell-1)} x^m t^n$$

$$\alpha(x, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} x^m t^n$$

The Cauchy product $\alpha(x, t)u_{\ell-1}(x, t)$ is given by

$$\left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} x^m t^n \right\} \cdot \left\{ \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} a_{\mu,\nu}^{(\ell-1)} x^\mu t^\nu \right\}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} a_{\mu,\nu}^{(\ell-1)}$$

$$L^{-1}(\cdot) = A_\ell(t) + x B_\ell(t) + \iint (\cdot) dx dx$$

$$A_\ell(t) = \sum_{n=0}^{\infty} A_n^{(\ell)} t^n$$

$$B_\ell(t) = \sum_{n=0}^{\infty} B_n^{(\ell)} t^n$$

$$u_\ell = A_\ell(t) + x B_\ell(t) - \iint \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} a_{\mu,\nu}^{(\ell-1)} dx dx$$

$$- \iint \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (n+1)(n+2) a_{m,n+2}^{(\ell-1)} x^m t^n dx dx$$

$$u_\ell = A_\ell(t) + x B_\ell(t)$$

$$- x^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n \frac{\left[(n+1)(n+2) a_{m,n+2}^{(\ell-1)} + \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} a_{\mu,\nu}^{(\ell-1)} \right]}{(m+1)(m+2)}$$

$$u_\ell(x_1, t) = 0$$

$$u_\ell(x_2, t) = 0$$

Let

$$A_\ell(t) + x_1 B_\ell(t) = \xi_1^{(\ell)}(t)$$

$$A_\ell(t) + x_2 B_\ell(t) = \xi_2^{(\ell)}(t)$$

where

$$\xi_1^{(\ell)}(t) = x_1^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x_1^m t^n \frac{\left[(n+1)(n+2) a_{m,n+2}^{(\ell-1)} + \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} a_{\mu,\nu}^{(\ell-1)} \right]}{(m+1)(m+2)}$$

$$\xi_2^{(\ell)}(t) = x_2^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x_2^m t^n \frac{\left[(n+1)(n+2) a_{m,n+2}^{(\ell-1)} + \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} \alpha_{\mu, \nu}^{(\ell-1)} \right]}{(m+1)(m+2)}$$

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \cdot \begin{bmatrix} A_\ell(t) \\ B_\ell(t) \end{bmatrix} = \begin{bmatrix} \xi_1^{(\ell)}(t) \\ \xi_2^{(\ell)}(t) \end{bmatrix}$$

Consequently,

$$A_\ell(t) = \frac{x_2 \xi_1^{(\ell)}(t) - x_1 \xi_2^{(\ell)}(t)}{x_2 - x_1}$$

$$B_\ell(t) = \frac{\xi_2^{(\ell)}(t) - \xi_1^{(\ell)}(t)}{x_2 - x_1}$$

We can now write

$$u_\ell = \sum_{m=0}^{\infty} a_m^{(\ell)}(t) x^m$$

with

$$a_0^{(\ell)}(t) = A_\ell(t)$$

$$a_1^{(\ell)}(t) = B_\ell(t)$$

$$a_{m-2}^{(\ell)}(t) = - \sum_{n=0}^{\infty} t^n \frac{\left[(n+1)(n+2) a_{m,n+2}^{(\ell-1)} + \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} \alpha_{\mu, \nu}^{(\ell-1)} \right]}{(m+1)(m+2)}$$

Since

$$A_\ell(t) = \sum_{n=0}^{\infty} A_n^{(\ell)} t^n$$

$$B_\ell(t) = \sum_{n=0}^{\infty} B_n^{(\ell)} t^n$$

Therefore,

$$u_\ell = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\ell)} x^m t^n$$

$$\phi_{\ell+1} = \phi_\ell + u_\ell$$

where

$$a_{0,n}^{(\ell)} = A_n^{(\ell)}$$

$$a_{1,n}^{(\ell)} = B_n^{(\ell)}$$

$$a_{m+2,n}^{(\ell)} = \frac{\left[(n+1)(n+2)a_{m,n+2}^{(\ell-1)} + \sum_{\mu=0}^m \sum_{v=0}^n \alpha_{m-\mu, n-v} \alpha_{\mu,v}^{(\ell-1)} \right]}{(m+1)(m+2)}$$

We can write the result as

$$u_\ell = \sum_{m=0}^{\infty} a_m^{(\ell)}(t) x^m$$

$$a_m^{(\ell)}(t) = \sum_{n=0}^{\infty} a_{m,n}^{(\ell)} t^n$$

SUMMARY:

$$u_0 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(0)} x^m t^n$$

$$u_1 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(1)} x^m t^n$$

$$\vdots$$

$$u_\ell = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\ell)} x^m t^n$$

The approximant $\phi_{\lambda+1}\{u\} = \sum_{\ell=0}^{\lambda} u_\ell$

$$\begin{aligned} \phi_{\lambda+1}\{u\} &= \sum_{\ell=0}^{\lambda} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\ell)} x^m t^n \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \sum_{\ell=0}^{\lambda} a_{m,n}^{(\ell)} \right\} x^m t^n \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n \phi_{\lambda+1}\{a_{m,n}\} \end{aligned}$$

We note that $\lim_{\lambda \rightarrow +\infty} \phi_{\lambda+1}\{u\} = u$ and that $\lim_{\lambda \rightarrow +\infty} \phi_{\lambda+1}\{a_{m,n}\} = a_{m,n}$. Since $u = \sum_{\ell=0}^{\infty} u_\ell$ and

$$u_\ell = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\ell)} x^m t^n$$

Substitution leads to

$$\begin{aligned}
u &= \sum_{\ell=0}^{\lambda} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\ell)} x^m t^n \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \sum_{\ell=0}^{\infty} a_{m,n}^{(\ell)} \right\} x^m t^n \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} x^m t^n
\end{aligned}$$

where $a_{m,n} = \sum_{\ell=0}^{\infty} a_{m,n}^{(\ell)}$.

TRANSFORMATION OF SPATIAL SOLUTION:

Having calculated u_{ℓ} , we know that

$$\begin{aligned}
\phi_{\lambda+1}\{A(t)\} &= \sum_{\ell=0}^{\lambda} A_{\ell}(t) \\
\phi_{\lambda+1}\{B(t)\} &= \sum_{\ell=0}^{\lambda} B_{\ell}(t) \\
\phi_{\lambda+1}\{u_0\} &= \phi_{\lambda+1}\{A(t)\} - x\phi_{\lambda+1}\{B(t)\}
\end{aligned}$$

We can now recast the problem into an initial-value problem format, accelerating convergence by simplified further calculation through avoidance of further boundary condition evaluations:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \alpha(x,t)u + \frac{\partial^2 u}{\partial t^2} &= \beta(x,t) \\
u(t,0) &= A(t) \\
\frac{\partial u(t,0)}{\partial x} &= B(t)
\end{aligned}$$

We proceed to desired accuracy by choosing λ :

$$\begin{aligned}
u(t,0) &\approx \phi_{\lambda+1}\{A(t)\} \\
\frac{\partial u(t,0)}{\partial x} &\approx \phi_{\lambda+1}\{B(t)\}
\end{aligned}$$

$L(\cdot) = \frac{\partial^2}{\partial x^2}(\cdot)$ and $L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) dx dx$. We emphasize that L^{-1} now represents definite integration. Upon substitution, we have

$$Lu = \beta(x, t) - \alpha(x, t)u - (\partial^2/\partial t^2)u$$

$$u = u_0 - L^{-1} \alpha(x, t)u - L^{-1}(\partial^2/\partial t^2)u$$

where

$$u_0 = u(t, 0) + x \frac{\partial u(t, 0)}{\partial x} + L^{-1} \beta(x, t)$$

$$u_0 = A(t) + xB(t) + \sum_{m=0}^{\infty} \frac{\beta_m(t)x^{m+2}}{(m+1)(m+2)}$$

with

$$A(t) = \sum_{n=0}^{\infty} A_n t^n$$

$$B(t) = \sum_{n=0}^{\infty} B_n t^n$$

Therefore we can write u_0 as

$$u_0 = \sum_{m=0}^{\infty} a_m^{(0)}(t) x^m$$

where

$$a_0^{(0)}(t) = A(t)$$

$$a_1^{(0)}(t) = B(t)$$

$$a_{m+2}^{(0)}(t) = \frac{\beta_m(t)}{(m+1)(m+2)}$$

We can also write

$$u_0 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(0)} x^m t^n$$

with

$$a_{0,n}^{(0)}(t) = A_n$$

$$a_{1,n}^{(0)}(t) = B_n$$

$$a_{m+2,n}^{(0)}(t) = \frac{\beta_{m,n}}{(m+1)(m+2)}$$

To calculate the m th component u_m of the decomposition of u , we have

$$u_m = -L^{-1} \alpha(x, t)u_{m-1} - L^{-1}(\partial^2/\partial t^2)u_{m-1}$$

where $L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) dx dx$. Thus,

$$u_1 = -L^{-1} \alpha(x, t) u_0 - L^{-1} (\partial^2 / \partial t^2) u_0$$

$$u_0 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(0)} x^m t^n$$

$$(\partial^2 / \partial t^2) u_0 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (n+1)(n+2) a_{m,n+2}^{(0)} x^m t^n$$

$$\alpha(x, t) u_0(x, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} a_{\mu, \nu}^{(0)}$$

so that

$$\begin{aligned} u_1 = & - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^{m+2} t^n \frac{\sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} a_{\mu, \nu}^{(0)}}{(m+1)(m+2)} \\ & - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^{m+2} t^n \frac{(n+1)(n+2) a_{m, n+2}^{(0)}}{(m+1)(m+2)} \\ u_1 = & x^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^2 a_{m,n}^{(1)} \end{aligned}$$

where

$$a_{m,n}^{(1)} = \frac{- \left[(n+1)(n+2) a_{m, n+2}^{(0)} + \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} a_{\mu, \nu}^{(0)} \right]}{(m+1)(m+2)}$$

⋮

$$u_{\ell > 0} = x^{2\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n a_{m,n}^{(\ell)}$$

with

$$a_{m,n}^{(\ell)} = \frac{- \left[(n+1)(n+2) a_{m, n+2}^{(\ell-1)} + \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu, n-\nu} a_{\mu, \nu}^{(\ell-1)} \right]}{(m+1)(m+2)}$$

The approximants $\phi_{\lambda-1} = \sum_{\ell=0}^{\lambda} u_{\ell}$ and $u = \sum_{\ell=0}^{\infty} u_{\ell}$ are computed as usual so that

$$u = \sum_{\ell=0}^{\infty} x^{2\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n a_{m,n}^{(\ell)}$$

Staggered summation can be applied at this point.

TEMPORAL FORMAT:

Consider the same partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \alpha(x, t)u + \frac{\partial^2 u}{\partial t^2} = \beta(x, t)$$

with the initial conditions:

$$u(x, 0) = \tau_1(x) = \sum_{m=0}^{\infty} \tau_m^{(1)} x^m$$

$$\frac{\partial u(x, 0)}{\partial t} = \tau_2(x) = \sum_{m=0}^{\infty} \tau_m^{(2)} x^m$$

and boundary conditions:

$$u(x_1, t) = \xi_1(t) = \sum_{n=0}^{\infty} \xi_n^{(1)} t^n$$

$$u(x_2, t) = \xi_2(t) = \sum_{n=0}^{\infty} \xi_n^{(2)} t^n$$

We will suppose that α and β are given in the form

$$\alpha(x, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} x^m t^n$$

$$\beta(x, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta_{m,n} x^m t^n = \sum_{n=0}^{\infty} \beta_n(x) t^n$$

where

$$\beta_n(x) = \sum_{m=0}^{\infty} \beta_{m,n} x^m$$

To derive the solution we define as usual $L = \partial^2 / \partial t^2$ and $L^{-1}(\cdot) = \int_0^t \int_0^t (\cdot) dt dt$, a (two-fold) definite integration operator. Then

$$L u + \alpha(x, t)u + (\partial^2 / \partial x^2)u = \beta(x, t)$$

$$L u = \beta(x, t) - \alpha(x, t)u - (\partial^2 / \partial x^2)u$$

and operating with L^{-1} we get

$$u = u_0 - L^{-1} \alpha(x, t)u - L^{-1}(\partial^2 / \partial x^2)u$$

where

$$u_0 = \tau_1(x) + t \tau_2(x) + \int_0^t \int_0^1 \beta(x, t) dt dt$$

Solving by decomposition, the solution u is written $u = \sum_{\ell=0}^{\infty} u_{\ell}$ and the approximant $\phi_{\lambda} = \sum_{\ell=0}^{\lambda-1} u_{\ell}$. Then

$$\phi_1 = u_0$$

$$\phi_{\lambda+1} = \phi_{\lambda} + u_{\lambda}$$

$$u_{\ell} = -L^{-1} \alpha(x, t)u_{\ell-1} - L^{-1}(\partial^2 / \partial x^2)u_{\ell-1}$$

Now we calculate the u_0 component (and ϕ_1 approximant).

$$u_0 = \tau_1(x) + t \tau_2(x) + L^{-1} \beta(x, t)$$

$$u_0 = \tau_1(x) + t \tau_2(x) + \sum_{n=0}^{\infty} \frac{\beta_n(x) t^{n+2}}{(n+1)(n+2)} = \phi_1$$

and we note that we can write

$$\tau_1(x) = \sum_{m=0}^{\infty} \tau_m^{(1)} x^m$$

$$\tau_2(x) = \sum_{m=0}^{\infty} \tau_m^{(2)} x^m$$

$$\beta(x, t) = \sum_{n=0}^{\infty} \beta_n(x) t^n$$

$$\beta_n(x) = \sum_{m=0}^{\infty} \beta_{m,n} x^m$$

and

$$u_0 = \sum_{n=0}^{\infty} a_n^{(0)}(x) t^n$$

where

$$a_0^{(0)}(x) = \tau_1(x)$$

$$a_1^{(0)}(x) = \tau_2(x)$$

$$a_{n+2}^{(0)}(x) = \frac{\beta_n(x)}{(n+1)(n+2)}$$

and

$$\begin{aligned}a_{m,0}^{(0)} &= \tau_m^{(1)} \\ a_{m,1}^{(0)} &= \tau_m^{(2)} \\ a_{m,n+2}^{(0)} &= \frac{\beta_{m,n}}{(n+1)(n+2)}\end{aligned}$$

Thus we write

$$u_0 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(0)} x^m t^n = \phi_1$$

or

$$u_0 = \sum_{n=0}^{\infty} a_n^{(0)}(x) t^n$$

where

$$a_n^{(0)}(x) = \sum_{m=0}^{\infty} a_{m,n}^{(0)} x^m$$

Now we can compute the u_1 component and ϕ_2 approximant:

$$\begin{aligned}u_1 &= -L^{-1} \alpha(x, t) u_0 - L^{-1} (\partial^2 / \partial x^2) u_0 \\ u_0 &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(0)} x^m t^n \\ \frac{\partial^2}{\partial x^2} u_0 &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+1)(m+2) a_{m+2,n}^{(0)} x^m t^n \\ \alpha(x, t) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} x^m t^n\end{aligned}$$

The Cauchy product $\alpha(x, t) u_0(x, t)$ is computed as

$$\begin{aligned}\alpha(x, t) u_0(x, t) &= \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} x^m t^n \right\} \cdot \left\{ \sum_{\mu=0}^{\infty} \sum_{v=0}^{\infty} a_{\mu,v}^{(0)} x^{\mu} t^v \right\} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n \sum_{\mu=0}^m \sum_{v=0}^n \alpha_{m-\mu, n-v} a_{\mu,v}^{(0)}\end{aligned}$$

(which can readily be programmed).

$$\begin{aligned}u_1 &= -\int_0^t \int_0^t \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n \sum_{\mu=0}^m \sum_{v=0}^n \alpha_{m-\mu, n-v} a_{\mu,v}^{(0)} dt dt \\ &\quad - \int_0^t \int_0^t \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+1)(m+2) a_{m+2,n}^{(0)} x^m t^n dt dt\end{aligned}$$

$$u_1 = -t^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n \frac{\left[(m+1)(m+2)a_{m+2,n}^{(0)} + \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu,n-\nu} a_{\mu,\nu}^{(0)} \right]}{(n+1)(n+2)}$$

which we can finally write as

$$u_1 = t^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(1)} x^m t^n$$

or

$$u_1 = t^2 \sum_{n=0}^{\infty} a_n^{(1)}(x) t^n$$

with

$$a_n^{(1)}(x) = \sum_{m=0}^{\infty} a_{m,n}^{(1)} x^m$$

We now have $\phi_2 = \phi_1 + u_1$ and can continue in the same manner to the u_2 component and the ϕ_{t+1} approximant.

$$u_t = -L^{-1} \alpha(x, t) u_{t-1} - L^{-1} (\partial^2 / \partial x^2) u_{t-1}$$

$$u_{t-1} = t^{2(\ell-1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\ell-1)} x^m t^n$$

$$\frac{\partial^2}{\partial x^2} u_{t-1} = t^{2(\ell-1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+1)(m+2) a_{m+2,n}^{(\ell-1)} x^m t^n$$

$$\alpha(x, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} x^m t^n$$

We next compute the Cauchy product indicated by $\alpha(x, t) u_{t-1}(x, t)$ or

$$\begin{aligned} & \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} x^m t^n \right\} \cdot \left\{ t^{2(\ell-1)} \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} a_{\mu,\nu}^{(\ell-1)} x^{\mu} t^{\nu} \right\} \\ &= t^{2(\ell-1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu,n-\nu} a_{\mu,\nu}^{(\ell-1)} \\ u_t &= -\int_0^t \int_0^1 t^{2(\ell-1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu,n-\nu} a_{\mu,\nu}^{(\ell-1)} dt dt \\ & \quad - \int_0^t \int_0^1 t^{2(\ell-1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n (m+1)(m+2) a_{m+2,n}^{(\ell-1)} dt dt \end{aligned}$$

$$u_\ell = t^{2\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n a_{m,n}^{(\ell)}$$

where

$$a_{m,n}^{(\ell)} = \frac{- \left[(m+1)(m+2)a_{m+2,n}^{(\ell-1)} + \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{m-\mu,n-\nu} a_{\mu,\nu}^{(\ell-1)} \right]}{(2\ell+n-1)(2\ell+n)}$$

$$\phi_{\ell-1} = \phi_\ell + u_\ell$$

SUMMARY:

$$u_0 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(0)} x^m t^n$$

$$u_1 = t^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(1)} x^m t^n$$

$$\vdots$$

$$u_\ell = t^{2\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\ell)} x^m t^n$$

$$\phi_{\lambda+1} = \sum_{\ell=0}^{\lambda} u_\ell \equiv \phi_{\lambda+1}\{u\}$$

$$\phi_{\lambda+1} = \sum_{\ell=0}^{\lambda} t^{2\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\ell)} x^m t^n$$

Staggered summation can now be applied.

COMPATIBILITY OF EQUATION AND CONDITIONS:

If the computed solution satisfies the equation and the conditions, we have the solution. If the physical problem is correctly modelled, no difficulty appears. One cannot arbitrarily assign conditions to an equation. The equations, conditions, and solution must be consistent. If attempts to model a physical system fail to give us correct conditions, one can get both temporal and spatial solutions and find that these solutions are different. If they are close over a finite region, then we realize the modelling needs improvement. This may allow us to develop a predictor-corrector methodology which we leave to future work.

If incorrect conditions are used for the decomposition solution, we do not have a solution. If the solution is correct, i.e., it satisfied the equation, we can do inverse operation, e.g., in $Lu + Nu = g$, we have $u = \Phi - L^{-1}Nu$ and can solve for F . If L is second order, we know $u_0 = \Phi - L^{-1}g$ where $\Phi = \alpha + \beta x$; hence, we know F and have a test for a, b .

NONLINEAR BOUNDARY-VALUE PROBLEMS:

We have two alternative, actually equivalent, approaches for boundary-value problems, whether ordinary or partial differential equations are involved. The first is to match each approximant ϕ_n for $n = 1, 2, \dots, n$ to the boundary conditions. In boundary-value problems modelled by ordinary linear differential equations, only one such matching is necessary. We can carry along the un-evaluated initial term without evaluating the integration constants by matching to the boundary conditions and only do the matching when the m -term approximant has been calculated. In nonlinear differential equations or (linear or nonlinear) partial differential equations, this is not possible. Then the matching must be done for each level of approximation.

The second (or double decomposition) procedure adds decomposition to the initial term. This allows a convenient match to the boundary conditions of the approximant ϕ_m because the constants from each integration are added to give a better initial term. As we will discover later, the solution can then be carried further, if more accuracy is needed, as an initial-value problem. The value of decomposition of the initial term is in the matching of the boundary conditions by adding all the integration constants $c_{0,m}$ and $c_{1,m}$ separately to form a new c_0 and c_1 and new initial term which is now close to a final value as $n \rightarrow \infty$ in ϕ_n . Now we can use this u_0 term without adding further constants of integration and matching to boundary conditions, since for high approximants to achieve accurate solutions, the computation is much less.

REFERENCE

1. G. Adomian and R. Rach, Analytic Solution of Nonlinear Boundary-value Problems in Several Dimensions, *J. Math. Anal. and Applic.*, 173, (118-137) (March 1993).

MODIFIED DECOMPOSITION

The modelling of physical problems can lead to ordinary or partial differential equations which are quite generally nonlinear. Examples include equations such as the Navier-Stokes equations in fluid mechanics, the Lane-Emden equation for stellar structure, nonlinear Schrödinger equations in quantum theory, soliton equations, etc.

We present here a variation of the decomposition method which can also be applied to such equations to obtain accurate quantitative solutions. A mathematical advantage of the various adaptations of decomposition is that linear equations are an easily solved special case and ordinary differential equations are a special case of the theory for partial differential equations, so we have a single unified field. This alternative formulation will be referred to as "modified decomposition" [1,2]. It requires the following result on transformation of series [3].

Normally we write $f(u) = \sum_{n=0}^{\infty} A_n(u_0, \dots, u_n)$. However, given a convergent series $u = \sum_{n=0}^{\infty} c_n x^n$ and $f(u)$, we can write

$$f(u) = \sum_{n=0}^{\infty} x^n A_n(c_0, \dots, c_n)$$

We can see this as follows. Let

$$u = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} c_n x^n$$

We wish to find a transformed series $f(u) = f\left(\sum_{n=0}^{\infty} c_n x^n\right)$. Since $f(u) = \sum_{n=0}^{\infty} A_n(u_0, \dots, u_n)$, $f(u)$ can also be written as

$$\sum_{n=0}^{\infty} A_n(c_0, \dots, c_n) x^n$$

Thus for $f(u) = u^2$, for example, we have

$$\begin{aligned}
 A_0(u_0) &= u_0^2 \\
 A_1(u_0, u_1) &= 2u_0u_1 \\
 A_2 &= u_1^2 + 2u_0u_2 \\
 &\vdots
 \end{aligned}$$

Since $u_0 = c_0$, $u_1 = c_1x$, $u_2 = c_2x^2, \dots$, we have

$$f(u) = \sum_{n=0}^{\infty} A_n(c_0, \dots, c_n)x^n$$

with $A_0 = c_0^2$, $A_1 = 2c_0c_1, \dots$. As an example consider $\tan^{-1}x$ and $f(u) = u^2$

$$\begin{aligned}
 \tan^{-1}x &= x - x^3/3 + x^5/5 \\
 A_0 = u_0^2 &= x^2 \\
 A_1 = 2u_0u_1 &= -2x^4/3 \\
 &\vdots \\
 (\tan^{-1}x)^2 &= x^2 - 2x^4/3 + \dots
 \end{aligned}$$

Power series solutions of linear homogeneous differential equations in initial-value problems yield simple recurrence relations for the coefficients but generally are not adequate for nonlinear equations.

Dealing, for example, with a simple linear inhomogeneous case $Lu + Ru = g$ with second order L , we let

$$\begin{aligned}
 g &= \sum_{n=0}^{\infty} g_n x^n \\
 u &= \alpha_0 + \alpha_1 x + \sum_{n=0}^{\infty} g_n x^2 - \rho I^2 u
 \end{aligned}$$

where I is an integration and I^2 will mean a two-fold integration. Thus

$$\sum_{n=0}^{\infty} c_n x^n = \alpha_0 + \alpha_1 x + \sum_{n=0}^{\infty} g_n x^{n+2} / (n+1)(n+2) - \rho \sum_{n=0}^{\infty} c_n x^{n+2} / (n+1)(n+2)$$

with $c_0 = \alpha_0$ and $c_1 = \alpha_1$. For $n \geq 2$

$$c_n = \frac{g_{n-2} - \rho c_{n-2}}{n(n-1)}$$

so we obtain coefficients from a recursion formula. The technique provides an interesting alternative for equations such as the Duffing equation and the Van der Pol equation.

ONE-DIMENSIONAL CASE:

Consider the nonlinear inhomogeneous ordinary differential equation $L_t u + Ru + Nu = g$ where $L_t u = \partial^2 / \partial t^2$, $R = \rho(t)$, $Nu = \alpha(t)f(u)$. We will view this as a special case in one dimension of a multi-dimensional partial differential equation. (In the following sections, we will consider equations in two, three, and four dimensions.) We can write

$$u = \sum_{n=0}^{\infty} a_n t^n$$

$$R = \rho(t) = \sum_{n=0}^{\infty} \rho_n t^n$$

$$g = g(t) = \sum_{n=0}^{\infty} g_n t^n$$

$$\alpha = \sum_{n=0}^{\infty} \alpha_n t^n$$

$$f = \sum_{n=0}^{\infty} A_n(a_0, \dots, a_n) t^n = \sum_{n=0}^{\infty} A_n t^n$$

$$u = \Phi_t + L_t^{-1} g - L_t^{-1} Ru - L_t^{-1} Nu$$

where $\Phi_t = \tau_0 + t\tau_1$ and $L_t^{-1} = \int_0^t \int_0^t (\cdot) dt dt$. The substitution yields

$$\begin{aligned} \sum_{n=0}^{\infty} a_n t^n = & \tau_0 + t\tau_1 + \int_0^t \int_0^t \sum_{n=0}^{\infty} g_n t^n dt dt - \int_0^t \int_0^t \left\{ \sum_{n=0}^{\infty} \rho_n t^n \right\} \cdot \left\{ \sum_{n=0}^{\infty} a_n t^n \right\} dt dt \\ & - \int_0^t \int_0^t \left\{ \sum_{n=0}^{\infty} \alpha_n t^n \right\} \cdot \left\{ \sum_{n=0}^{\infty} A_n t^n \right\} dt dt \end{aligned}$$

Multiplying and collecting like powers of t ,

$$\left\{ \sum_{n=0}^{\infty} \rho_n t^n \right\} \cdot \left\{ \sum_{n=0}^{\infty} a_n t^n \right\} = \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \rho_v a_{n-v}$$

and

$$\left\{ \sum_{n=0}^{\infty} \alpha_n t^n \right\} \cdot \left\{ \sum_{n=0}^{\infty} A_n t^n \right\} = \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \alpha_v A_{n-v}$$

Replacing the above quantities in brackets with the equivalent expressions on the right side,

$$\begin{aligned} \sum_{n=0}^{\infty} a_n t^n &= \tau_0 + t\tau_1 + \int_0^t \int_0^t \sum_{n=0}^{\infty} g_n t^n dt dt \\ &\quad - \int_0^t \int_0^t \left\{ \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \rho_v a_{n-v} \right\} dt dt \\ &\quad - \int_0^t \int_0^t \left\{ \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \alpha_v A_{n-v} \right\} dt dt \end{aligned}$$

Carrying out the integrations, we have

$$\begin{aligned} \sum_{n=0}^{\infty} a_n t^n &= \tau_0 + t\tau_1 + \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} g_n \\ &\quad - \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} \sum_{v=0}^n \rho_v a_{n-v} \\ &\quad - \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} \sum_{v=0}^n \alpha_v A_{n-v} \end{aligned}$$

In the summations on the right, n can be replaced by $n-2$ to write

$$\begin{aligned} \sum_{n=0}^{\infty} a_n t^n &= \tau_0 + t\tau_1 + \sum_{n=2}^{\infty} \frac{t^n}{n(n-1)} g_{n-2} \\ &\quad - \sum_{n=2}^{\infty} \frac{t^n}{n(n-1)} \sum_{v=0}^{n-2} \rho_v a_{n-2-v} \\ &\quad - \sum_{n=2}^{\infty} \frac{t^n}{n(n-1)} \sum_{v=0}^{n-2} \alpha_v A_{n-2-v} \end{aligned}$$

Finally, we can equate coefficients of like powers of t on the left side and on the right side to arrive at recurrence relations for the coefficients. Thus

$$a_0 = \tau_0$$

$$a_1 = \tau_1$$

and for $n \geq 2$

$$a_n = \frac{g_{n-2} - \sum_{v=0}^{n-2} \rho_v a_{n-2-v} - \sum_{v=1}^{n-2} \alpha_v A_{n-2-v}}{n(n-1)}$$

This solution, of course, is

$$u(t) = \sum_{n=0}^{\infty} a_n t^n$$

TWO-DIMENSIONAL CASE—NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS:

$$L_t u + L_x u + Ru + Nu = g$$

We now have two linear operators L_t and L_x . Let $L_t = \partial^2 / \partial t^2$ and $L_x = \partial^2 / \partial x^2$. Assume that we can write

$$u = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k} t^n x^k$$

or

$$u = \sum_{n=0}^{\infty} t^n \left[\sum_{k=0}^{\infty} a_{n,k} x^k \right] = \sum_{n=0}^{\infty} a_n(x) t^n$$

If we have $u(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n} x^m y^n$, we get terms

$$c_{0,0}, c_{0,1} y, c_{0,2} y^2, \dots, c_{1,0} x, c_{1,1} xy, c_{1,2} xy^2, \dots, c_{2,0} x^2, c_{2,1} x^2 y, \dots$$

The first group can be written $x^0 \sum_{n=0}^{\infty} c_{0,n} y^n$, the second group as $x^1 \sum_{n=0}^{\infty} c_{1,n} y^n$, the third group as $x^2 \sum_{n=0}^{\infty} c_{2,n} y^n$ etc. Thus

$$u(x, y) = \sum_{m=0}^{\infty} c_m(y) x^m$$

where $c_m(y) = \sum_{n=0}^{\infty} c_{m,n} y^n$ so that the double series is collapsed into a single series. We suppose the operator R is

$$\begin{aligned}
 R &= \rho(t, x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \rho_{n,k} t^n x^k \\
 &= \sum_{n=0}^{\infty} t^n \left[\sum_{k=0}^{\infty} x^k \rho_{n,k} \right] = \sum_{n=0}^{\infty} \rho_n(x) t^n
 \end{aligned}$$

Let the nonlinear term $Nu = \alpha(t, x) f(u)$ and

$$\alpha(t, x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \alpha_{n,k} t^n x^k = \sum_{n=0}^{\infty} t^n \left[\sum_{k=0}^{\infty} \alpha_{n,k} x^k \right] = \sum_{n=0}^{\infty} \alpha_n(x) t^n$$

Write $f(u) = \sum_{n=0}^{\infty} t^n A_n(a_0(x), \dots, a_n(x)) = \sum_{n=0}^{\infty} t^n A_n(x)$. Let

$$g = g(t, x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g_{n,k} t^n x^k = \sum_{n=0}^{\infty} t^n \left[\sum_{k=0}^{\infty} g_{n,k} x^k \right] = \sum_{n=0}^{\infty} t^n g_n(x)$$

The decomposition solution using the t partial solution is given by:

$$u = \Phi_t + L_t^{-1} g - L_t^{-1} L_x u - L_t^{-1} Ru - L_t^{-1} Nu$$

where

$$\Phi_t = \tau_0(x) + t\tau_1(x) = u(t=0, x) + t \partial u / \partial t (t=0, x)$$

and $L_t^{-1} = \int_0^t \int_0^1 (\cdot) dt dt$. Substituting for u , $f(u)$, g , and ρ , we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} a_n(x) t^n &= \tau_0(x) + t\tau_1(x) + \int_0^t \int_0^1 \sum_{n=0}^{\infty} g_n(x) t^n dt dt \\
 &\quad - \int_0^t \int_0^1 (\partial^2 / \partial x^2) \sum_{n=0}^{\infty} a_n(x) t^n dt dt \\
 &\quad - \int_0^t \int_0^1 \left\{ \sum_{n=0}^{\infty} \rho_n(x) t^n \right\} \cdot \left\{ \sum_{n=0}^{\infty} a_n(x) t^n \right\} dt dt \\
 &\quad - \int_0^t \int_0^1 \left\{ \sum_{n=0}^{\infty} \alpha_n(x) t^n \right\} \cdot \left\{ \sum_{n=0}^{\infty} A_n(x) t^n \right\} dt dt
 \end{aligned}$$

The bracketed products are

$$\left\{ \sum_{n=0}^{\infty} \rho_n(x) t^n \right\} \cdot \left\{ \sum_{n=0}^{\infty} a_n(x) t^n \right\} = \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \rho_v(x) a_{n-v}(x)$$

$$\left\{ \sum_{n=0}^{\infty} \alpha_n(x) t^n \right\} \cdot \left\{ \sum_{n=0}^{\infty} A_n(x) t^n \right\} = \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \alpha_v(x) A_{n-v}(x)$$

Substituting the above products

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(x) t^n &= \tau_0(x) + t\tau_1(x) + \int_0^t \int_0^t \sum_{n=0}^{\infty} g_n(x) t^n dt dt \\ &\quad - \int_0^t \int_0^t \left(\partial^2 / \partial x^2 \right) \left\{ \sum_{n=0}^{\infty} a_n(x) t^n dt dt \right\} \\ &\quad - \int_0^t \int_0^t \left\{ \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \rho_v(x) \bar{\alpha}_{n-v}(x) \right\} dt dt \\ &\quad - \int_0^t \int_0^t \left\{ \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \alpha_v(x) A_{n-v}(x) \right\} dt dt \end{aligned}$$

We now carry out the above integrations to write

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(x) t^n &= \tau_0(x) + t\tau_1(x) + \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} g_n(x) \\ &\quad - \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} \frac{\partial^2}{\partial x^2} a_n(x) \\ &\quad - \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} \sum_{v=0}^n \rho_v(x) a_{n-v}(x) \\ &\quad - \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} \sum_{v=0}^n \alpha_v(x) A_{n-v}(x) \end{aligned}$$

Let $n \rightarrow n-2$ on the right side. Then

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(x) t^n &= \tau_0(x) + t\tau_1(x) + \sum_{n=2}^{\infty} \frac{t^n}{n(n-1)} g_{n-2}(x) \\ &\quad - \sum_{n=2}^{\infty} \frac{t^n}{n(n-1)} \frac{\partial^2}{\partial x^2} a_{n-2}(x) \end{aligned}$$

$$\begin{aligned}
& - \sum_{n=2}^{\infty} \frac{t^n}{n(n-1)} \sum_{v=0}^{n-2} \rho_v(x) a_{n-2-v}(x) \\
& - \sum_{n=2}^{\infty} \frac{t^n}{n(n-1)} \sum_{v=0}^{n-2} \alpha_v(x) A_{n-2-v}(x)
\end{aligned}$$

Finally, equating coefficients of like powers of t , we derive the recursion formula for the coefficients

$$a_0(x) = \tau_0(x)$$

$$a_1(x) = \tau_1(x)$$

and for $n \geq 2$,

$$a_n(x) = \frac{g_{n-2}(x) - (\partial^2 / \partial x^2) a_{n-2}(x) - \sum_{v=0}^{n-2} \{ \rho_v(x) a_{n-2-v}(x) + \alpha_v(x) A_{n-2-v}(x) \}}{n(n-1)}$$

The final solution is now given by $u(t, x) = \sum_{n=0}^{\infty} a_n(x) t^n$. (Whether modified or regular decomposition is used, we can still apply Padé approximants, Shanks, Wynn, Euler, or Van Wijngaarden transforms to accelerate convergence.)

THREE-DIMENSIONAL CASE—NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS:

Consider $L_t u + L_x u + L_y u + Ru + Nu = g$ where we let $L_t = \partial^2 / \partial t^2$, $L_x = \partial^2 / \partial x^2$, $L_y = \partial^2 / \partial y^2$, $R = \rho(t, x, y)$, $Nu = \alpha(t, x, y)f(u)$. We now assume

$$\begin{aligned}
u &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} a_{n,k,\ell} t^n x^k y^\ell \\
&= \sum_{n=0}^{\infty} t^n \left[\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} a_{n,k,\ell} x^k y^\ell \right] = \sum_{n=0}^{\infty} a_n(x, y) t^n \\
R &= \rho(t, x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \rho_{n,k,\ell} t^n x^k y^\ell \\
&= \sum_{n=0}^{\infty} t^n \left[\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \rho_{n,k,\ell} x^k y^\ell \right] = \sum_{n=0}^{\infty} \rho_n(x, y) t^n
\end{aligned}$$

$$\begin{aligned}\alpha &= \alpha(t, x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \alpha_{n,k,\ell} t^n x^k y^\ell \\ &= \sum_{n=0}^{\infty} t^n \left[\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \alpha_{n,k,\ell} x^k y^\ell \right] = \sum_{n=0}^{\infty} \alpha_n(x, y) t^n\end{aligned}$$

$$f(u) = \sum_{n=0}^{\infty} t^n A_n(a_0(x, y), \dots, a_n(x, y)) = \sum_{n=0}^{\infty} t^n A_n(x, y)$$

$$\begin{aligned}g &= g(t, x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} g_{n,k,\ell} t^n x^k y^\ell \\ &= \sum_{n=0}^{\infty} t^n \left[\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} g_{n,k,\ell} x^k y^\ell \right] = \sum_{n=0}^{\infty} t^n g_n(x, y)\end{aligned}$$

The t partial solution is

$$u = \Phi_t + L_t^{-1} g - L_t^{-1} L_x u - L_t^{-1} L_y u - L_t^{-1} R u - L_t^{-1} N u$$

where

$$\Phi_t = u(t=0, x, y) + t \frac{\partial u}{\partial t}(t=0, x, y) = \tau_0(x, y) + t \tau_1(x, y)$$

We can now write

$$\begin{aligned}\sum_{n=0}^{\infty} a_n(x, y) t^n &= \tau_0(x, y) + t \tau_1(x, y) \\ &\quad + \int_0^t \int_0^t \sum_{n=0}^{\infty} t^n g_n(x, y) dt dt \\ &\quad - \int_0^t \int_0^t (\partial^2 / \partial x^2) \sum_{n=0}^{\infty} t^n a_n(x, y) dt dt \\ &\quad - \int_0^t \int_0^t (\partial^2 / \partial y^2) \sum_{n=0}^{\infty} t^n a_n(x, y) dt dt \\ &\quad - \int_0^t \int_0^t \left\{ \sum_{n=0}^{\infty} t^n \rho_n(x, y) \right\} \cdot \left\{ \sum_{n=0}^{\infty} t^n a_n(x, y) \right\} dt dt \\ &\quad - \int_0^t \int_0^t \left\{ \sum_{n=0}^{\infty} t^n \alpha_n(x, y) \right\} \cdot \left\{ \sum_{n=0}^{\infty} t^n A_n(x, y) \right\} dt dt\end{aligned}$$

The bracketed quantities are

$$\left\{ \sum_{n=0}^{\infty} t^n \rho_n(x, y) \right\} \cdot \left\{ \sum_{n=0}^{\infty} t^n a_n(x, y) \right\} = \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \rho_v(x, y) a_{n-v}(x, y)$$

$$\left\{ \sum_{n=0}^{\infty} t^n \alpha_n(x, y) \right\} \cdot \left\{ \sum_{n=0}^{\infty} t^n A_n(x, y) \right\} = \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \alpha_v(x, y) A_{n-v}(x, y)$$

which we can substitute to obtain

$$\begin{aligned} \sum_{n=0}^{\infty} t^n a_n(x, y) &= \tau_0(x, y) + t\tau_1(x, y) + \int_0^t \int_0^t \sum_{n=0}^{\infty} t^n g_n(x, y) dt dt \\ &\quad - \int_0^t \int_0^t (\partial^2 / \partial x^2) \sum_{n=0}^{\infty} t^n a_n(x, y) dt dt \\ &\quad - \int_0^t \int_0^t (\partial^2 / \partial y^2) \sum_{n=0}^{\infty} t^n a_n(x, y) dt dt \\ &\quad - \int_0^t \int_0^t \left\{ \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \rho_v(x, y) a_{n-v}(x, y) \right\} dt dt \\ &\quad - \int_0^t \int_0^t \left\{ \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \alpha_v(x, y) A_{n-v}(x, y) \right\} dt dt \end{aligned}$$

which we now integrate to get

$$\begin{aligned} \sum_{n=0}^{\infty} t^n a_n(x, y) &= \tau_0(x, y) + t\tau_1(x, y) + \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} g_n(x, y) \\ &\quad - \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} \frac{\partial^2}{\partial x^2} a_n(x, y) \\ &\quad - \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} \frac{\partial^2}{\partial y^2} a_n(x, y) \\ &\quad - \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} \sum_{v=0}^n \rho_v(x, y) a_{n-v}(x, y) \\ &\quad - \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} \sum_{v=0}^n \alpha_v(x, y) A_{n-v}(x, y) \end{aligned}$$

We now replace n by $n-2$ on the right to get

$$\begin{aligned}
\sum_{n=0}^{\infty} t^n a_n(x, y) &= \tau_0(x, y) + t\tau_1(x, y) + \sum_{n=2}^{\infty} \frac{t^n}{n(n-1)} g_{n-2}(x, y) \\
&\quad - \sum_{n=2}^{\infty} \frac{t^n}{n(n-1)} \frac{\partial^2}{\partial x^2} a_{n-2}(x, y) - \sum_{n=2}^{\infty} \frac{t^n}{n(n-1)} \frac{\partial^2}{\partial y^2} a_{n-2}(x, y) \\
&\quad - \sum_{n=2}^{\infty} \frac{t^n}{n(n-1)} \sum_{v=0}^{n-2} \rho_v(x, y) a_{n-2-v}(x, y) \\
&\quad - \sum_{n=2}^{\infty} \frac{t^n}{n(n-1)} \sum_{v=0}^{n-2} \alpha_v(x, y) A_{n-2-v}(x, y)
\end{aligned}$$

By equating coefficients of like powers of t , we have $a_0(x, y) = \tau_0(x, y)$, $a_1(x, y) = \tau_1(x, y)$ and for $n \geq 2$,

$$\begin{aligned}
a_n(x, y) &= \left\{ g_{n-2}(x, y) - \left(\partial^2 / \partial x^2 \right) a_{n-2}(x, y) - \left(\partial^2 / \partial y^2 \right) a_{n-2}(x, y) \right. \\
&\quad \left. - \sum_{v=0}^{n-2} \left[\rho_v(x, y) a_{n-2-v}(x, y) + \alpha_v(x, y) A_{n-2-v}(x, y) \right] \right\} / n(n-1)
\end{aligned}$$

The final solution is now given by $u(t, x, y) = \sum_{n=0}^{\infty} a_n(x, y) t^n$.

FOUR-DIMENSIONAL CASE—NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS:

$$L_t u + L_x u + L_y u + L_z u + Ru + Nu = g$$

where we let $L_t = \partial^2 / \partial t^2$, $L_x = \partial^2 / \partial x^2$, $L_y = \partial^2 / \partial y^2$, $L_z = \partial^2 / \partial z^2$. We can assume

$$\begin{aligned}
u &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} a_{n,k,\ell,m} t^n x^k y^\ell z^m \\
&= \sum_{n=0}^{\infty} t^n \left[\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} x^k y^\ell z^m a_{n,k,\ell,m} \right] \\
&= \sum_{n=0}^{\infty} t^n a_n(x, y, z)
\end{aligned}$$

$$\begin{aligned}
R &= \rho(t, x, y, z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \rho_{n,k,\ell,m} t^n x^k y^\ell z^m \\
&= \sum_{n=0}^{\infty} t^n \left[\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} x^k y^\ell z^m \rho_{n,k,\ell,m} \right] = \sum_{n=0}^{\infty} t^n \rho_n(x, y, z) \\
Nu &= \alpha(t, x, y, z) f(u) \\
\alpha &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} t^n x^k y^\ell z^m \alpha_{n,k,\ell,m} \\
&= \sum_{n=0}^{\infty} t^n \left[\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} x^k y^\ell z^m \alpha_{n,k,\ell,m} \right] = \sum_{n=0}^{\infty} t^n \alpha_n(x, y, z) \\
f(u) &= \sum_{n=0}^{\infty} t^n A_n(a_0(x, y, z), \dots, a_n(x, y, z)) = \sum_{n=0}^{\infty} t^n A_n(x, y, z) \\
g &= g(t, x, y, z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} t^n x^k y^\ell z^m g_{n,k,\ell,m} \\
&= \sum_{n=0}^{\infty} t^n \left[\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} x^k y^\ell z^m g_{n,k,\ell,m} \right] = \sum_{n=0}^{\infty} t^n g_n(x, y, z)
\end{aligned}$$

The t equation is

$$u = \Phi_t + L_t^{-1} g - L_t^{-1} [L_x + L_y + L_z] u - L_t^{-1} R u - L_t^{-1} \dot{N} u$$

with $\Phi_t = u(t=0, x, y, z) - t \partial u / \partial t(t=0, x, y, z) = \tau_0(x, y, z) + t \tau_1(x, y, z)$ and we can proceed as before with substitutions and integrations to get recursion formulas for the coefficients.

REMARK: We have seen that nonlinear partial differential equations are solvable by the modified decomposition procedure using concepts of the decomposition method (partial solutions, the A_n and transformations of series using the A_n). We have seen previously that such equations are solvable by straightforward decomposition also. Comparisons can now be made; in general, decomposition solutions converge faster but the solution is identical. A simple example is $L_t u + L_x u + f(u) = 0$ where $L_t = \partial / \partial t$, $L_x = \partial / \partial x$, $f(u) = u^2$, $u(t=0, x) = 1/2 x$. The t partial solution is

$$u = \Phi_t - L_t^{-1} L_x u - L_t^{-1} f(u)$$

where $\Phi_1 = u(t=0) = 1/2 x$. Then

$$\begin{aligned} u_1 &= -L_t^{-1}(\partial/\partial x)u_0 - L_t^{-1}A_0 \\ u_2 &= -L_t^{-1}(\partial/\partial x)u_1 - L_t^{-1}A_1 \\ &\vdots \end{aligned}$$

where the A_n are defined for u^2 . We get the (decomposition) solution

$$\begin{aligned} u &= \frac{1}{2x} \left[1 + \frac{t}{2x} + \frac{t^2}{4x^2} + \dots \right] \\ &= \frac{1}{2x-t} \quad \text{if} \quad \left| \frac{t}{2x} \right| < 1 \end{aligned}$$

Using modified decomposition, let

$$\begin{aligned} u &= \sum_{n=0}^{\infty} t^n a_n(x) \\ Nu &= \sum_{n=0}^{\infty} t^n A_n(x) \end{aligned}$$

hence

$$\begin{aligned} \sum_{n=0}^{\infty} t^n A_n(x) &= \tau_0(x) - \int_0^t \sum_{n=0}^{\infty} t^n (\partial/\partial x) a_n(x) dt \\ &\quad - \int_0^t \sum_{n=0}^{\infty} t^n \sum_{v=0}^n a_{n-v}(x) dt - \int_0^t \sum_{n=0}^{\infty} t^n \sum_{v=0}^n A_{n-v}(x) dt \\ \sum_{n=0}^{\infty} t^n a_n(x) &= \tau_0(x) - \sum_{n=0}^{\infty} (t^{n+1}/n+1) (\partial/\partial x) a_n(x) \\ &\quad - \sum_{n=0}^{\infty} (t^{n+1}/n+1) \sum_{v=0}^n a_{n-v}(x) - \sum_{n=0}^{\infty} (t^{n+1}/n+1) \sum_{v=0}^n A_{n-v}(x) \\ \sum_{n=0}^{\infty} t^n a_n &= \tau_0 - \sum_{n=1}^{\infty} (t^n/n) (\partial/\partial x) a_{n-1} \\ &\quad - \underbrace{\sum_{n=1}^{\infty} (t^n/n) \sum_{v=0}^{n-1} a_{n-1-v}}_{\text{محصلة}} - \sum_{n=1}^{\infty} (t^n/n) \sum_{v=0}^{n-1} A_{n-1-v} \end{aligned}$$

from which we get

$$a_0 = \tau_0 = 1/2 x$$

$$a_1 = -(\partial/\partial x)a_0 - \underbrace{a_0}_{\text{constant}} - A_0(a_0)$$

$$\begin{aligned} a_2 &= -(\partial/\partial x)a_1 - \underbrace{(1/2)(a_1 - a_0)}_{\text{constant}} - (1/2)(A_1 - A_0) \\ &= 1/2 x^3 - \underbrace{(1/2)(1/4 x^2 - 1/2 x)}_{\text{constant}} - (1/2)(A_1 - A_0) \end{aligned}$$

so that

$$u = \frac{1}{2x} \left[1 + \frac{t}{2x} + \frac{t^2}{4x^3} + \dots \right]$$

which is the same solution obtained by decomposition. For homogeneous equations with a constant coefficient of $f(u)$ as in this example, the rate of convergence is equal. For the general case, we find that the decomposition method converges faster.

Suppose we consider the solution of a nonlinear equation where the nonlinearity and the excitation are given as a power series, or equivalently as graphs from which series are derived by curve-fitting techniques. Let's write u and $f(u)$ in series form

$$u = \sum_{n=0}^{\infty} a_n t^n$$

$$f(u) = \sum_{v=0}^{\infty} \alpha_v u^v$$

$$f(u) = \sum_{v=0}^{\infty} \alpha_v \left\{ \sum_{n=0}^{\infty} a_n t^n \right\}^v$$

Computing $\left\{ \sum_{n=0}^{\infty} a_n t^n \right\}^v$ we see that

$$\left\{ \sum_{n=0}^{\infty} a_n t^n \right\}^v = \sum_{n=0}^{\infty} t^n A_n [u^v] \Big|_{u_\lambda = a_\lambda}$$

for all λ and for $v > 0$. (Of course, for $v = 0$ the quantity on the left is clearly equal to 1.) Thus, if we define

$$\begin{aligned}
 b_0 &= \alpha_0 + \sum_{v=0}^{\infty} \alpha_v A_0[u^v] \Big|_{u_\lambda=a_\lambda} \\
 b_n &= \sum_{v=1}^{\infty} \alpha_v A_n[u^v] \Big|_{u_\lambda=a_\lambda} \quad \text{for } n > 0
 \end{aligned} \tag{1}$$

The A_n are easily evaluated. For example,

$$\begin{aligned}
 A_0[u^v] &= u_0^v \\
 A_1[u^v] &= v u_1 u_0^{v-1} \\
 &\vdots
 \end{aligned}$$

We observe that $A_n[u^{v+1}] = u_n$, i.e., the linear case, then

$$A_n[f(u)] \Big|_{u_v=a_v} = A_n[u^v] \Big|_{u_\lambda=a_\lambda} = A_n(a_0, \dots, a_n)$$

The previous result on transformation of series stated that if $u = \sum_{n=0}^{\infty} a_n t^n$,

$$f(u) = \sum_{n=0}^{\infty} t^n A_n\{f(u)\} \Big|_{u_v=a_v}$$

From (1)

$$b_0 = \alpha_0(1) + \alpha_1(a_0) + \alpha_2(a_0^2) + \alpha_3(a_0^3) + \dots + \alpha_m(a_0^m) + \dots$$

$$b_0 = \sum_{n=0}^{\infty} \alpha_n(a_0^n)$$

$$b_1 = \alpha_1(a_1) + \alpha_2(2a_0a_1) + \alpha_3(3a_0^2a_1) + \dots + \alpha_m(ma_1a_0^{m-1})$$

$$b_1 = \sum_{n=1}^{\infty} \alpha_n(na_1a_0^{n-1})$$

$$\text{اضافى} \left\{ \begin{aligned}
 &b_0 = \alpha_0(1) + \alpha_1(a_0) + \alpha_2(a_0^2) + \alpha_3(a_0^3) + \dots + \alpha_m(a_0^m) + \dots \\
 &b_0 = \sum_{n=0}^{\infty} \alpha_n(a_0^n) \\
 &b_1 = \alpha_1(a_1) + \alpha_2(2a_0a_1) + \alpha_3(3a_0^2a_1) + \dots + \alpha_m(ma_1a_0^{m-1}) \\
 &b_1 = \sum_{n=1}^{\infty} \alpha_n(na_1a_0^{n-1})
 \end{aligned} \right.$$

The α_n dominate the convergence. For simplicity of notation, let

$$A_m^{(n)} \equiv A_m[u^n] \Big|_{u_v = a_v} = A_m^{(n)}(a_0, \dots, a_n)$$

Then

$$b_0 = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n A_0^{(n)}(a_0) = \sum_{n=0}^{\infty} \alpha_n a_n^0$$

$$b_1 = \sum_{n=1}^{\infty} \alpha_n A_1^{(n)}(a_0, a_1)$$

$$b_2 = \sum_{n=1}^{\infty} \alpha_n A_2^{(n)}(a_0, a_1, a_2)$$

$$\vdots$$

$$b_m = \sum_{n=1}^{\infty} \alpha_n A_m^{(n)}(a_0, \dots, a_m)$$

$$f(u) = \sum_{n=0}^{\infty} b_n t^n = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n A_0^{(n)}(a_0)$$

$$+ t \sum_{n=1}^{\infty} \alpha_n A_1^{(n)}(a_0, a_1) + t^2 \sum_{n=1}^{\infty} \alpha_n A_2^{(n)}(a_0, a_1, a_2)$$

$$+ t^3 \sum_{n=1}^{\infty} \alpha_n A_3^{(n)}(a_0, a_1, a_2, a_3) + \dots$$

where

$$b_0 = \left\{ \alpha_0 + \sum_{n=1}^{\infty} \alpha_n A_0^{(n)}(a_0) \right\}$$

$$f(u) = \alpha_0 + \sum_{m=0}^{\infty} t^m \sum_{n=1}^{\infty} \alpha_n A_m^{(n)}(a_0, \dots, a_m)$$

$$f(u) = \sum_{n=0}^{\infty} b_n t^n$$

if $u = \sum_{n=0}^{\infty} a_n t^n$ and $f(u) = \sum_{n=0}^{\infty} \alpha_n u^n$.

THEOREM: If $\sum_{n=0}^{\infty} a_n t^n$ and $f(u) = \sum_{n=0}^{\infty} \alpha_n u^n$ are convergent, then $f(u) = \sum_{n=0}^{\infty} b_n t^n$ is convergent where

$$b_0 = \alpha_0 + \sum_{\nu=1}^{\infty} \alpha_{\nu} A_0(u^{\nu}) \Big|_{u_k=a_k}$$

$$b_{n>0} = \sum_{\nu=1}^{\infty} \alpha_{\nu} A_n(u^{\nu}) \Big|_{u_k=a_k}$$

This is useful in the following problem. Consider (using modified decomposition) the nonlinear equation,

$$d^2u/dt^2 + Nu = g(t) = \sum_{n=0}^{\infty} g_n t^n$$

with $u(0) = c_0$ and $u'(0) = c_1$ and $Nu = \sum_{n=0}^{\infty} \alpha_n u^n$

$$\frac{d^2u}{dt^2} + \sum_{n=0}^{\infty} \alpha_n u^n = \sum_{n=0}^{\infty} g_n t^n$$

Substituting

$$u = \sum_{n=0}^{\infty} a_n t^n$$

$$\frac{du}{dt} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

$$\frac{d^2u}{dt^2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} t^n$$

and using the theorem above $\sum_{n=0}^{\infty} \alpha_n u^n = \sum_{n=0}^{\infty} b_n t^n$ with the given formulas for b_n ,

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} t^n + \sum_{n=0}^{\infty} b_n t^n = \sum_{n=0}^{\infty} g_n t^n$$

Let $g_n = (n+1)(n+2) a_{n+2} + b_n$ so that we have found the coefficients

$$a_0 = c_0$$

$$a_1 = c_1$$

$$a_{n+2} = \frac{g_n - b_n}{(n+1)(n+2)}$$

with $u = \sum_{n=0}^{\infty} a_n t^n$. (Of course, we can solve the problem by decomposition as well as by modified decomposition.)

Let us consider, as a generic example, a partial differential equation in the operator form of the decomposition method:

$$L_x u + L_y u + Nu = 0$$

where L_x is a linear differential with respect to x , L_y is a linear differential with respect to y , and Nu is a nonlinear term. The x -dimension partial solution is

$$u = \Phi_x - L_x^{-1} \{L_y u + Nu\}$$

where $L_x \Phi_x = 0$. Assume the solution in the form

$$u(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} x^m y^n$$

$$u = \sum_{m=0}^{\infty} \xi_m(y) x^m$$

where $\xi_m = \sum_{n=0}^{\infty} a_{m,n} y^n$. Write also $Nu = f(u) = \sum_{m=0}^{\infty} A_m(y) x^m$ where the $A_m(y) = A_m(\xi_0(y), \dots, \xi_m(y))$ are our polynomials. Substituting, we have

$$\sum_{m=0}^{\infty} \xi_m(y) x^m = \Phi_x - L_x^{-1} L_y \sum_{m=0}^{\infty} \xi_m(y) x^m - L_x^{-1} \sum_{m=0}^{\infty} A_m(y) x^m$$

Since

$$L_x^{-1} \sum_{m=0}^{\infty} \xi_m(y) x^m = \sum_{m=0}^{\infty} \xi_m(y) x^{m+2} / (m+1)(m+2)$$

$$= \sum_{m=2}^{\infty} \xi_{m-2}(y) x^m / m(m-1)$$

and

$$L_x^{-1} \sum_{m=0}^{\infty} A_m(y) x^m = \sum_{m=2}^{\infty} A_{m-2}(y) x^m / m(m-1)$$

we can write

$$\sum_{m=0}^{\infty} \xi_m(y) x^m = \Phi_x - \sum_{m=2}^{\infty} L_y \xi_{m-2}(y) x^m / m(m-1) - \sum_{m=2}^{\infty} A_{m-2}(y) x^m / m(m-1)$$

Assuming L_x is a second-order differential operator,

$$\phi_x = k_0(y) + x k_1(y)$$

where the k_0 and k_1 can be determined from the given conditions. We finally get recurrence relations for the coefficients ξ_m

$$\xi_0(y) = k_0(y)$$

$$\xi_1(y) = k_1(y)$$

and for $m \geq 2$

$$\xi_m(y) = -\{L_y \xi_{m-2}(y) + A_{m-2}(y)\} / m(m-1)$$

where $A_m(y) = A_m(\xi_0(y), \dots, \xi_m(y))$. The y -dimension partial solution is similarly obtained. Assuming

$$L_y = \partial^2 / \partial y^2$$

$$u(x, y) = \sum_{n=0}^{\infty} \eta_n(x) y^n$$

$$f(u) = \sum_{n=0}^{\infty} A_n(x) y^n$$

where $A_n(x) = A_n(\eta_0(x), \dots, \eta_n(x))$.

$$\sum_{n=0}^{\infty} \eta_n(x) y^n = \Phi_y - L_y^{-1} L_x \sum_{n=0}^{\infty} \eta_n(x) y^n - L_y^{-1} \sum_{n=0}^{\infty} A_n(x) y^n$$

Proceeding as for the x partial solution, we now get the recurrence relation

$$\eta_0(x) = c_0(x)$$

$$\eta_1(x) = c_1(x)$$

and for $n \geq 2$

$$\eta_n(x) = -\{L_x \eta_{n-2}(x) + A_{n-2}(x)\} / n(n-1)$$

where $A_n(x) = A_n(\eta_0(x), \dots, \eta_n(x))$.

REMARK: When $k_0(y)$ and $k_1(y)$ are expressed as power series in y , and $c_0(x)$ and $c_1(x)$ are expressed in power series in x , the series solutions

$$u = \sum_{m=0}^{\infty} \xi_m(y) x^m$$

$$u = \sum_{n=0}^{\infty} \eta_n(x) y^n$$

can be written as

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n} x^m y^n$$

Consider as an example the equation

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 + \rho(x, y)u + \alpha(x, y)f(u) = g(x, y)$$

Since we have chosen a two-dimensional case with the conditions,

$$u(0, y) = C_0(y) \quad \text{and} \quad \frac{\partial u(0, y)}{\partial x} = C_1(y)^*$$

we assume the solution in the form

$$u = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} x^m y^n$$

and similarly write

$$\rho = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \rho_{n,m} x^m y^n$$

$$\alpha = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha_{n,m} x^m y^n$$

(An ordinary differential equation such as $d^2u/dx^2 - \rho(x)u + \alpha(x)f(u) = g(x)$ becomes a special case as do linear cases of both the ordinary and partial differential equations. We use a single series. For a three-dimensional series, we use a triple series.) We write the differential equation in our usual decomposition form as

$$L_x u + L_y u + Ru + Nu = g$$

*This could also be solved with boundary conditions.

We can use either the x or the y partial solution. Using the x partial solution for which we have stated conditions, we operate on both sides with L_x^{-1} to write

$$L_x^{-1} L_x u = u - u(0, y) - x \frac{\partial u(0, y)}{\partial x} = L_x^{-1} g - L_x^{-1} L_y u - L_x^{-1} Ru - L_x^{-1} Nu$$

where L_x^{-1} is the two-fold definite integration from 0 to x . We now have

$$u = C_0(y) + xC_1(y) + L_x^{-1} g - L_x^{-1} L_y u - L_x^{-1} Ru - L_x^{-1} Nu$$

Computing $L_x^{-1} g$ we get

$$\begin{aligned} L_x^{-1} g &= L_x^{-1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g_{n,m} x^n y^m \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g_{n,m} x^{n+2} y^m / (n+1)(n+2) \\ &= \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} g_{n-2,m} x^n y^m / n(n-1) \end{aligned}$$

Computing $L_y u$, we have

$$\begin{aligned} L_y u &= (\partial^2 / \partial y^2) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} x^n y^m \\ &= \sum_{n=0}^{\infty} \sum_{m=2}^{\infty} m(m-1) c_{n,m} x^n y^{m-2} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (m+1)(m+2) c_{n,m+2} x^n y^m \end{aligned}$$

Now computing $L_x^{-1} L_y u$ we have

$$\begin{aligned} L_x^{-1} L_y u &= L_x^{-1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (m+1)(m+2) c_{n,m+2} x^n y^m \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(m+1)(m+2)}{(n+1)(n+2)} c_{n,m+2} x^{n+2} y^m \\ &= \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} \frac{(m+1)(m+2)}{n(n-1)} c_{n-2,m+2} x^n y^m \end{aligned}$$

Computing Ru ,

$$\begin{aligned} Ru &= \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \rho_{n,m} x^n y^m \right\} \cdot \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} x^n y^m \right\} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \sum_{v=0}^n \sum_{\mu=0}^m \rho_{v,\mu} c_{n-v,m-\mu} \right\} x^n y^m \end{aligned}$$

Next,

$$L_x^{-1} Ru = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \sum_{v=0}^n \sum_{\mu=0}^m \rho_{v,\mu} c_{n-2-v,m-\mu} / n(n-1) \right\} x^n y^m$$

Also

$$f(u) = f \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n,m} x^n y^m \right) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{n,m} x^n y^m$$

where the polynomials $A_{n,m}(c_{0,0}, \dots, c_{n,m})$ are obtained by transformation of series. We have

$$\begin{aligned} Nu &= \alpha f(u) = \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha_{n,m} x^n y^m \right\} \cdot \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{n,m} x^n y^m \right\} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \sum_{v=0}^n \sum_{\mu=0}^m \alpha_{v,\mu} A_{n-v,m-\mu} \right\} x^n y^m \end{aligned}$$

Then

$$\begin{aligned} L_x^{-1} Nu &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \sum_{v=0}^n \sum_{\mu=0}^m \alpha_{v,\mu} A_{n-v,m-\mu} / (n+1)(n+2) \right\} x^{n+2} y^m \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \sum_{v=0}^{n-2} \sum_{\mu=0}^m \alpha_{v,\mu} A_{n-2-v,m-\mu} / n(n-1) \right\} x^n y^m \end{aligned}$$

We also write $C_0(y) = \sum_{m=0}^{\infty} c_{0,m} y^m$ and $C_1(y) = \sum_{m=0}^{\infty} c_{1,m} y^m$. We know $c_{0,0}, c_{0,1}, c_{0,2}, \dots$. Similarly, we know $c_{1,0}, c_{1,1}, c_{1,2}, \dots$.

$$u = C_0(y) + x C_1(y) + L_x^{-1} g - L_x^{-1} L_y u - L_x^{-1} Ru - L_x^{-1} Nu$$

and we can write

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} x^n y^m &= \sum_{m=0}^{\infty} c_{0,m} y^m + x \sum_{m=0}^{\infty} c_{1,m} y^m + \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} \frac{g_{n-2,m}}{n(n-1)} x^n y^m \\
&\quad - \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} \frac{(m+1)(m+2)}{n(n-1)} c_{n-2,m+2} x^n y^m \\
&\quad - \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} \left\{ \sum_{v=0}^{n-2} \sum_{\mu=0}^{\infty} \rho_{v,\mu} c_{n-2-v,m-\mu} / n(n-1) \right\} x^n y^m \\
&\quad - \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} \left\{ \sum_{v=0}^{n-2} \sum_{\mu=0}^{\infty} \alpha_{v,\mu} A_{n-2-v,m-\mu} / n(n-1) \right\} x^n y^m
\end{aligned}$$

We now equate coefficients of like powers. We are using the x partial solution; hence, we are particularly interested in powers of x .

$$\text{For } n=0, \quad c_{0,m} = C_{0,m}$$

$$\text{For } n=1, \quad c_{1,m} = C_{1,m}$$

For $n \geq 2$, we have for like powers of y^m a recurrence relation yielding the coefficients

$$\begin{aligned}
c_{n,m} &= \frac{g_{n-2,m}}{n(n-1)} - \frac{(m+1)(m+2)}{n(n-1)} c_{n-2,m+2} \\
&\quad - \sum_{v=0}^{n-2} \sum_{\mu=0}^m \frac{\rho_{v,\mu} c_{n-2-v,m-\mu}}{n(n-1)} - \sum_{v=0}^{n-2} \sum_{\mu=0}^m \frac{\alpha_{v,\mu} A_{n-2-v,m-\mu}}{n(n-1)}
\end{aligned}$$

and can now write $u(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} x^n y^m$ since the coefficients are determined and the A_n can be found. Since $c_{0,0}, c_{0,1}, c_{0,2}$ are known by decomposition of $C_0(y)$ we can find other components, e.g., $c_{2,0}$ depends on $c_{0,2}$. Similarly $C_1(y)$ yields components $c_{1,m}$ for all m , so $c_{3,1}$, for example, is found from $c_{1,3}$.

The linear cases ($\alpha = 0$) are considerably easier since the A_n become unnecessary. Also the recurrence relation simplifies if $g = 0$ or $\rho = 0$. For example, if we consider the equation $\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0$ we have

$$c_{n,m} = - \frac{(m+1)(m+2)}{n(n-1)} c_{n-2,m+2}$$

MODIFIED DECOMPOSITION SERIES AND BOUNDARY-VALUE PROBLEMS:

The "modified decomposition" series solutions have been found for initial-value problems by incorporating and adapting ideas of the decomposition method. Now using the double decomposition technique discussed in recent papers, the procedure can be further generalized to treat initial-value and boundary-value problems in a similar and computationally efficient formulation with an acceleration of convergence. We will consider some progressively more complicated problems.

LINEAR (HOMOGENEOUS) ORDINARY DIFFERENTIAL EQUATIONS:

Consider the example $d^2u/dx^2 + \rho u = 0$ for Dirichlet conditions $u(x = \xi_i) = b_i$ for $i = 1, 2$. Let ρ be a constant (to simplify the discussion) and seek the solution in the form of a Maclaurin series

$$u = \sum_{n=0}^{\infty} a_n x^n$$

$$u = \sum_{n=0}^{\infty} u_n$$

$$u_n = a_n x^n$$

In the usual operator form for decomposition solutions, this equation is written $Lu + Ru = 0$ where, in this case, $L = d^2/dx^2$ and $R = \rho$. (Of course, the method was developed for more general equations.) Now write $L^{-1}Lu = -L^{-1}Ru$ where L^{-1} is a two-fold indefinite integration yielding $u - c_0 - c_1x$; hence

$$\begin{aligned} u = \sum_{n=0}^{\infty} a_n x^n &= c_0 + xc_1 - \rho \iint \sum_{n=0}^{\infty} a_n x^n dx dx \\ &= c_0 + xc_1 - \rho \sum_{n=0}^{\infty} a_n x^{n+2} / (n+1)(n+2) \\ &= c_0 + xc_1 - \rho \sum_{n=2}^{\infty} a_{n-2} x^n / n(n-1) \end{aligned}$$

Equating coefficients, $a_0 = c_0$ and $a_1 = c_1$. For $n \geq 2$ we have the recurrence relation

$$a_n = -\rho a_{n-2} / n(n-1)$$

Using double decomposition,

$$c_0 = \sum_{m=0}^{\infty} c_0^{(m)}$$

$$c_1 = \sum_{m=0}^{\infty} c_1^{(m)}$$

$$a_n = \sum_{m=0}^{\infty} a_n^{(m)}$$

Also $a_0^{(m)} = c_0^{(m)}$, $a_1^{(m)} = c_1^{(m)}$, and for $n \geq 2$, $a_n^{(m)} = -\rho a_{n-2}^{(m)} / n(n-1)$, achieving a decomposition of the recurrence relation

$$u = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)} = \sum_{n=0}^{\infty} \sum_{m=0}^n u_m^{(n-m)}$$

i.e.,

$$u_n = \sum_{m=0}^n u_m^{(n-m)}$$

by staggered summation. (See Appendix II.) We use the staggered summation to rearrange the double decomposition components of u into a new series to achieve a decomposition suitable for boundary-value problems. (The switching of n and m makes the result computable; only the sum matters — the components are not unique.)

We must now determine $c_0^{(m)}$ and $c_1^{(m)}$ by use of the boundary conditions. Instead of $u(x = \xi_1) = b_1$ and $u(x = \xi_2) = b_2$, we use the approximant ϕ_{m+1} to the correct solution u . We can write this as $\phi_{m+1}\{u\}$, an operator on u . Similarly, the approximant to a_n is

$$\phi_m\{a_n\} = \sum_{v=0}^{m-1} a_n^{(v)}$$

Then

$$\phi_1\{a_0\} = a_0^{(0)}$$

$$\phi_1\{a_1\} = a_1^{(0)}$$

We will have determined the solution if we can also compute the values of $c_n^{(m)}$ and $c_1^{(m)}$ matching the (approximant to the) solution with the boundary

conditions

$$\phi_{m+1}\{u\}(x = \xi_1) = b_1$$

$$\phi_{m+1}\{u\}(x = \xi_2) = b_2$$

For the staggered series of u

$$u_0 = a_0^{(0)} + a_1^{(0)} x$$

$$u_1 = a_0^{(1)} + a_1^{(1)} x + a_2^{(0)} x^2 + a_3^{(0)} x^3$$

$$u_2 = a_0^{(2)} + a_1^{(2)} x + a_2^{(1)} x^2 + a_3^{(1)} x^3 + a_4^{(0)} x^4 + a_5^{(0)} x^5$$

$$\vdots$$

$$u_m = a_0^{(m)} + a_1^{(m)} x + \dots + a_{2m}^{(0)} x^{(2m)} + a_{2m+1}^{(0)} x^{2m+1}$$

Equivalently,

$$u_m = \sum_{n=0}^{2m+1} a_n^{(m-[n/2])}$$

where $[n/2]$ is defined as the greatest integer value less than $n/2$. Thus the staggered summation has resulted in a different decomposition of u suitable for boundary-value problems.

$$u = \sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} \sum_{n=0}^{2m+1} a_n^{(m-[n/2])} x^n$$

REMARK: The greatest integer function used here is for a second-order equation. For a third-order equation, we will have $[n/3]$ and for fourth-order, we use $[n/4]$. Next, we derive the approximant for the staggered series of the solution; thus $\phi_{m-1}\{u\}$ is given by:

$$\begin{aligned} \phi_{m-1}\{u\} &= u_0 + u_1 + \dots + u_m = \sum_{n=0}^m a_0^{(n)} + x \sum_{n=0}^m a_1^{(n)} + x^2 \sum_{n=0}^{m-1} a_2^{(n)} \\ &\quad + x^3 \sum_{n=0}^{m-1} a_3^{(n)} + \dots + x^{2m} a_{2m}^{(0)} + x^{2m+1} a_{2m+1}^{(0)} \\ \phi_{m-1}\{u\} &= \phi_{m-1}\{a_0\} + x \phi_{m-1}\{a_1\} + x^2 \phi_{m-1}\{a_2\} \\ &\quad + x^3 \phi_{m-1}\{a_3\} + \dots + x^{2m} \phi_{m-1}\{a_{2m}\} + x^{2m+1} \phi_{m-1}\{a_{2m+1}\} \end{aligned}$$

or equivalently

$$\phi_{m+1}\{u\} = \sum_{n=0}^{2m+1} \phi_{m+1-\lfloor n/2 \rfloor} \{a_n\} x^n$$

We use the approximation of the boundary conditions

$$\phi_{m+1}\{u\}(x = \xi_1) = b_1$$

$$\phi_{m+1}\{u\}(x = \xi_2) = b_2$$

in order to compute the constants $c_0^{(m)}$ and $c_1^{(m)}$ and consequently the components $a_0^{(m)}$ and $a_1^{(m)}$ of the MacLaurin series for the solution, thus determining the solution u .

SUMMARY:

The basic steps are:

- 1) Compute $a_0^{(m)}$ and $a_1^{(m)}$ by matching the solution approximants to the boundaries, i.e.,

$$\phi_{m+1}\{u\}(x = \xi_1) = b_1$$

$$\phi_{m+1}\{u\}(x = \xi_2) = b_2$$

- 2) Using the recurrence relations for $n \geq 2$, i.e., $a_n^{(m)} = -\rho a_{n-2}^{(m)} / n(n-1)$, compute more components $a_n^{(m)}$ to improve the accuracy of the solution approximants. Thus we have

$$a_n = \sum_{m=0}^{\infty} a_n^{(m)}$$

and therefore

$$u = \sum_{n=0}^{\infty} a_n x^n$$

satisfying both the equation $d^2u/dx^2 + \rho u = 0$ and the conditions

$$u(x = \xi_1) = b_1$$

$$u(x = \xi_2) = b_2$$

We can now accelerate convergence by going to an initial-value formatted solution without further matching of solution to boundary conditions. Let

$$a_0 = \phi_{m+1}\{a_0\}$$

$$a_1 = \phi_{m+1}\{a_1\}$$

and for $n \geq 2$

$$a_n = -\rho a_{n-2} / n(n-1)$$

which gives us a new and improved u_0 to start as an initial-value problem

$$\phi_{m+1}\{u_0^{(i)}\} = \phi_{m+1}\{a_0\} + \phi_{m+1}\{a_1\}x$$

where $u_0^{(i)}$ is the approximate initial value. Now $u = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} a_n x^n$.

NONLINEAR ORDINARY DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS:

Consider as a specific example $d^2u/dx^2 + \alpha f(u) = 0$ given the conditions

$$u(x = \xi_1) = b_1$$

$$u(x = \xi_2) = b_2$$

We seek Maclaurin series solution $u = \sum_{n=0}^{\infty} a_n x^n$. The equation in operator form is $Lu + Nu = 0$ with $L = d^2/dx^2$ and $Nu = \alpha f(u)$. Operating with L^{-1} , we have

$$u = c_0 + xc_1 - \alpha \iint f(u) dx dx$$

Using the result for transformation of series

$$f(u) = f\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} A_n(a_0, \dots, a_n) x^n$$

where the A_n are functions of $a_0, \dots, a_n$ rather than $u_0, \dots, u_n$. Now calculating the integral

$$\iint f(u) dx dx$$

or

$$\iint \sum_{n=0}^{\infty} A_n x^n dx dx = \sum_{n=0}^{\infty} A_n x^{n+2} / (n+1)(n+2) = \sum_{n=2}^{\infty} A_{n-2} x^n / n(n-1)$$

Therefore

$$\sum_{n=0}^{\infty} a_n x^n = c_0 + x c_1 - \alpha \sum_{n=2}^{\infty} A_{n-2} x^n / n(n-1)$$

Equating coefficients,

$$a_0 = c_0$$

$$a_1 = c_1$$

and for $n \geq 2$

$$a_n = -\alpha A_{n-2} / n(n-1)$$

To match the boundary conditions, we apply decomposition to the integration constants and double decomposition to the coefficients of the Maclaurin series solution.

$$c_0 = \sum_{m=0}^{\infty} c_0^{(m)}$$

$$c_1 = \sum_{m=0}^{\infty} c_1^{(m)}$$

$$a_n = \sum_{m=0}^{\infty} a_n^{(m)}$$

Substituting into the recurrence relations

$$\overline{a_0^{(m)}} = c_0^{(m)}$$

$$a_1^{(m)} = c_1^{(m)}$$

and for $n \geq 2$

$$a_n^{(m)} = -\alpha A_{n-2}^{(m)} / n(n-1)$$

We have achieved a decomposition of the recurrence relations which will determine the solution once the components $c_0^{(m)}$ and $c_1^{(m)}$ are found.

Next, we organize the solution into a form suitable for matching at the boundaries by rearranging the double decomposition components of the solution into a staggered series. This organizes the solution into the boundary-value format so that $c_0^{(m)}$ and $c_1^{(m)}$ components are calculable by use of the approximants to the boundary conditions

$$\phi_{m+1}\{u\}(x = \xi_1) = b_1$$

$$\phi_{m+1}\{u\}(x = \xi_2) = b_2$$

We have

$$u = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n^{(m)} x^n$$

To stagger the series,

$$\begin{aligned} u_0 &= a_0^{(0)} + a_1^{(0)} x \\ u_1 &= a_0^{(1)} + a_1^{(1)} x + a_2^{(0)} x^2 + a_3^{(0)} x^3 \\ u_2 &= a_0^{(2)} + a_1^{(2)} x + a_2^{(1)} x^2 + a_3^{(1)} x^3 + a_4^{(0)} x^4 + a_5^{(0)} x^5 \\ &\vdots \\ u_m &= a_0^{(m)} + a_1^{(m)} x + \dots + a_{2m}^{(0)} x^{2m} + a_{2m+1}^{(0)} x^{2m+1} \end{aligned}$$

which we can write

$$u_m = \sum_{n=0}^{2m+1} a_n^{(m-[n/2])} x^n$$

where $[n/2]$ is an integer greater than $n/2$ (for second-order equations). For third order, we write $[n/3]$ and for fourth order, $[n/4]$. We now have a different decomposition of u which is suitable for boundary-value problems:

$$u = \sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} \sum_{n=0}^{2m+1} a_n^{(m-[n/2])} x^n$$

Now we derive the solution approximants for the staggered approximants

$$\begin{aligned} \phi_{m+1}\{u\} &= \sum_{n=0}^m u_n = \sum_{n=0}^m a_0^{(n)} + x \sum_{n=0}^m a_1^{(n)} + x^2 \sum_{n=0}^{m-1} a_2^{(n)} \\ &\quad + x^3 \sum_{n=0}^{m-1} a_3^{(n)} + \dots + x^{2m} a_{2m}^{(0)} + x^{2m+1} a_{2m+1}^{(0)} \end{aligned}$$

Hence

$$\begin{aligned} \phi_{m+1}\{u\} &= \phi_{m+1}\{a_0\} + x \phi_{m+1}\{a_1\} + x^2 \phi_m\{a_2\} \\ &\quad + x^3 \phi_m\{a_3\} + \dots + x^{2m} \phi_1\{a_{2m}\} + x^{2m+1} \phi_1\{a_{2m+1}\} \end{aligned}$$

or

$$\phi_{m+1}\{u\} = \sum_{n=0}^{2m+1} \phi_{m+1-[n/2]}\{a_n\} x^n$$

Now using the boundary condition approximants

$$\phi_{m+1}\{u\}(x = \xi_1) = b_1$$

$$\phi_{m+1}\{u\}(x = \xi_2) = b_2$$

we can compute the constants $c_0^{(m)}$, $c_1^{(m)}$ and $a_0^{(m)}$, $a_1^{(m)}$, the coefficients for the Maclaurin series determining the solution.

SUMMARY:

The basic steps are:

- 1) Compute $a_0^{(m)}$ and $a_1^{(m)}$ by matching the solution approximants to the boundaries.
- 2) Compute components $a_n^{(m)}$ to improve the accuracy of the solution approximants using the recurrence relations for $n \geq 2$. We get

$$a_n = \sum_{m=0}^{\infty} a_n^{(m)}$$

and finally

$$u = \sum_{n=0}^{\infty} a_n x^n$$

- 3) Finally we can transpose from the boundary-value format to the initial-value format

$$\phi_{m+1}\{u_0^{(i)}\} = \phi_{m+1}\{a_0\} + \phi_{m+1}\{a_1\} x$$

with

$$a_0 = \phi_{m+1}\{a_0\}$$

$$a_1 = \phi_{m+1}\{a_1\}$$

and for $n \geq 2$ $a_n = -\alpha A_{n-2} / n(n-1)$. (Note this is for the case of zero input.) This avoids further need to match the solution to the boundary conditions. Now $u = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} a_n x^n$ and convergence is accelerated over that of the boundary-value formatted solution.

LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS:

We consider the example $d^2u / dx^2 + \rho(x)u = 0$ with the conditions

$$u(x = \xi_1) = b_1$$

$$u(x = \xi_2) = b_2$$

Let

$$\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

and seek the solution in the form $u = \sum_{n=0}^{\infty} a_n x^n$. The solution is

$$u = c_0 + xc_1 - \iint \rho(x) u dx dx$$

Let

$$\begin{aligned} \rho(x)u &= \left\{ \sum_{n=0}^{\infty} \rho_n x^n \right\} \cdot \left\{ \sum_{n=0}^{\infty} a_n x^n \right\} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{v=0}^n \rho_v a_{n-v} \right\} x^n \end{aligned}$$

Now

$$\begin{aligned} \iint \rho(x)u dx dx &= \iint \sum_{n=0}^{\infty} \left\{ \sum_{v=0}^n \rho_v a_{n-v} \right\} x^n dx dx \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{v=0}^n \rho_v a_{n-v} \right\} x^{n+2} / (n+1)(n+2) \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{v=0}^{n-2} \rho_v a_{n-2-v} \right\} x^n / n(n-1) \end{aligned}$$

We now have

$$\sum_{n=0}^{\infty} a_n x^n = c_0 + xc_1 - \sum_{n=2}^{\infty} \left\{ \sum_{v=0}^{n-2} \rho_v a_{n-2-v} \right\} x^n / n(n-1)$$

Consequently $a_0 = c_0$ and $a_1 = c_1$ and for $n \geq 2$

$$a_n = - \sum_{v=0}^{n-2} \rho_v a_{n-2-v} / n(n-1)$$

We conclude with the solution $a_0^{(m)} = c_0^{(m)}$ and $a_1^{(m)} = c_1^{(m)}$. The $c_0^{(m)}$ and $c_1^{(m)}$ are determined using the boundary conditions

$$\phi_{m+1}\{u\}(x = \xi_1) = b_1$$

$$\phi_{m+1}\{u\}(x = \xi_2) = b_2$$

and for $n \geq 2$, $a_n^{(m)} = -\sum_{v=0}^{n-2} \rho_v a_{n-2-v}^{(m)} / n(n-1)$ achieving a decomposition of the recurrence relations. Then computing $a_n = \sum_{m=0}^{\infty} a_n^{(m)}$, we have the solution $u_n = \sum_{m=0}^{\infty} a_n x^n$.

HOMOGENEOUS NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS:

Consider the example $d^2u/dx^2 + \alpha(x)f(u) = 0$ with the conditions

$$u(x = \xi_1) = b_1$$

$$u(x = \xi_2) = b_2$$

Let $\alpha(x) = \sum_{n=0}^{\infty} \alpha_n x^n$ and seek the solution in the form $u = \sum_{n=0}^{\infty} a_n x^n$, i.e., a Maclaurin series. Let $Nu = \alpha(x)f(u)$ and write $Lu + Nu = 0$ which leads to the solution $u = c_0 + xc_1 - \iint \alpha(x)f(u) dx dx$. To calculate this integral (which is an indefinite integration for every iteration), we first use the result for transformation of series

$$f(u) = f\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} A_n x^n$$

where $A_n = A_n(a_0, \dots, a_n)$. Now

$$\alpha(x)f(u) = \left\{\sum_{n=0}^{\infty} \alpha_n x^n\right\} \cdot \left\{\sum_{n=0}^{\infty} A_n x^n\right\} = \sum_{n=0}^{\infty} \left\{\sum_{v=0}^n \alpha_v A_{n-v}\right\} x^n$$

and

$$\begin{aligned} \iint \alpha(x)f(u) dx dx &= \iint \sum_{n=0}^{\infty} \left\{\sum_{v=0}^n \alpha_v A_{n-v}\right\} x^n dx dx \\ &= \sum_{n=1}^{\infty} \left\{\sum_{v=0}^n \alpha_v A_{n-v}\right\} x^{n+2} / (n+1)(n+2) \\ &= \sum_{n=1}^{\infty} \left\{\sum_{v=1}^{n+1} \alpha_v A_{n+1-v}\right\} x^{n+1} / n(n+1) \end{aligned}$$

We now have

$$\sum_{n=0}^{\infty} a_n x^n = c_0 + x c_1 - \sum_{n=2}^{\infty} \left\{ \sum_{v=0}^{n-2} \alpha_v A_{n-2-v} \right\} x^n / n(n-1)$$

Equating coefficients $a_0 = c_0$ and $a_1 = c_1$ and for $n \geq 2$

$$a_n = \sum_{v=0}^{n-2} \alpha_v A_{n-2-v} / n(n-1)$$

Hence, $a_0^{(m)} = c_0^{(m)}$, $a_1^{(m)} = c_1^{(m)}$. Using

$$\phi_{m+1}\{u\}(x = \xi_1) = b_1$$

$$\phi_{m+1}\{u\}(x = \xi_2) = b_2$$

we determine $c_0^{(m)}$ and $c_1^{(m)}$ and for $n \geq 2$

$$a_n^{(m)} = - \sum_{v=0}^{\infty} \alpha_v A_{n-2-v}^{(m)} / n(n-1)$$

Then, computing $a_n = \sum_{m=0}^{\infty} a_n^{(m)}$, we get $u = \sum_{n=0}^{\infty} a_n x^n$.

HOMOGENEOUS NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS FOR LINEAR AND NONLINEAR TERMS:

Consider a specific example with Dirichlet conditions

$$d^2 u / dx^2 + \rho(x) u + \alpha(x) f(u) = 0$$

$$u(x = \xi_1) = b_1$$

$$u(x = \xi_2) = b_2$$

We seek a solution in Maclaurin series form $u = \sum_{n=0}^{\infty} a_n x^n$ satisfying the boundary conditions. In the operator format of the decomposition method, the above equation is written $Lu + Ru + Nu = 0$ with $L = d^2 / dx^2$, $R = \rho(x)$ and $Nu = \alpha(x) f(u)$. Operating with L^{-1} ,

$$u = c_0 + xc_1 - \iint \rho(x)u \, dx \, dx - \alpha(x)f(u) \, dx \, dx$$

where this is understood to be an indefinite integration for every iteration, since it is a nonlinear function. Following the earlier examples, we can write

$$a_0^{(m)} = c_0^{(m)}$$

$$a_1^{(m)} = c_1^{(m)}$$

with $c_0^{(m)}$ and $c_1^{(m)}$ to be determined by the specified conditions

$$\phi_{m-1}\{u\}(x = \xi_1) = b_1$$

$$\phi_{m+1}\{u\}(x = \xi_2) = b_2$$

and for $n \geq 2$

$$a_n^{(m)} = - \sum_{v=0}^{n-2} \rho_v a_{n-2-v}^{(m)} - \sum_{v=0}^{n-2} \alpha_v A_{n-2-v}^{(m)} / n(n-1)$$

which gives us a decomposition of the recurrence relation. Then, computing

$$a_n = \sum_{m=0}^{\infty} a_n^{(m)}, \text{ we have the solution } u = \sum_{n=0}^{\infty} a_n x^n.$$

COMMENTS:

Comparison of initial-value format and boundary-value format:

1) Initial-value problem, formatted solution $u^{(I.V.)}$

$$u = \sum_{n=0}^{\infty} u_n^{(I.V.)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)} x^n$$

$$\phi_{n\gamma}^{(I.V.)} = \sum_{m=0}^{n-1} u_m^{(I.V.)}$$

2) Boundary-value problem formatted solution $u^{(B.V.)}$

$$u = \sum_{m=0}^{\infty} u_m^{(B.V.)} = \sum_{m=0}^{\infty} \sum_{n=0}^{2m+1} a_n^{(m-[n/2])} x^n$$

where

$$u_m^{(B.V.)} = \sum_{n=0}^{2m+1} a_n^{(m-[n/2])} x^n$$

and where

$$\phi_{m+1}^{(B,V)} = \sum_{n=0}^m u_n^{(B,V)}$$

We remark further that $u_n^{(I,V)} \neq u_n^{(B,V)}$ and $\phi_{n+1}^{(I,V)} \neq \phi_{n+1}^{(B,V)}$, but as n become sufficiently large, $\phi_n^{(I,V)}$ becomes numerically equal to $\phi_n^{(B,V)}$, i.e., $u = \lim_{n \rightarrow \infty} \phi_n^{(I,V)} = \lim_{n \rightarrow \infty} \phi_n^{(B,V)}$.

Thus the evident difference between the two is in the organization of the components of the double decomposition, i.e., the staggering of the double series summations makes it possible to calculate the matching coefficients. The procedure is quite general and will work for a wide class of problems.

It is to be emphasized that the series resulting from decomposition is not a Maclaurin series. It is actually a generalized Taylor series about a function rather than about a point, which reduces in trivial cases to the well-known series. Despite the improved applicability of the Maclaurin series with the use of the A_n polynomials and decomposition techniques, the decomposition series is still superior in convergence properties to the modified decomposition series.

SOLVING EQUATIONS WITH DIFFICULT NONLINEARITIES:

Consider a second-order (nonlinear) ordinary differential equation in the form $u'' + \alpha(t)\Gamma(u) = \beta(t)$ with initial conditions $u(0) = c_0$ and $u'(0) = c_1$. We suppose $\Gamma(u)$ is obtained by curve-fitting or leads to difficult computation of the A_n polynomials. Or in a function such as $\Gamma(u) = \sin u$ we may prefer to work with the powers of u . We, therefore, write

$$\Gamma(u) = \sum_{m=0}^{\infty} \gamma_m u^m$$

Using modified decomposition, let

$$\begin{aligned} u &= \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} u_n & u' &= \sum_{n=0}^{\infty} u'_n = \sum_{n=0}^{\infty} (n+1)a_{n+1}t^n \\ \alpha &= \sum_{n=0}^{\infty} \alpha_n t^n & u'' &= \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}t^n \\ \beta &= \sum_{n=0}^{\infty} \beta_n t^n \end{aligned}$$

$\Gamma(u)$ can be written $\sum_{n=0}^{\infty} A_n(u_0, \dots, u_n)$ and we have assumed that u is written in a convergent series $u = \sum_{n=0}^{\infty} a_n t^n$ so that

$$\Gamma(u) = \Gamma\left(\sum_{n=0}^{\infty} a_n t^n\right)$$

with $u_n = a_n t^n$. Hence

$$A_n(u_0, \dots, u_n) = t^n A_n(a_0, \dots, a_n)$$

so

$$\Gamma(u) = \sum_{n=0}^{\infty} A_n(a_0, \dots, a_n) t^n$$

If we write for clarity $A_n\{f(u)\}$ for the A_n representing $f(u)$,

$$\Gamma(u) = \sum_{n=0}^{\infty} A_n\{\Gamma(u)\} t^n = \sum_{n=0}^{\infty} A_n(a_0, \dots, a_n) t^n$$

i.e., the $A_n\{\Gamma(u)\}$ are functions of $a_0, \dots, a_n$. Since $\Gamma(u) = \sum_{n=0}^{\infty} \gamma_n u^n$ and $u^n = \sum_{m=0}^{\infty} A_m\{u^n\} t^m$, the $A_m\{u^n\}$ are also functions of $a_0, \dots, a_m$. Substituting, we have

$$\begin{aligned} \Gamma(u) &= \sum_{m=0}^{\infty} \gamma_m u^m = \sum_{m=0}^{\infty} \gamma_m \sum_{\mu=0}^{\infty} A_{\mu}(u^m) t^{\mu} \\ &= \sum_{m=0}^{\infty} t^m \sum_{\mu=0}^{\infty} \gamma_{\mu} A_{\mu}(u^m) \end{aligned}$$

which we compare with $\Gamma(u) = \sum_{m=0}^{\infty} A_m\{\Gamma(u)\} t^m$ so that

$$A_m\{\Gamma(u)\} = \sum_{\mu=0}^{\infty} \gamma_{\mu} A_{\mu}\{u^m\}$$

Returning to the differential equation $u'' + \alpha \Gamma(u) = \beta$, and substituting the respective series, we have

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}t^n + \left\{ \sum_{n=0}^{\infty} \alpha_n t^n \right\} \cdot \left\{ \sum_{n=0}^{\infty} A_n \{\Gamma(u)\} t^n \right\} = \sum_{n=0}^{\infty} \beta_n t^n$$

Performing the Cauchy product

$$\left\{ \sum_{n=0}^{\infty} \alpha_n t^n \right\} \cdot \left\{ \sum_{n=0}^{\infty} A_n \{\Gamma(u)\} t^n \right\} = \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \alpha_{n-v} A_v \{\Gamma(u)\}$$

so that

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}t^n + \sum_{n=0}^{\infty} \left[\sum_{v=0}^n \alpha_{n-v} A_v \{\Gamma(u)\} \right] t^n = \sum_{n=0}^{\infty} \beta_n t^n$$

where we can now substitute

$$A_v \{\Gamma(u)\} = \sum_{\mu=0}^{\infty} \gamma_{\mu} A_v \{u^{\mu}\}$$

which makes it easy to compute the A_n for $\Gamma(u)$. We obtain

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}t^n + \sum_{n=0}^{\infty} \left\{ \sum_{v=0}^n \alpha_{n-v} \sum_{\mu=0}^{\infty} \gamma_{\mu} A_v \{u^{\mu}\} \right\} t^n = \sum_{n=0}^{\infty} \beta_n t^n$$

where the $A_v = A_v(a_0, \dots, a_v)$ are functions of the coefficients for the series for u .

Equating coefficients of like powers of the independent variable t ,

$$(n+1)(n+2)a_{n+2} + \sum_{v=0}^n a_{n-v} \sum_{\mu=0}^{\infty} \gamma_{\mu} A_v \{u^{\mu}\} = \beta_n$$

and solving for a_{n+2} we have

$$a_{n+2} = \left[\beta_n - \sum_{v=0}^n a_{n-v} \sum_{\mu=0}^{\infty} \gamma_{\mu} A_v \{u^{\mu}\} \right] / (n+1)(n+2)$$

with $a_0 = c_0$ and $a_1 = c_1$ so we can solve the differential equation for u ; the A_n can be obtained by a rapid computer calculation by decomposition into simple integral powers of u , just as the choice of L in the decomposition method led to simple integrations, avoiding difficult Green's functions.

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CHAPTER 6

APPLICATIONS OF MODIFIED DECOMPOSITION

We now consider some applications of modified decomposition. The Duffing equation is an interesting example of an ordinary differential equation; it has important applications further discussed in Chapters 11 and 12. The equation is given as

$$u'' + \alpha u' + \beta u + \gamma u^3 = g(t)$$

CONSTANT COEFFICIENT CASE:

We assume that the solution, as well as the excitation, is in Maclaurin series form. Then.

$$g(t) = \sum_{m=0}^{\infty} g_m t^m$$

$$u = \sum_{m=0}^{\infty} a_m t^m$$

$$u' = du/dt = \sum_{m=0}^{\infty} (m+1)a_{m+1} t^m$$

$$u'' = d^2u/dt^2 = \sum_{m=0}^{\infty} (m+1)(m-2)a_{m-2} t^m$$

$$u^3 = \left\{ \sum_{n=0}^{\infty} a_n t^n \right\} \left\{ \sum_{v=0}^{\infty} a_v t^v \right\} \left\{ \sum_{\mu=0}^{\infty} a_{\mu} t^{\mu} \right\} = \sum_{m=0}^{\infty} A_m(a_0, \dots, a_m) t^m$$

$$A_n(a_0, \dots, a_n) = \sum_{v=0}^n \sum_{\mu=0}^v a_{n-v-\mu} a_{v-\mu} a_{\mu}$$

$$u(0) = k_0 = a_0 = \sum_{m=0}^{\infty} a_m t^m \Big|_{t=0}$$

$$u'(0) = k_1 = a_1 = \sum_{m=0}^{\infty} (m+1)a_{m+1} t^m \Big|_{t=0}$$

Substituting

$$\sum_{m=0}^{\infty} (m+1)(m+2)a_{m+2}t^m + \alpha \sum_{m=0}^{\infty} (m+1)a_{m+1}t^m \\ + \beta \sum_{m=0}^{\infty} a_m t^m + \gamma \sum_{m=0}^{\infty} A_m(a_0, \dots, a_m)t^m = \sum_{m=0}^{\infty} g_m t^m$$

Then

$$(m+1)(m+2)a_{m+2} + \alpha(m+1)a_{m+1} + \beta a_m + \gamma A_m = g_m$$

We can now write the recursion relations

$$a_0 = u(0)$$

$$a_1 = u'(0)$$

$$a_{m+2} = \frac{g_m - \alpha(m+1)a_{m+1} - \beta a_m - \gamma A_m}{(m+1)(m+2)}$$

so the a_m are determined (dependent on the A_m) and we can write

$$u(t) = \sum_{n=0}^{\infty} a_n t^n. \text{ Thus } u_0 = a_0, u_1 = a_1 t, \dots,$$

$$a_0 = u(0)$$

$$a_1 = u'(0)$$

$$a_2 = (g_0 - (1)\alpha a_1 - \beta a_0 - \gamma A_0)/(1)(2)$$

$$a_3 = (g_1 - (2)\alpha a_2 - \beta a_1 - \gamma A_1)/(2)(3)$$

$$a_4 = (g_2 - (3)\alpha a_3 - \beta a_2 - \gamma A_2)/(3)(4)$$

$$a_5 = (g_3 - (4)\alpha a_4 - \beta a_3 - \gamma A_3)/(4)(5)$$

$$a_6 = (g_4 - (5)\alpha a_5 - \beta a_4 - \gamma A_4)/(5)(6)$$

$$a_7 = (g_5 - (6)\alpha a_6 - \beta a_5 - \gamma A_5)/(6)(7)$$

$$a_8 = (g_6 - (7)\alpha a_7 - \beta a_6 - \gamma A_6)/(7)(8)$$

$$a_9 = (g_7 - (8)\alpha a_8 - \beta a_7 - \gamma A_7)/(8)(9)$$

$$a_{10} = (g_8 - (9)\alpha a_9 - \beta a_8 - \gamma A_8)/(9)(10)$$

⋮

The approximants to the solution will be given by

$$\begin{aligned}
\phi_1 &= a_0 \\
\phi_2 &= a_0 + a_1 t \\
\phi_3 &= a_0 + a_1 t + a_2 t^2 \\
&\vdots \\
\phi_{m+1} &= \sum_{n=0}^m a_n t^n
\end{aligned}$$

For convenience of the reader we list the Adomian polynomials for the nonlinearity in the Duffing equation:

$$\begin{aligned}
A_0 &= a_0^3 \\
A_1 &= 3a_0^2 a_1 \\
A_2 &= 3a_0^2 a_2 + 3a_1^2 a_0 \\
A_3 &= a_1^3 + 3a_0^2 a_3 + 6a_0 a_1 a_2 \\
A_4 &= 3a_0^2 a_4 + 3a_1^2 a_2 + 3a_2^2 a_0 + 6a_0 a_1 a_3 \\
A_5 &= 3a_0^2 a_5 + 3a_1^2 a_3 + 3a_2^2 a_1 + 6a_0 a_1 a_4 + 6a_0 a_2 a_3 \\
A_6 &= a_3^3 + 3a_0^2 a_6 + 3a_1^2 a_4 + 3a_2^2 a_0 + 6a_0 a_1 a_5 \\
&\quad + 6a_0 a_2 a_4 + 6a_1 a_2 a_3 \\
A_7 &= 3a_0^2 a_7 + 3a_1^2 a_5 + 3a_2^2 a_3 + 3a_3^2 a_1 + 6a_0 a_1 a_6 \\
&\quad + 6a_0 a_2 a_5 + 6a_0 a_3 a_4 + 6a_1 a_2 a_4 \\
A_8 &= 3a_0^2 a_8 + 3a_1^2 a_6 + 3a_2^2 a_4 + 3a_3^2 a_2 + 3a_4^2 a_0 \\
&\quad - 6a_0 a_1 a_7 + 6a_0 a_2 a_6 + 6a_0 a_3 a_5 - 6a_1 a_2 a_5 + 6a_1 a_3 a_4 \\
A_9 &= a_3^3 + 3a_0^2 a_9 + 3a_1^2 a_7 + 3a_2^2 a_5 + 3a_3^2 a_1 + 6a_0 a_1 a_8 \\
&\quad + 6a_0 a_2 a_7 + 6a_0 a_3 a_6 + 6a_0 a_4 a_5 + 6a_1 a_2 a_6 \\
&\quad + 6a_1 a_3 a_5 + 6a_2 a_3 a_4 \\
A_{10} &= 3a_0^2 a_{10} + 3a_1^2 a_8 + 3a_2^2 a_6 + 3a_3^2 a_4 + 3a_4^2 a_2 + 3a_5^2 a_0 \\
&\quad + 6a_0 a_1 a_9 + 6a_0 a_2 a_8 + 6a_0 a_3 a_7 + 6a_0 a_4 a_6 \\
&\quad + 6a_1 a_2 a_7 + 6a_1 a_3 a_6 + 6a_1 a_4 a_5 - 6a_2 a_3 a_5
\end{aligned}$$

We can substitute the expressions for the A_n into the expressions for the a_n , but it is unnecessary, since it is more practical to numerically evaluate the A_n beforehand and then to determine the a_n . If one prefers to work with the final expressions for the a_n , they are:

$$a_0 = u(0)$$

$$a_1 = u'(0)$$

$$a_2 = (g_0 - \alpha(1)a_1 - \beta a_0 - \gamma a_0^3)/(1)(2)$$

$$a_3 = (g_1 - \alpha(2)a_2 - \beta a_1 - \gamma(3a_0^3 a_1))/(2)(3)$$

$$a_4 = (g_2 - \alpha(3)a_3 - \beta a_2 - \gamma(3a_0^2 a_2 + 3a_1^2 a_0))/(3)(4)$$

$$a_5 = (g_3 - \alpha(4)a_4 - \beta a_3 - \gamma(a_1^3 + 3a_0^2 a_3 + 6a_0 a_1 a_2))/(4)(5)$$

$$a_6 = (g_4 - \alpha(5)a_5 - \beta a_4 - \gamma(3a_0^2 a_4 + 3a_1^2 a_2 + 3a_2^2 a_0 + 6a_0 a_1 a_3))/(5)(6)$$

$$a_7 = (g_5 - \alpha(6)a_6 - \beta a_5 - \gamma(3a_0^2 a_5 + 3a_1^2 a_3 + 3a_2^2 a_1 + 6a_0 a_1 a_4 + 6a_0 a_2 a_3))/(6)(7)$$

$$a_8 = (g_6 - \alpha(7)a_7 - \beta a_6 - \gamma(a_3^2 + 3a_0^2 a_6 + 3a_1^2 a_4 + 3a_2^2 a_2 + 6a_0 a_1 a_5 + 6a_0 a_2 a_4 + 6a_1 a_2 a_3))/(7)(8)$$

$$a_9 = (g_7 - \alpha(8)a_8 - \beta a_7 - \gamma(3a_0^2 a_7 + 3a_1^2 a_5 + 3a_2^2 a_3 + 3a_3^2 a_1 + 6a_0 a_1 a_6 + 6a_0 a_2 a_5 + 6a_0 a_3 a_4 + 6a_1 a_2 a_4))/(8)(9)$$

$$a_{10} = (g_8 - \alpha(9)a_9 - \beta a_8 - \gamma(3a_0^2 a_8 + 3a_1^2 a_6 + 3a_2^2 a_4 + 3a_3^2 a_2 + 3a_4^2 a_0 + 6a_0 a_1 a_7 + 6a_0 a_2 a_6 + 6a_0 a_3 a_5 + 6a_1 a_2 a_5 + 6a_1 a_3 a_4))/(9)(10)$$

VARIABLE COEFFICIENT CASE OF THE DUFFING EQUATION:

Write

$$u'' + \left\{ \sum_{m=0}^{\infty} \alpha_m t^m \right\} u' + \left\{ \sum_{m=0}^{\infty} \beta_m t^m \sum_{m=0}^{\infty} \right\} u + \left\{ \sum_{m=0}^{\infty} \gamma_m t^m \right\} u^3 = \sum_{m=0}^{\infty} g_m t^m$$

$$u = \sum_{m=0}^{\infty} a_m t^m$$

$$u' = \sum_{m=0}^{\infty} (m+1) a_{m+1} t^m$$

$$u'' = \sum_{m=0}^{\infty} (m+1)(m+2) a_{m+2} t^m$$

$$u^3 = \sum_{m=0}^{\infty} A_m(a_0, \dots, a_m) t^m = \sum_{m=0}^{\infty} t^m \sum_{v=0}^m \sum_{\mu=0}^v a_{m-v} a_{v-\mu} a_{\mu}$$

$$u(0) = k_0 = a_0 = \sum_{m=0}^{\infty} a_m t^m \Big|_{t=0}$$

$$u'(0) = k_1 = a_1 = \sum_{m=0}^{\infty} (m+1) a_{m+1} t^m \Big|_{t=0}$$

$$\begin{aligned} \sum_{m=0}^{\infty} (m+1)(m+2)a_{m+2}t^m + \left[\sum_{m=0}^{\infty} \alpha_m t^m \right] \cdot \left[\sum_{m=0}^{\infty} (m+1)a_{m+1}t^m \right] \\ + \left[\sum_{m=0}^{\infty} \beta_m t^m \right] \cdot \left[\sum_{m=0}^{\infty} a_m t^m \right] + \left[\sum_{m=0}^{\infty} \gamma_m t^m \right] \\ \left[\sum_{m=0}^{\infty} A_m(a_0, \dots, a_m)t^m \right] = \sum_{m=0}^{\infty} g_m t^m \end{aligned}$$

Carrying out the indicated Cauchy products,

$$\begin{aligned} \sum_{m=0}^{\infty} (m+1)(m+2)a_{m+2}t^m + \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^m \alpha_{m-n}(n+1)a_{n+1} \right\} t^m + \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^m \beta_{m-n} a_n \right\} t^m \\ + \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^m \gamma_{m-n} A_n(a_0, \dots, a_n) \right\} t^m = \sum_{m=0}^{\infty} g_m t^m \end{aligned}$$

Consequently,

$$\begin{aligned} (m+1)(m+2)a_{m+2} + \sum_{n=0}^m \alpha_{m-n}(n+1)a_{n+1} \\ + \sum_{n=0}^m \beta_{m-n} a_n + \sum_{n=0}^m \gamma_{m-n} A_n(a_0, \dots, a_n) = g_m \end{aligned}$$

so recursion relations can be given

$$a_0 = u(0)$$

$$a_1 = u'(0)$$

$$a_{m+2} = \frac{g_m - \sum_{n=0}^m \{ \alpha_{m-n}(n+1)a_{n+1} + \beta_{m-n} a_n + \gamma_{m-n} A_n \}}{(m+1)(m+2)}$$

so that $u(t) = \sum_{m=0}^{\infty} a_m t^m$ is determined since the A_n are known. The computation is easily programmable. We can list the a_m as follows:

$$a_0 = u(0)$$

$$a_1 = u'(0)$$

$$a_2 = (g_2 - \alpha_2(1)a_1 - \beta_2 a_0 - \gamma_2 A_0)/(1)(2)$$

$$a_3 = (g_3 - \alpha_3(1)a_1 - \alpha_3(2)a_2 - \beta_3 a_0 - \beta_3 a_1 - \gamma_3 A_0 - \gamma_3 A_1)/(2)(3)$$

$$a_4 = (g_4 - \alpha_4(1)a_1 - \alpha_4(2)a_2 - \alpha_4(3)a_3 - \beta_4 a_0 - \beta_4 a_1 - \beta_4 a_2$$

$$- \gamma_4 A_0 - \gamma_4 A_1 - \gamma_4 A_2)/(3)(4)$$

$$a_5 = (g_5 - \alpha_5(1)a_1 - \alpha_5(2)a_2 - \alpha_5(3)a_3 - \alpha_5(4)a_4$$

$$- \beta_5 a_0 - \beta_5 a_1 - \beta_5 a_2 - \beta_5 a_3 - \gamma_5 A_0 - \gamma_5 A_1 - \gamma_5 A_2 - \gamma_5 A_3)/(4)(5)$$

$$a_6 = (g_6 - \alpha_6(1)a_1 - \alpha_6(2)a_2 - \alpha_6(3)a_3 - \alpha_6(4)a_4 - \alpha_6(5)a_5$$

$$- \beta_6 a_0 - \beta_6 a_1 - \beta_6 a_2 - \beta_6 a_3 - \beta_6 a_4$$

$$- \gamma_6 A_0 - \gamma_6 A_1 - \gamma_6 A_2 - \gamma_6 A_3 - \gamma_6 A_4)/(5)(6)$$

$$a_7 = (g_7 - \alpha_7(1)a_1 - \alpha_7(2)a_2 - \alpha_7(3)a_3 - \alpha_7(4)a_4 - \alpha_7(5)a_5 - \alpha_7(6)a_6$$

$$- \beta_7 a_0 - \beta_7 a_1 - \beta_7 a_2 - \beta_7 a_3 - \beta_7 a_4 - \beta_7 a_5$$

$$- \gamma_7 A_0 - \gamma_7 A_1 - \gamma_7 A_2 - \gamma_7 A_3 - \gamma_7 A_4 - \gamma_7 A_5)/(6)(7)$$

$$a_8 = (g_8 - \alpha_8(1)a_1 - \alpha_8(2)a_2 - \alpha_8(3)a_3 - \alpha_8(4)a_4 - \alpha_8(5)a_5 - \alpha_8(6)a_6$$

$$- \alpha_8(7)a_7 - \beta_8 a_0 - \beta_8 a_1 - \beta_8 a_2 - \beta_8 a_3 - \beta_8 a_4 - \beta_8 a_5 - \beta_8 a_6$$

$$- \gamma_8 A_0 - \gamma_8 A_1 - \gamma_8 A_2 - \gamma_8 A_3 - \gamma_8 A_4 - \gamma_8 A_5 - \gamma_8 A_6)/(7)(8)$$

$$a_9 = (g_9 - \alpha_9(1)a_1 - \alpha_9(2)a_2 - \alpha_9(3)a_3 - \alpha_9(4)a_4 - \alpha_9(5)a_5 - \alpha_9(6)a_6$$

$$- \alpha_9(7)a_7 - \alpha_9(8)a_8 - \beta_9 a_0 - \beta_9 a_1 - \beta_9 a_2 - \beta_9 a_3 - \beta_9 a_4$$

$$- \beta_9 a_5 - \beta_9 a_6 - \beta_9 a_7 - \gamma_9 A_0 - \gamma_9 A_1 - \gamma_9 A_2 - \gamma_9 A_3 - \gamma_9 A_4$$

$$- \gamma_9 A_5 - \gamma_9 A_6 - \gamma_9 A_7)/(8)(9)$$

$$a_{10} = (g_{10} - \alpha_{10}(1)a_1 - \alpha_{10}(2)a_2 - \alpha_{10}(3)a_3 - \alpha_{10}(4)a_4 - \alpha_{10}(5)a_5 - \alpha_{10}(6)a_6$$

$$- \alpha_{10}(7)a_7 - \alpha_{10}(8)a_8 - \alpha_{10}(9)a_9 - \beta_{10} a_0 - \beta_{10} a_1 - \beta_{10} a_2 - \beta_{10} a_3$$

$$- \beta_{10} a_4 - \beta_{10} a_5 - \beta_{10} a_6 - \beta_{10} a_7 - \beta_{10} a_8 - \gamma_{10} A_0 - \gamma_{10} A_1 - \gamma_{10} A_2 - \gamma_{10} A_3$$

$$- \gamma_{10} A_4 - \gamma_{10} A_5 - \gamma_{10} A_6 - \gamma_{10} A_7 - \gamma_{10} A_8)/(9)(10)$$

⋮

$$u(t) = \sum_{m=0}^{\infty} a_m t^m$$

$$\phi_1 = a_0$$

$$\phi_2 = a_0 + a_1 t$$

$$\phi_{m+1} = \sum_{n=0}^m a_n t^n$$

REMARK: Possible areas of further investigation include regions of convergence, numerical algorithms for computation, application of convergence acceleration transforms*, and stochastic versions of the Duffing equation where we solve for first- and second-order statistics of u . We can, of course, consider special cases of our solution such as

- (i) $u(0) = k$, and $u'(0) = 0$ with $g = 0$
- (ii) $u(0) = 0$ and $g = 0$ with $u'(0) = k$
- (iii) $u(0) = 0$ and $u'(0) = 0$ with $g = g_0$, a constant
- (iv) $u(0) = u'(0) = 0$, $g = g_0 + g_1 t$ or $g = g_0 + g_1 t + g_2 t^2$
- (v) $u(0) = k_0$, $u'(0) = k_1$, $g = \sum_{n=0}^{\infty} g_n t^n$

We might then use double decomposition and write

$$\begin{aligned} u_0 &= \sum_{n=0}^{\infty} u_{0,n} = c_{0,n} \\ &\vdots \\ u_m &= \sum_{n=0}^{\infty} u_{m,n} \end{aligned}$$

with

$$\begin{aligned} \phi_1 &= u_0 = \sum_{n=0}^{\infty} u_{0,n} \\ \phi_2 &= \phi_1 + u_1 = \phi_1 + \sum_{n=0}^{\infty} u_{1,n} \\ &\vdots \end{aligned}$$

We emphasize that these problems are also solvable by usual decomposition. Also, we note that the rate of convergence of the modified decomposition only approaches that of decomposition when the excitation approaches zero. The reason for this is that the initial term contains only the first term of the series for g ; only when we go to sufficient terms of u will we have enough of the input to have as good an approximation.

* Such as Padé approximants, Shanks and Wynn transforms, and the Euler and Van Wijngaarden transforms.

APPLICATION TO LINEAR PARTIAL DIFFERENTIAL EQUATIONS:

Suppose we begin with the equation $L_x u + L_y u = 0$ and, to be specific, choose $L_x = \partial^2 / \partial x^2$ and $L_y = \partial^2 / \partial y^2$. Following the decomposition procedure, we write the equation for the x partial solution :

$$u = \Phi_x - L_x^{-1} L_y u$$

where $\Phi_x = \xi_0(y) + x\xi_1(y)$ must be found from the given boundary conditions and L_x^{-1} is defined as the two-fold (indefinite) integral $\iint (\cdot) dx dx$. It has been demonstrated previously that solutions are easily determined by decomposition. Now, however, we use the modified decomposition method which we have discussed for ordinary differential equations. Thus we let

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n} x^m y^n$$

or

$$u = \sum_{m=0}^{\infty} a_m(y) x^m$$

where

$$a_m(y) = \sum_{n=0}^{\infty} c_{m,n} y^n$$

We now have

$$\begin{aligned} \sum_{m=0}^{\infty} a_m(y) x^m &= \xi_0(y) + x\xi_1(y) - \iint (\partial^2 / \partial y^2) \sum_{m=0}^{\infty} a_m(y) x^m dx dx \\ \sum_{m=0}^{\infty} a_m(y) x^m &= \xi_0(y) + x\xi_1(y) - \sum_{m=0}^{\infty} \frac{x^{m+2}}{(m+1)(m+2)} \frac{\partial^2}{\partial y^2} a_m(y) \\ \sum_{m=0}^{\infty} a_m(y) x^m &= \xi_0(y) + x\xi_1(y) - \sum_{m=2}^{\infty} \frac{x^m}{m(m-1)} \frac{\partial^2}{\partial y^2} a_{m-2}(y) \end{aligned}$$

The coefficients are identified by

$$a_0(y) = \xi_0(y)$$

$$a_1(y) = \xi_1(y)$$

and for $m \geq 2$ by the recurrence relation

$$a_m(y) = -\partial^2/\partial y^2 a_{m-2}(y)/m(m-1)$$

We can equally well consider the y partial solution by writing

$$u = \Phi_y - L_y^{-1} L_x u$$

where $\Phi_y = \eta_0(x) + y\eta_1(x)$ and $u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n} x^m y^n$

$$u = \sum_{n=0}^{\infty} b_n(x) y^n$$

where $b_n(x) = \sum_{m=0}^{\infty} c_{m,n} x^m$ and $L_y^{-1} = \iint (\cdot) dy dy$ is an indefinite integration operator. Now

$$\begin{aligned} \sum_{n=2}^{\infty} b_n(x) y^n &= \eta_0(x) + y\eta_1(x) - \iint (\partial^2/\partial x^2) \sum_{n=0}^{\infty} b_n(x) y^n dy dy \\ &= \eta_0(x) + y\eta_1(x) \sum_{n=0}^{\infty} \frac{y^{n+2}}{(n+1)(n+2)} \frac{\partial^2}{\partial x^2} b_n(x) \\ &= \eta_0(x) + y\eta_1(x) \sum_{n=2}^{\infty} \frac{y^{n-2}}{n(n-1)} \frac{\partial^2}{\partial x^2} b_{n-2}(x) \end{aligned}$$

We get immediately

$$b_0(x) = \eta_0(x)$$

$$b_1(x) = \eta_1(x)$$

and for $n \geq 2$,

$$b_n(x) = (-\partial^2/\partial x^2) b_{n-2}(x)/n(n-1)$$

We now consider the more general linear form

$$L_x u + L_y u + Ru = 0$$

again with $L_x = \partial^2/\partial x^2$ and $L_y = \partial^2/\partial y^2$. The x partial solution is given by

$$u = \Phi_x - L_x^{-1} L_y u - L_x^{-1} Ru$$

For simplicity, we choose $R = \rho$.

$$\Phi_x = \xi_0(y) + x\xi_1(y)$$

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n} x^m y^n$$

$$u = \sum_{m=0}^{\infty} a_m(y) x^m$$

where $a_m(y) = \sum_{n=0}^{\infty} c_{m,n} y^n$ and $L_x^{-1} = \iint (\cdot) dx dx$ is an indefinite integration operator. Now

$$\begin{aligned} \sum_{m=0}^{\infty} a_m(y) x^m &= \xi_0(y) + x\xi_1(y) \\ &\quad - \iint (\partial^2 / \partial y^2) \sum_{m=0}^{\infty} a_m(y) x^m dx dx \\ &\quad - \iint \rho \sum_{m=0}^{\infty} a_m(y) dx dx \\ \sum_{m=0}^{\infty} a_m(y) x^m &= \xi_0(y) + x\xi_1(y) \\ &\quad - \sum_{m=0}^{\infty} \frac{x^{m+2}}{(m+1)(m+2)} \frac{\partial^2}{\partial y^2} a_m(y) \\ &\quad - \sum_{m=0}^{\infty} \rho \frac{x^{m+2}}{(m+1)(m+2)} a_m(y) \\ \sum_{m=0}^{\infty} a_m(y) x^m &= \xi_0(y) + x\xi_1(y) \\ &\quad - \sum_{m=2}^{\infty} \frac{x^m}{m(m-1)} \frac{\partial^2}{\partial y^2} a_{m-2}(y) \\ &\quad - \sum_{m=2}^{\infty} \rho \frac{x^m}{m(m-1)} a_{m-2}(y) \end{aligned}$$

so that

$$a_0(y) = \xi_0(y)$$

$$a_1(y) = \xi_1(y)$$

and for $m \geq 2$

$$a_m(y) = \frac{(-\partial^2 / \partial y^2) a_{m-2}(y) - \rho a_{m-2}(y)}{m(m-1)}$$

The y partial solution follows similarly:

$$b_0(x) = \eta_0(x)$$

$$b_1(x) = \eta_1(x)$$

and for $n \geq 2$

$$b_n(x) = \frac{(-\partial^2 / \partial y^2) b_{n-2}(x) - \rho b_{n-2}(x)}{n(n-1)}$$

We consider $L_x u + L_y u + Ru = 0$. Let $R = \rho(x, y)$ with L_x and L_y as before.

$$\rho(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho_{m,n} x^m y^n$$

$$\rho(x, y) = \sum_{m=0}^{\infty} \rho_m(y) x^m$$

where

$$\rho_m(y) = \sum_{n=0}^{\infty} \rho_{m,n} y^n$$

$$u = \Phi_x - L_x^{-1} L_y u - L_x^{-1} Ru$$

$$\Phi_x = \xi_0(y) + x \xi_1(y)$$

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n} x^m y^n$$

$$u = \sum_{m=0}^{\infty} a_m(y) x^m$$

where

$$a_m(y) = \sum_{n=0}^{\infty} c_{m,n} y^n$$

Now

$$\begin{aligned} \sum_{m=0}^{\infty} a_m(y) x^m &= \xi_0(y) + x \xi_1(y) \\ &\quad - \iint (\partial^2 / \partial y^2) \sum_{m=0}^{\infty} a_m(y) x^m \, dx \, dx \\ &\quad - \iint \left\{ \sum_{m=0}^{\infty} \rho_m(y) x^m \right\} \left\{ \sum_{m=0}^{\infty} a_m(y) x^m \right\} dx \, dx \end{aligned}$$

We rewrite the bracketed quantities

$$\left\{ \sum_{m=0}^{\infty} \rho_m(y) x^m \right\} \left\{ \sum_{m=0}^{\infty} a_m(y) x^m \right\} = \sum_{m=0}^{\infty} x^m \sum_{\mu=0}^{\infty} \rho_{\mu}(y) a_{m-\mu}(y)$$

Now

$$\begin{aligned} \sum_{m=0}^{\infty} a_m(y) x^m &= \xi_0(y) + x \xi_1(y) \\ &\quad - \sum_{m=0}^{\infty} \frac{x^{m+2}}{(m+1)(m+2)} \frac{\partial^2}{\partial y^2} a_m(y) \\ &\quad - \sum_{m=0}^{\infty} \frac{x^{m+2}}{(m+1)(m+2)} \sum_{\mu=0}^m \rho_{\mu}(y) a_{m-\mu}(y) \\ \sum_{m=0}^{\infty} a_m(y) x^m &= \xi_0(y) + x \xi_1(y) \\ &\quad - \sum_{m=2}^{\infty} \frac{x^m}{m(m-1)} \frac{\partial^2}{\partial y^2} a_{m-2}(y) \\ &\quad - \sum_{m=2}^{\infty} \frac{x^m}{m(m-1)} \sum_{\mu=0}^{m-2} \rho_{\mu}(y) a_{m-2-\mu}(y) \end{aligned}$$

so that

$$a_0(y) = \xi_0(y)$$

$$a_1(y) = \xi_1(y)$$

and for $m \geq 2$

$$a_m(y) = \frac{(-\partial^2 / \partial y^2) a_{m-2}(y) - \sum_{\mu=0}^{m-2} \rho_{\mu}(y) a_{m-2-\mu}(y)}{m(m-1)}$$

The y partial solution follows similarly

$$b_0(x) = \eta_0(x)$$

$$b_i(x) = \eta_i(x)$$

and for $n \geq 2$

$$b_n(x) = \frac{(-\partial^2 / \partial x^2) b_{n-2}(x) - \sum_{v=0}^{n-2} \rho_v(x) b_{n-2-v}(x)}{n(n-1)}$$

We have used

$$R = \rho(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho_{m,n} x^m y^n = \sum_{n=0}^{\infty} \rho_n(x) y^n$$

where

$$\rho_n(x) = \sum_{m=0}^{\infty} \rho_{m,n} x^m$$

Also, $u = \sum_{n=0}^{\infty} b_n(x) y^n$ with $b_n(x) = \sum_{m=0}^{\infty} c_{m,n} x^m$ and

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n} x^m y^n$$

APPLICATION TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS:

Consider $L_x u + L_y u + Nu = 0$ and the x partial solution with $L_x = \partial^2 / \partial x^2$, $L_y = \partial^2 / \partial y^2$, and $L_x^{-1} = \iint (\cdot) dx dx$. We let $Nu = \alpha f(u)$. (We have shown algorithms for considering functions such as $f(u, v)$ or $f(u, v, w)$ in Chapter 3.) We have now

$$u = \Phi_x - L_x^{-1} L_y u - L_x^{-1} Nu$$

where

$$\Phi_x = \xi_0(y) + x \xi_1(y).$$

Let

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n} x^m y^n \quad u = \sum_{n=0}^{\infty} a_n(y) x^n$$

or

$$a_m(y) = \sum_{n=0}^{\infty} c_{m,n} y^n$$

We have generally written $f(u) = \sum_{n=0}^{\infty} A_n$; however, we showed in the

previous results on the transformation of series that if $u = \sum_{m=0}^{\infty} a_m x^m$ we can then write

$$f(u) = \sum_{m=0}^{\infty} x^m A_m(y)$$

where $A_m(y) = A_m(a_0(y), \dots, a_m(y))$ and we do so now. We now have

$$\begin{aligned} \sum_{m=0}^{\infty} a_m(y) x^m &= \xi_0(y) + x \xi_1(y) \\ &\quad - \iint (\partial^2 / \partial y^2) \sum_{m=0}^{\infty} a_m(y) x^m dx dx \\ &\quad - \iint \alpha \sum_{m=0}^{\infty} x^m A_m(y) dx dx \\ \sum_{m=0}^{\infty} a_m(y) x^m &= \xi_0(y) + x \xi_1(y) \\ &\quad - \sum_{m=0}^{\infty} \frac{x^{m+2}}{(m+1)(m+2)} \frac{\partial^2}{\partial y^2} a_m(y) \\ &\quad - \sum_{m=0}^{\infty} \alpha \frac{x^{m+2}}{(m+1)(m+2)} A_m(y) \\ \sum_{m=0}^{\infty} a_m(y) x^m &= \xi_0(y) + x \xi_1(y) \\ &\quad - \sum_{m=2}^{\infty} \frac{x^m}{m(m-1)} \frac{\partial^2}{\partial y^2} a_{m-2}(y) \\ &\quad - \sum_{m=2}^{\infty} \alpha \frac{x^m}{m(m-1)} A_{m-2}(y) \end{aligned}$$

so that

$$a_0(y) = \xi_0(y) \quad a_1(y) = \xi_1(y)$$

and for $m \geq 2$

$$a_m(y) = \frac{(-\partial^2 / \partial y^2) a_{m-2}(y) - \alpha A_{m-2}(y)}{m(m-1)}$$

The y partial solution is similarly obtained, letting $u = \sum_{n=0}^{\infty} b_n(x) y^n$ where $b_n(x) = \sum_{m=0}^{\infty} c_{m,n} x^m$ or $u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n} x^m y^n$. Let

$$f(u) = \sum_{n=0}^{\infty} y^n B_n(x)$$

with $B_n(x) = B_n(b_0(x), \dots, b_n(x))$ where we have now used B_n instead of the usual A_n only to distinguish it from the previous set of polynomials.

GENERAL INHOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS:

Consider the linear case:

$$L_x u + L_y u + Ru = g$$

$$u = \sum_{m=0}^{\infty} a_m(y) x^m$$

$$a_0(y) = \xi_0(y)$$

$$a_1(y) = \xi_1(y)$$

and for $m \geq 2$

$$a_m(y) = \frac{\gamma_{m-2}(y) - (\partial^2 / \partial y^2) a_{m-2}(y) - \sum_{\mu=0}^{m-2} \rho_{\mu}(y) a_{m-2-\mu}(y)}{m(m-1)}$$

where

$$R = \rho(x, y) = \sum_{m=0}^{\infty} \rho_m(y) x^m$$

$$\rho_m(y) = \sum_{n=0}^{\infty} \rho_{m,n} y^n$$

$$g = g(x, y) = \sum_{m=0}^{\infty} \gamma_m(y) x^m$$

$$\gamma_m(y) = \sum_{n=0}^{\infty} g_{m,n} y^n$$

$$\Phi_x = \xi_0(y) + x \xi_1(y)$$

Now the nonlinear case:

$$L_x u + L_y u + Ru - Nu = g$$

$$u = \sum_{m=0}^{\infty} a_m(y) x^m$$

$$Nu = \alpha(x, y) f(u)$$

$$a_0(y) = \xi_0(y)$$

$$a_1(y) = \xi_1(y)$$

and for $m \geq 2$

$$a_m(y) = \left\{ \gamma_{m-2}(y) - (\partial^2 / \partial y^2) a_{m-2}(y) - \sum_{\mu=0}^{m-2} \rho_\mu(y) a_{m-2-\mu}(y) - \sum_{\mu=0}^{m-2} \alpha_\mu(y) A_{m-2-\mu}(y) \right\} / m(m-1)$$

where

$$A_m(y) = A_m(a_0(y), \dots, a_m(y))$$

and

$$\alpha = \alpha(x, y) = \sum_{m=0}^{\infty} \alpha_m(y) x^m$$

The solution is

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} t^m x^n$$

Define

$$a_{0,n} = c_n^{(0)}$$

$$a_{1,n} = c_n^{(1)}$$

$$a_{m+2,n} = \frac{\gamma_{m,n} - \alpha a_{m,n} - \beta(n+1)(n+2)a_{m,n+2}}{(m+1)(m+2)}$$

Consider the example

$$\frac{\partial^2 u}{\partial t^2} + \alpha u + \beta \frac{\partial^2 u}{\partial x^2} = \gamma(t, x)$$

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{m,n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} t^m x^n$$

$$u_{m,n} = a_{m,n} t^m x^n$$

$$\frac{\partial^2 u}{\partial t^2} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+1)(m+2) a_{m+2,n} t^m x^n$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (n+1)(n+2) a_{m,n+2} t^m x^n$$

$$\gamma(t, x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \gamma_{m,n} t^m x^n$$

$$u(0, x) = \delta(x) = \sum_{n=0}^{\infty} \delta_n x^n$$

$$\frac{\partial u}{\partial t}(0, x) = \epsilon(x) = \sum_{n=0}^{\infty} \epsilon_n x^n$$

The solution is

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} t^m x^n$$

$$a_{0,n} = \delta_n$$

$$a_{1,n} = \epsilon_n$$

$$a_{m+2,n} = \frac{\gamma_{m,n} - \alpha a_{m,n} - \beta(n+1)(n+2)a_{m,n-2}}{(m+1)(m+2)}$$

The algorithm has a duality property since we can calculate either the t partial solution as we have or the x partial solution. Let's take $\beta = 1$ to write

$$a_{m+2,n} = \frac{\gamma_{m,n} - \alpha a_{m,n} - (n+1)(n+2)a_{m,n-2}}{(m+1)(m+2)}$$

for the x partial solution. If we have a boundary-value problem, we use the double decomposition technique for a few terms to get a good initial value. Then rearrange and solve as an initial-value problem.

Summarizing, the t -coordinate partial solution—we might call it the temporal format—is

$$u = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} a_{m,n} x^n \right) t^m = \sum_{m=0}^{\infty} a_m(x) t^m$$

with the μ th approximant

$$\phi_{\mu} = \sum_{m=0}^{\mu-1} \left(\sum_{n=0}^{\infty} a_{m,n} x^n \right) t^m = \sum_{m=0}^{\mu-1} a_m(x) t^m$$

while the spatial format of the solution (or x -coordinate partial solution) is

$$u = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} a_{m,n} t^m \right) x^n = \sum_{n=0}^{\infty} a_n(t) x^n$$

with the v th approximant

$$\phi_v = \sum_{n=0}^{v-1} \left(\sum_{m=0}^{\infty} a_{m,n} t^m \right) x^n = \sum_{n=0}^{v-1} a_n(t) x^n$$

Each sequence of coefficients in the two formats, i.e., the $a_m(x)$ and the $a_n(t)$ has its own radius of convergence, and moreover, the solution itself is unique. As another example of a linear partial differential equation, consider

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} + \beta u + \gamma \frac{\partial u}{\partial x} - \delta \frac{\partial^2 u}{\partial x^2} &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon_{m,n} t^m x^n \\ u(0, x) &= \sum_{n=0}^{\infty} \rho_n x^n \\ \frac{\partial u(0, x)}{\partial t} &= \sum_{n=0}^{\infty} \sigma_n x^n & \alpha, \beta, \gamma, \delta, \varepsilon \text{ constant} \\ u &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} t^m x^n \\ a_{0,n} &= \rho_n \\ a_{1,n} &= \sigma_n \\ a_{m+2,n} &= \frac{\varepsilon_{m,n} - \alpha(m+1)a_{m-1,n} - \beta a_{m,n} - \gamma(n+1)a_{m,n+1} - \delta(n+1)(n+2)a_{m,n-2}}{(m+1)(m+2)} \end{aligned}$$

We can now write the approximants $\phi_{m,n} = \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{n-1} a_{\mu,\nu} t^\mu x^\nu$.

LINEAR PARTIAL DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} + \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} t^m x^n \right\} \frac{\partial u}{\partial t} + \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta_{m,n} t^m x^n \right\} u + \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \gamma_{m,n} t^m x^n \right\} \frac{\partial u}{\partial x} \\ + \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \delta_{m,n} t^m x^n \right\} \frac{\partial^2 u}{\partial x^2} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon_{m,n} t^m x^n \\ u(0, x) = \sum_{n=0}^{\infty} \rho_n x^n \\ \frac{\partial u(0, x)}{\partial t} = \sum_{n=0}^{\infty} \sigma_n x^n \end{aligned}$$

Using modified decomposition we write

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} t^m x^n$$

LINEAR PARTIAL DIFFERENTIAL EQUATION IN TWO SPATIAL DIMENSIONS AND ONE TEMPORAL DIMENSION:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} + \beta u + \gamma \frac{\partial u}{\partial x} + \delta \frac{\partial u}{\partial y} + \zeta \frac{\partial^2 u}{\partial x^2} + \eta \frac{\partial^2 u}{\partial y^2} \\ = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon_{\ell,m,n} t^{\ell} x^m y^n \\ u(0, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho_{m,n} x^m y^n \\ \frac{\partial}{\partial t} u(0, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sigma_{m,n} x^m y^n \end{aligned}$$

The solution by modified decomposition is

$$\begin{aligned} u &= \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{\ell,m,n} t^{\ell} x^m y^n \\ a_{0,m,n} &= \rho_{m,n} \\ a_{1,m,n} &= \sigma_{m,n} \\ a_{\ell+2,m,n} &= \left\{ \varepsilon_{\ell,m,n} - \alpha(\ell+1)a_{\ell+1,m,n} \right. \\ &\quad - \beta a_{\ell,m,n} - \gamma(m+1)a_{\ell,m-1,n} \\ &\quad - \delta(n+1)a_{\ell,m,n+1} \\ &\quad - \zeta(m+1)(m+2)a_{\ell,m-2,n} \\ &\quad \left. - \eta(n+1)(n+2)a_{\ell,m,n-2} \right\} / \{(\ell+1)(\ell+2)\} \end{aligned}$$

LINEAR PARTIAL DIFFERENTIAL EQUATION IN THREE SPATIAL DIMENSIONS AND ONE TEMPORAL DIMENSION:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} + \beta u + \gamma \frac{\partial u}{\partial x} + \delta \frac{\partial u}{\partial y} + \xi \frac{\partial u}{\partial z} \\ - \eta \frac{\partial^2 u}{\partial x^2} + \zeta \frac{\partial^2 u}{\partial y^2} + \omega \frac{\partial^2 u}{\partial z^2} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon_{k,\ell,m,n} t^k x^{\ell} y^m z^n \end{aligned}$$

$$u(0, x, y, z) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho_{\ell, m, n} x^{\ell} y^m z^n$$

$$\frac{\partial u}{\partial t}(0, x, y, z) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sigma_{\ell, m, n} x^{\ell} y^m z^n$$

Solution by modified decomposition:

$$u = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{k, \ell, m, n} t^k x^{\ell} y^m z^n$$

$$a_{0, \ell, m, n} = \rho_{\ell, m, n}$$

$$a_{1, \ell, m, n} = \sigma_{\ell, m, n}$$

$$a_{k+2, \ell, m, n} = \left\{ \varepsilon_{k, \ell, m, n} - \alpha(k+1)a_{k+1, \ell, m, n} \right. \\
- \beta a_{k, \ell, m, n} - \gamma(\ell+1)a_{k, \ell+1, m, n} - \delta(m+1)a_{k, \ell, m-1, n} \\
- \zeta(n+1)a_{k, \ell, m, n+1} - \eta(\ell+1)(\ell+2)a_{k, \ell-2, m, n} \\
- \zeta(m+1)(m+2)a_{k, \ell, m+2, n} \\
\left. - \omega(n+1)(n+2)a_{k, \ell, m, n+2} \right\} / \{(k+1)(k+2)\}$$

NONLINEAR PARTIAL DIFFERENTIAL EQUATION:

$$\frac{\partial^2 u}{\partial t^2} \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} + \beta u + \gamma \frac{\partial u}{\partial x} + \delta \frac{\partial^2 u}{\partial x^2} + \psi u^2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon_{m, n} t^m x^n$$

$$u(0, x) = \sum_{n=0}^{\infty} \rho_n x^n$$

$$\frac{\partial u(0, x)}{\partial t} = \sum_{n=0}^{\infty} \sigma_n x^n$$

Assuming constant coefficients, write

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m, n} t^m x^n$$

$$u^2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m, n} t^m x^n$$

where the $A_{m, n}$ are our polynomials.

$$a_{0,n} = \rho_n$$

$$a_{1,n} = \sigma_n$$

$$a_{m+2,n} = \left\{ \varepsilon_{m,n} - \alpha(m+1)a_{m+1,n} - \beta a_{m,n} - \gamma(n+1)a_{m,n+1} \right. \\ \left. - \delta(n+1)(n+2)a_{m,n+2} - \psi A_{m,n} \right\} / (m+1)(m+2)$$

The approximants will be

$$\phi_{m,n} = \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{n-1} a_{\mu,\nu} t^{\mu} x^{\nu}$$

NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} + \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} t^m x^n \right\} \frac{\partial u}{\partial t} + \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta_{m,n} t^m x^n \right\} u + \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \gamma_{m,n} t^m x^n \right\} \frac{\partial u}{\partial x} \\ + \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \delta_{m,n} t^m x^n \right\} \frac{\partial^2 u}{\partial x^2} + \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \psi_{m,n} t^m x^n \right\} u^2 \\ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon_{m,n} t^m x^n \end{aligned}$$

$$u(0, x) = \sum_{n=0}^{\infty} \rho_n x^n$$

$$\frac{\partial}{\partial t} u(0, x) = \sum_{n=0}^{\infty} \sigma_n x^n$$

For the solution by modified decomposition, we write

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} t^m x^n$$

$$u^2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n} t^m x^n$$

where the $A_{m,n}$ are our polynomials.

$$a_{0,n} = \rho_n$$

$$a_{1,n} = \sigma_n$$

$$\begin{aligned}
a_{m+2,n} = & \left\{ \varepsilon_{m,n} - \sum_{\mu=0}^m \sum_{v=0}^n \alpha_{m-\mu,n-v} (\mu+1) a_{\mu+1,v} \right. \\
& - \sum_{\mu=0}^m \sum_{v=0}^n \beta_{m-\mu,n-v} a_{\mu,v} - \sum_{\mu=0}^m \sum_{v=0}^n \gamma_{m-\mu,n-v} (v+1) a_{\mu,v+1} \\
& - \sum_{\mu=0}^m \sum_{v=0}^n \delta_{m-\mu,n-v} (v+1)(v+2) a_{\mu,v-2} \\
& \left. - \sum_{\mu=0}^m \sum_{v=0}^n \psi_{m-\mu,n-v} A_{\mu,v} \right\} / \{(m+1)(m+2)\}
\end{aligned}$$

APPLICATION TO COUPLED PARTIAL DIFFERENTIAL EQUATIONS:

Consider the coupled partial differential equations

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} + \alpha v &= \beta(x, t) \\
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial t^2} + \gamma u &= \delta(x, t)
\end{aligned}$$

with (uncoupled) boundary conditions**

$$\begin{aligned}
u(x, 0) &= \xi_1(t) & u(x, 0) &= \tau_1(x) \\
u(x_2, t) &= \xi_2(t) & \frac{\partial u(x, 0)}{\partial t} &= \tau_2(x) \\
v(x, t) &= \eta_1(t) & v(x, 0) &= \sigma_1(x) \\
v(x_2, t) &= \eta_2(t) & \frac{\partial v(x, 0)}{\partial t} &= \sigma_2(x)
\end{aligned}$$

We will use the spatial format, hence the initial conditions are not used in this example. We assume α and γ are constants and

$$\begin{aligned}
\beta(x, t) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta_{m,n} x^m t^n \\
\delta(x, t) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \delta_{m,n} x^m t^n
\end{aligned}$$

** Note that the case of coupled boundary equations is solvable.

$$\xi_1(t) = \sum_{n=0}^{\infty} \xi_n^{(1)} t^n$$

$$\xi_2(t) = \sum_{n=0}^{\infty} \xi_n^{(2)} t^n$$

$$\eta_1(t) = \sum_{n=0}^{\infty} \eta_n^{(1)} t^n$$

$$\eta_2(t) = \sum_{n=0}^{\infty} \eta_n^{(2)} t^n$$

Define

$$L(\cdot) = \frac{\partial^2}{\partial x^2} (\cdot)$$

$$L^{-1}(\cdot) = A(t) + xB(t) + \iint (\cdot) dx dx$$

for the u solution and $C(t) + D(t) + \iint (\cdot) dx dx$ for the v solution. (We point out also that we can use double decomposition and recast this as an initial-value or temporal format problem to accelerate convergence.) We begin with

$$Lu = \beta(x, t) - (\partial^2 / \partial t^2) u - \alpha v$$

$$Lv = \delta(x, t) - (\partial^2 / \partial t^2) v - \gamma u$$

or

$$u = u_0 - L^{-1}(\partial^2 / \partial t^2) u - L^{-1} \alpha v$$

$$v = v_0 - L^{-1}(\partial^2 / \partial t^2) v - L^{-1} \gamma u$$

with

$$u_0 = A_0(t) + xB_0(t) + \iint \beta(x, t) dx dx$$

$$v_0 = C_0(t) + xD_0(t) + \iint \delta(x, t) dx dx$$

Now we can write the decompositions $u = \sum_{\ell=0}^{\infty} u_{\ell}$ and $v = \sum_{\ell=0}^{\infty} v_{\ell}$ and the approximants

$$\phi_{\lambda+1}\{u\} = \sum_{\ell=0}^{\lambda} u_{\ell}$$

$$\phi_{\lambda+1}\{v\} = \sum_{\ell=0}^{\lambda} v_{\ell}$$

The general decomposition components u_ℓ and v_ℓ are:

$$\begin{aligned} u_\ell &= A_\ell(t) + xB_\ell(t) - \iint (\partial^2/\partial t^2) u_{\ell-1} dx dx \\ &\quad - \iint \alpha v_{\ell-1} dx dx \\ v_\ell &= C_\ell(t) + xD_\ell(t) - \iint (\partial^2/\partial t^2) v_{\ell-1} dx dx \\ &\quad - \iint \gamma u_{\ell-1} dx dx \end{aligned}$$

We have, of course,

$$\begin{aligned} \phi_1\{u\} &= u_0 & \phi_1\{v\} &= v_0 \\ \phi_{\lambda+1}\{u\} &= \sum_{\ell=0}^{\lambda} u_\ell & \phi_{\lambda+1}\{v\} &= \sum_{\ell=0}^{\lambda} v_\ell \\ u &= \sum_{\ell=0}^{\infty} u_\ell & v &= \sum_{\ell=0}^{\infty} v_\ell \end{aligned}$$

The approximants must satisfy the boundary conditions; hence

$$\begin{aligned} \phi_1\{u(x_1, t)\} &= \xi_1(t) \\ \phi_1\{u(x_2, t)\} &= \xi_2(t) \\ \phi_1\{v(x_1, t)\} &= \eta_1(t) \\ \phi_1\{v(x_2, t)\} &= \eta_2(t) \end{aligned}$$

which implies

$$\begin{aligned} u_0(x_1, t) &= \xi_1(t) \\ u_0(x_2, t) &= \xi_2(t) \\ v_0(x_1, t) &= \eta_1(t) \\ v_0(x_2, t) &= \eta_2(t) \end{aligned}$$

Also

$$\begin{aligned} \phi_\lambda\{u(x_1, t)\} &= \xi_1(t) & \phi_{\lambda+1}\{u(x_1, t)\} &= \xi_1(t) \\ \phi_\lambda\{u(x_2, t)\} &= \xi_2(t) & \phi_{\lambda+1}\{u(x_2, t)\} &= \xi_2(t) \\ \phi_\lambda\{v(x_1, t)\} &= \eta_1(t) & \phi_{\lambda+1}\{v(x_1, t)\} &= \eta_1(t) \\ \phi_\lambda\{v(x_2, t)\} &= \eta_2(t) & \phi_{\lambda+1}\{v(x_2, t)\} &= \eta_2(t) \end{aligned}$$

Since

$$\phi_{\lambda+1}\{u\} = \phi_{\lambda}\{u\} + u_{\lambda}$$

$$\phi_{\lambda+1}\{v\} = \phi_{\lambda}\{v\} + v_{\lambda}$$

we clearly have

$$u_m(x_1, t) = u_m(x_2, t) = 0$$

$$v_m(x_1, t) = v_m(x_2, t) = 0$$

Summarizing,

$$u_0(x_1, t) = \xi_1(t) \quad u_0(x_2, t) = \xi_2(t)$$

$$v_0(x_1, t) = \eta_1(t) \quad v_0(x_2, t) = \eta_2(t)$$

and for $m \geq 1$, $u_m(x_1, t) = u_m(x_2, t) = v_m(x_1, t) = v_m(x_2, t) = 0$. Write

$$A_0(t) + x_1 B_0(t) = \xi_1^{(0)}(t)$$

$$A_0(t) + x_2 B_0(t) = \xi_2^{(0)}(t)$$

$$C_0(t) + x_1 D_0(t) = \eta_1^{(0)}(t)$$

$$C_0(t) + x_2 D_0(t) = \eta_2^{(0)}(t)$$

and

$$A_{\ell}(t) + x_1 A_{\ell}(t) = \xi_1^{(\ell)}(t)$$

$$A_{\ell}(t) + x_2 B_{\ell}(t) = \xi_2^{(\ell)}(t)$$

$$C_{\ell}(t) + x_1 D_{\ell}(t) = \eta_1^{(\ell)}(t)$$

$$C_{\ell}(t) + x_2 D_{\ell}(t) = \eta_2^{(\ell)}(t)$$

Now

$$\xi_1^{(0)}(t) = \xi_1(t) - \iint \beta(x_1, t) dx_1 dx_2$$

$$\xi_2^{(0)}(t) = \xi_2(t) - \iint \beta(x_2, t) dx_2 dx_1$$

$$\eta_1^{(0)}(t) = \eta_1(t) - \iint \delta(x_1, t) dx_1 dx_2$$

$$\eta_2^{(0)}(t) = \eta_2(t) - \iint \delta(x_2, t) dx_2 dx_1$$

and

$$\begin{aligned} \xi_1^{(\ell)}(t) = & \iint (\partial^2 / \partial t^2) u_{\ell-1}(x_1, t) dx_1 dx_2 \\ & + \iint \alpha v_{\ell-1}(x_1, t) dx_1 dx_2 \end{aligned}$$

$$\begin{aligned} \xi_2^{(\ell)}(t) = & \iint (\partial^2 / \partial t^2) u_{\ell-1}(x_2, t) dx_2 dx_1 \\ & + \iint \alpha v_{\ell-1}(x_2, t) dx_2 dx_1 \end{aligned}$$

$$\begin{aligned}\eta_1^{(\ell)}(t) &= \iint (\partial^2 / \partial t^2) v_{\ell-1}(x_1, t) dx_1 dx_1 \\ &\quad + \iint \gamma u_{\ell-1}(x_1, t) dx_1 dx_1 \\ \eta_2^{(\ell)}(t) &= \iint (\partial^2 / \partial t^2) v_{\ell-1}(x_2, t) dx_2 dx_2 \\ &\quad + \iint \gamma u_{\ell-1}(x_2, t) dx_2 dx_2\end{aligned}$$

We can write

$$\xi_1^{(0)}(t) = \sum_{n=0}^{\infty} \xi_{1,n}^{(0)} t^n$$

$$\xi_2^{(0)}(t) = \sum_{n=0}^{\infty} \xi_{2,n}^{(0)} t^n$$

$$\eta_1^{(0)}(t) = \sum_{n=0}^{\infty} \eta_{1,n}^{(0)} t^n$$

$$\eta_2^{(0)}(t) = \sum_{n=0}^{\infty} \eta_{2,n}^{(0)} t^n$$

and

$$\xi_1^{(\ell)}(t) = \sum_{n=0}^{\infty} \xi_{1,n}^{(\ell)} t^n$$

$$\xi_2^{(\ell)}(t) = \sum_{n=0}^{\infty} \xi_{2,n}^{(\ell)} t^n$$

$$\eta_1^{(\ell)}(t) = \sum_{n=0}^{\infty} \eta_{1,n}^{(\ell)} t^n$$

$$\eta_2^{(\ell)}(t) = \sum_{n=0}^{\infty} \eta_{2,n}^{(\ell)} t^n$$

Consequently, we write the matrix equations

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \cdot \begin{pmatrix} A_0(t) \\ B_0(t) \end{pmatrix} = \begin{pmatrix} \xi_1^{(0)}(t) \\ \xi_2^{(0)}(t) \end{pmatrix}$$

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \cdot \begin{pmatrix} C_0(t) \\ D_0(t) \end{pmatrix} = \begin{pmatrix} \eta_1^{(0)}(t) \\ \eta_2^{(0)}(t) \end{pmatrix}$$

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \cdot \begin{pmatrix} A_\ell(t) \\ B_\ell(t) \end{pmatrix} = \begin{pmatrix} \xi_1^{(\ell)}(t) \\ \xi_2^{(\ell)}(t) \end{pmatrix}$$

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \cdot \begin{pmatrix} C_\ell(t) \\ D_\ell(t) \end{pmatrix} = \begin{pmatrix} \eta_1^{(\ell)}(t) \\ \eta_2^{(\ell)}(t) \end{pmatrix}$$

Thus, we find

$$A_0(t) = \frac{x_2 \xi_1^{(0)}(t) - x_1 \xi_2^{(0)}(t)}{x_2 - x_1}$$

$$B_0(t) = \frac{\xi_2^{(0)}(t) - \xi_1^{(0)}(t)}{x_2 - x_1}$$

$$A_n^{(0)} = \frac{x_2 \xi_{1,n}^{(0)} - x_1 \xi_{2,n}^{(0)}}{x_2 - x_1}$$

$$B_n^{(0)} = \frac{\xi_{2,n}^{(0)} - \xi_{1,n}^{(0)}}{x_2 - x_1}$$

$$C_0(t) = \frac{x_2 \eta_1^{(0)}(t) - x_1 \eta_2^{(0)}(t)}{x_2 - x_1}$$

$$D_0(t) = \frac{\eta_2^{(0)}(t) - \eta_1^{(0)}(t)}{x_2 - x_1}$$

$$C_n^{(0)}(t) = \frac{x_2 \eta_{1,n}^{(0)} - x_1 \eta_{2,n}^{(0)}}{x_2 - x_1}$$

$$D_n^{(0)}(t) = \frac{\eta_{2,n}^{(0)} - \eta_{1,n}^{(0)}}{x_2 - x_1}$$

$$A_\ell = \frac{x_2 \xi_1^{(\ell)} - x_1 \xi_2^{(\ell)}}{x_2 - x_1}$$

$$B_\ell = \frac{\xi_2^{(\ell)} - \xi_1^{(\ell)}}{x_2 - x_1}$$

$$A_n^{(\ell)} = \frac{x_2 \xi_{1,n}^{(\ell)} - x_1 \xi_{2,n}^{(\ell)}}{x_2 - x_1}$$

$$B_n^{(\ell)} = \frac{\xi_{2,n}^{(\ell)} - \xi_{1,n}^{(\ell)}}{x_2 - x_1}$$

$$C_\ell(t) = \frac{x_2 \eta_1^{(\ell)}(t) - x_1 \eta_2^{(\ell)}(t)}{x_2 - x_1}$$

$$D_\ell(t) = \frac{\eta_2^{(\ell)}(t) - \eta_1^{(\ell)}(t)}{x_2 - x_1}$$

$$C_n^{(\ell)} = \frac{x_2 \eta_{1,n}^{(\ell)} - x_1 \eta_{2,n}^{(\ell)}}{x_2 - x_1}$$

$$D_n^{(\ell)} = \frac{\eta_{2,n}^{(\ell)} - \eta_{1,n}^{(\ell)}}{x_2 - x_1}$$

Now we write the u_0, v_0 components of u and v and the ϕ_1 approximants for u and v

$$u_0 = A_0(t) + xB_0(t) + \iint \beta(x,t) dx dx$$

Substituting

$$A_0(t) = \sum_{n=0}^{\infty} A_n^{(0)} t^n$$

$$B_0(t) = \sum_{n=0}^{\infty} B_n^{(0)} t^n$$

$$\beta(x,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta_{m,n} x^m t^n$$

we have

$$u_0 = A_0(t) + xB_0(t) + x^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\beta_{m,n} x^m t^n}{(m+1)(m+2)} = \phi_1\{u\}$$

Similarly,

$$v_0 = C_0(t) + xD_0(t) + \iint \delta(x,t) dx dx$$

with

$$C_0(t) = \sum_{n=0}^{\infty} C_n^{(0)} t^n$$

$$D_0(t) = \sum_{n=0}^{\infty} D_n^{(0)} t^n$$

$$\delta(x,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \delta_{m,n} x^m t^n$$

so that

$$v_0 = C_0(t) + xD_0(t) + x^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\delta_{m,n} x^m t^n}{(m+1)(m+2)} = \phi_1\{v\}$$

We can now write u_0, v_0 in a convenient form as

$$u_0 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(0)} x^m t^n$$

$$v_0 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n}^{(0)} x^m t^n$$

where

$$a_{0,n}^{(0)} = A_n^{(0)}$$

$$a_{1,n}^{(0)} = B_n^{(0)}$$

$$a_{m+2,n}^{(0)} = \frac{\beta_{m,n}}{(m+1)(m+2)}$$

and

$$b_{0,n}^{(0)} = C_n^{(0)}$$

$$b_{1,n}^{(0)} = D_n^{(0)}$$

$$b_{m+2,n}^{(0)} = \frac{\delta_{m,n}}{(m+1)(m+2)}$$

Next, we calculate u_1 and v_1 for the $\phi_2\{u\}$ and $\phi_2\{v\}$ approximants

$$u_1 = A_1(t) + xB_1(t) - \iint (\partial^2/\partial t^2) u_0 \, dx \, dx$$

$$- \iint \alpha v_0 \, dx \, dx$$

$$v_1 = C_1(t) + xD_1(t) - \iint (\partial^2/\partial t^2) v_0 \, dx \, dx$$

$$- \iint \gamma u_0 \, dx \, dx$$

Let

$$A_1(t) = \sum_{n=0}^{\infty} A_n^{(1)} t^n$$

$$B_1(t) = \sum_{n=0}^{\infty} B_n^{(1)} t^n$$

$$C_1(t) = \sum_{n=0}^{\infty} C_n^{(1)} t^n$$

$$D_1(t) = \sum_{n=0}^{\infty} D_n^{(1)} t^n$$

$$\begin{aligned}
u_1 &= A_1(t) + xB_1(t) - x^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{(m+1)(m+2)} a_{m,n+2}^{(0)} x^m t^n \\
&\quad - x^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha b_{m,n}^{(0)}}{(m+1)(m+2)} x^m t^n \\
v_1 &= C_1(t) + xD_1(t) - x^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{(m+1)(m+2)} b_{m,n+2}^{(0)} x^m t^n \\
&\quad - x^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\gamma a_{m,n}^{(0)}}{(m+1)(m+2)} x^m t^n
\end{aligned}$$

We can now write

$$\begin{aligned}
u_1 &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(1)} x^m t^n \\
v_1 &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n}^{(1)} x^m t^n
\end{aligned}$$

where

$$\begin{aligned}
a_{0,n}^{(1)} &= A_n^{(1)} & a_{l,n}^{(1)} &= B_n^{(1)} \\
a_{m+2,n}^{(1)} &= -\frac{(n+1)(n+2)a_{m,n+2}^{(0)} - \alpha b_{m,n}^{(0)}}{(m+1)(m+2)}
\end{aligned}$$

and

$$\begin{aligned}
b_{0,n}^{(1)} &= C_n^{(1)} & b_{l,n}^{(1)} &= D_n^{(1)} \\
b_{m+2,n}^{(1)} &= -\frac{(n+1)(n+2)b_{m,n+2}^{(0)} - \gamma a_{m,n}^{(0)}}{(m+1)(m+2)}
\end{aligned}$$

Now we have the approximants $\phi_2\{u\}$ and $\phi_2\{v\}$

$$\begin{aligned}
\phi_2\{u\} &= \phi_1\{u\} + u_1 \\
\phi_2\{v\} &= \phi_1\{v\} + v_1
\end{aligned}$$

Now we write the general components u_ℓ and v_ℓ and the approximants $\phi_{\ell+1}\{u\}$ and $\phi_{\ell+1}\{v\}$

$$\begin{aligned}
u_\ell &= A_\ell(t) + xB_\ell(t) - \iint (\partial^2/\partial t^2) u_{\ell-1} dx dx - \iint \alpha v_{\ell-1} dx dx \\
v_\ell &= C_\ell(t) + xD_\ell(t) - \iint (\partial^2/\partial t^2) v_{\ell-1} dx dx - \iint \gamma u_{\ell-1} dx dx
\end{aligned}$$

$$A_\ell(t) = \sum_{n=0}^{\infty} A_n^{(\ell)} t^n$$

$$B_\ell(t) = \sum_{n=0}^{\infty} B_n^{(\ell)} t^n$$

$$C_\ell(t) = \sum_{n=0}^{\infty} C_n^{(\ell)} t^n$$

$$D_\ell(t) = \sum_{n=0}^{\infty} D_n^{(\ell)} t^n$$

Substituting

$$u_{\ell-1} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\ell-1)} x^m t^n$$

$$v_{\ell-1} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n}^{(\ell-1)} x^m t^n$$

we have

$$\begin{aligned} u_\ell &= A_\ell(t) + x B_\ell(t) - x^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{(m+1)(m+2)} a_{m,n+2}^{(\ell-1)} x^m t^n \\ &\quad - x^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha b_{m,n}^{(\ell-1)}}{(m+1)(m+2)} x^m t^n \\ v_\ell &= C_\ell(t) + x D_\ell(t) - x^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{(m+1)(m+2)} b_{m,n+2}^{(\ell-1)} x^m t^n \\ &\quad - x^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\gamma a_{m,n}^{(\ell-1)}}{(m+1)(m+2)} x^m t^n \end{aligned}$$

and finally

$$\begin{aligned} u_\ell &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\ell)} x^m t^n \\ v_\ell &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n}^{(\ell)} x^m t^n \end{aligned}$$

where

$$\begin{aligned} a_{0,n}^{(\ell)} &= A_n^{(\ell)} & a_{1,0}^{(\ell)} &= B_0^{(\ell)} \\ a_{m+2,n}^{(\ell)} &= -\frac{(n+1)(n+2)a_{m,n+2}^{(\ell-1)} - \alpha b_{m,n}^{(\ell-1)}}{(m+1)(m+2)} \end{aligned}$$

$$b_{0,n}^{(\ell)} = C_n^{(\ell)}, \quad b_{1,n}^{(\ell)} = D_n^{(\ell)}$$

$$b_{m+2,n}^{(\ell)} = -\frac{(n+1)(n+2)b_{m,n+2}^{(\ell-1)} - \gamma a_{m,n}^{(\ell-1)}}{(m+1)(m+2)}$$

Now the approximants $\phi_{\ell+1}\{u\} = \phi_{\ell}\{u\} + u_{\ell}$ and $\phi_{\ell+1}\{v\} = \phi_{\ell}\{v\} + v_{\ell}$.

If we write $\phi_{\lambda+1}\{u\} = \sum_{\ell=0}^{\lambda} u_{\ell}$ and $\phi_{\lambda+1}\{v\} = \sum_{\ell=0}^{\lambda} v_{\ell}$ and substitute

$$u_{\ell} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{\ell} x^m t^n$$

$$v_{\ell} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n}^{\ell} x^m t^n$$

we can write

$$\begin{aligned} \phi_{\lambda+1}\{u\} &= \sum_{\ell=0}^{\lambda} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{\ell} x^m t^n \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \sum_{\ell=0}^{\lambda} a_{m,n}^{(\ell)} \right\} x^m t^n \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \phi_{\lambda+1}\{a_{m,n}\} x^m t^n \\ \phi_{\lambda+1}\{v\} &= \sum_{\ell=0}^{\lambda} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n}^{\ell} x^m t^n \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \sum_{\ell=0}^{\lambda} b_{m,n}^{(\ell)} \right\} x^m t^n \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \phi_{\lambda+1}\{b_{m,n}\} x^m t^n \end{aligned}$$

and we note that

$$\lim_{\lambda \rightarrow \infty} \phi_{\lambda+1}\{u\} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lim_{\lambda \rightarrow \infty} \phi_{\lambda+1}\{a_{m,n}\} x^m t^n$$

or

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} x^m t^n$$

and analogously

$$v = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n} x^m t^n$$

To summarize, $u = \sum_{\ell=0}^{\infty} u_{\ell}$ and $v = \sum_{\ell=0}^{\infty} v_{\ell}$

$$u_{\ell} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\ell)} x^m t^n$$

$$v_{\ell} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n}^{(\ell)} x^m t^n$$

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} x^m t^n$$

$$v = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n} x^m t^n$$

$$a_{m,n} = \sum_{\ell=0}^{\infty} a_{m,n}^{(\ell)}$$

$$b_{m,n} = \sum_{\ell=0}^{\infty} b_{m,n}^{(\ell)}$$

COUPLED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} + \alpha uv = \beta(x, t)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial t^2} + \gamma uv = \delta(x, t)$$

with uncoupled boundary conditions***

$$u(x_1, t) = \xi_1(t)$$

$$u(x_2, t) = \xi_2(t)$$

$$v(x_1, t) = \eta_1(t)$$

$$v(x_2, t) = \eta_2(t)$$

In operator form

$$Lu = \beta(x, t) - (\partial^2 / \partial t^2)u - \alpha uv$$

$$Lv = \delta(x, t) - (\partial^2 / \partial t^2)v - \gamma uv$$

***We note that the case of coupled boundary conditions is also solvable by the decomposition method.

Operating with the inverse operator (an indefinite integral operator)

$$u = u_0 - L^{-1}(\partial^2/\partial t^2)u - L^{-1}\alpha uv$$

$$v = v_0 - L^{-1}(\partial^2/\partial t^2)v - L^{-1}\gamma uv$$

where

$$u_0 = A_0(t) + xB_0(t) + \iint \beta(x,t) dx dx$$

$$v_0 = C_0(t) + xD_0(t) + \iint \delta(x,t) dx dx$$

Let

$$u = \sum_{\ell=0}^{\infty} u_{\ell} \qquad v = \sum_{\ell=0}^{\infty} v_{\ell}$$

$$uv = \left\{ \sum_{\ell=0}^{\infty} u_{\ell} \right\} \cdot \left\{ \sum_{\lambda=0}^{\infty} v_{\lambda} \right\} = \sum_{\ell=0}^{\infty} \sum_{\lambda=0}^{\ell} u_{\ell-\lambda} v_{\lambda}$$

This nonlinearity can be expressed in terms of the A_n polynomials

$$\hat{A}_{\ell} = \sum_{\lambda=0}^{\ell} u_{\ell-\lambda} v_{\lambda}$$

where

$$\hat{A}_{\ell} = A_{\ell}(u_0, \dots, u_{\ell}; v_0, \dots, v_{\ell})$$

We have changed the notation of the polynomials slightly, i.e. to $\hat{A}_{\ell}$, so that it will not be confused with the integration constant $A_{\ell}(t)$.

The approximants are $\phi_{\lambda+1}\{u\} = \sum_{\ell=0}^{\lambda} u_{\ell}$ and $\phi_{\lambda+1}\{v\} = \sum_{\ell=0}^{\lambda} v_{\ell}$ and the components u_{ℓ} and v_{ℓ} are given by:

$$u_{\ell} = A_{\ell}(t) + xB_{\ell}(t) - \iint (\partial^2/\partial t^2)u_{\ell-1} dx dx$$

$$- \iint \alpha \left[\sum_{\lambda=0}^{\ell-1} u_{\ell-1-\lambda} v_{\lambda} \right] dx dx$$

$$v_{\ell} = C_{\ell}(t) + xD_{\ell}(t) - \iint (\partial^2/\partial t^2)v_{\ell-1} dx dx$$

$$- \iint \gamma \left[\sum_{\lambda=0}^{\ell-1} u_{\ell-1-\lambda} v_{\lambda} \right] dx dx$$

Since

$$u_\ell = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\ell)} x^m t^n$$

$$v_\ell = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n}^{(\ell)} x^m t^n$$

the term $u_{\ell-1-\lambda} v_\lambda$ can be written as

$$\left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}^{(\ell-1-\lambda)} x^m t^n \right\} \left\{ \sum_{\mu=0}^{\infty} \sum_{v=0}^{\infty} b_{\mu,v}^{(\lambda)} x^\mu t^v \right\}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m t^n \sum_{\mu=0}^m \sum_{v=0}^n a_{m-\mu, n-v}^{(\ell-1-\lambda)} b_{\mu,v}^{(\lambda)}$$

which is substituted into the equations for the u_ℓ and v_ℓ components.

EXERCISE: Generalize the algorithm for the A_n polynomials to products such as $u^M v^N$ observing that $u = \sum_{\ell=0}^{\infty} u_\ell$ and

$$u^2 = \sum_{\ell=0}^{\infty} \sum_{\lambda=0}^{\ell} u_{\ell-\lambda} u_\lambda$$

$$u^3 = \sum_{\ell=0}^{\infty} \sum_{\lambda=0}^{\ell} \sum_{\mu=0}^{\lambda} u_{\ell-\lambda} u_{\lambda-\mu} u_\mu$$

$$\vdots$$

$$\hat{A}[u^2] = \sum_{\lambda=0}^{\ell} u_{\ell-\lambda} u_\lambda$$

$$\hat{A}[u^3] = \sum_{\lambda=0}^{\ell} \sum_{\mu=0}^{\lambda} u_{\ell-\lambda} u_{\lambda-\mu} u_\mu$$

$$\vdots$$

EXERCISES:

1) Show the solution of the anharmonic oscillator

$$u'' + \alpha u^2 = \beta(t) = \sum_{n=0}^{\infty} \beta_n t^n$$

with $u(0) = c_0$, $u'(0) = c_1$ is given by

$$u = \sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} t^{2m} \sum_{n=0}^{\infty} a_n^{(m)} t^n$$

with

$$\begin{aligned} a_0^{(0)} &= c_0, \quad a_1^{(0)} = c_1 \\ a_{n+2}^{(0)} &= \beta_n / (n+1)(n+2) \end{aligned}$$

for $m=0$ and for $m>0$ by

$$a_n^{(m)} = \frac{-\alpha A_n^{(m-1)}}{(n+m+1)(n+m+2)}$$

where $A_n^{(m)} = \sum_{u=0}^m \sum_{v=0}^n a_{n-v}^{(m-u)} a_v^{(u)}$.

- 2) Generalize the algorithm for the A_n polynomials to products such as $u^M v^M$ observing that $u = \sum_{\ell=0}^{\infty} u_{\ell}$ and

$$\begin{aligned} u^2 &= \sum_{\ell=0}^{\infty} \sum_{\lambda=0}^{\ell} u_{\ell-\lambda} u_{\lambda} \\ u^3 &= \sum_{\ell=0}^{\infty} \sum_{\lambda=0}^{\ell} \sum_{\mu=0}^{\lambda} u_{\ell-\lambda} u_{\lambda-\mu} u_{\mu} \\ &\vdots \\ \hat{A}_{\ell}[u^2] &= \sum_{\lambda=0}^{\ell} u_{\ell-\lambda} u_{\lambda} \quad \dots \\ &\vdots \\ \hat{A}_{\ell}[u^3] &= \sum_{\lambda=0}^{\ell} \sum_{\mu=0}^{\lambda} u_{\ell-\lambda} u_{\lambda} u_{\mu} \\ &\vdots \end{aligned}$$

DECOMPOSITION SOLUTIONS FOR NEUMANN BOUNDARY CONDITIONS

For simplicity, consider a linear differential equation $Lu + Ru = g$ where $L = d^2/dx^2$ (and R can involve no differentiations higher than first-order). Assume conditions are given as

$$du/dx \Big|_{x=b_1} = \beta_1$$

$$du/dx \Big|_{x=b_2} = \beta_2$$

The decomposition solution is $u = \Phi + L^{-1}g - L^{-1}Ru$ where Φ satisfies $L\Phi = 0$ and L^{-1} is a pure two-fold integration (not involving constants). The derivative du/dx or u' is given by

$$u' = \Phi' + Ig - IRu'$$

where $L\Phi' = 0$ and I is a single pure integration and is, of course, not equal to the two-fold integration operator L^{-1} . Returning now to the solution u , we have by decomposition:

$$u = \Phi + L^{-1}g - L^{-1}Ru$$

$$\sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} \Phi_m + L^{-1}g - L^{-1}R \sum_{m=0}^{\infty} u_m$$

where we note the decomposition not only of u but also of Φ . Other decompositions are also sometimes useful. For example, when integrations such as $L^{-1}Ru$ or of $L^{-1}g$ become difficult, R or g can be decomposed into a convenient series so that the individual terms become simpler integrations. Now $u_m = \Phi_m + L^{-1}g$ and

$$u_m = \Phi_m - L^{-1}Ru_{m-1} = \sum_{n=0}^{\infty} (-L^{-1}R)^n \Phi_{m-n} + (-L^{-1}R)^m L^{-1}g$$

so that

$$\begin{aligned}
u &= \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^m (-L^{-1}R)^n \Phi_{m-n} + (-L^{-1}R)^m L^{-1}g \right\} \\
&= \sum_{m=0}^{\infty} \left\{ (-L^{-1}R)^m \sum_{n=0}^{\infty} \Phi_n + (-L^{-1}R)^m L^{-1}g \right\} \\
u &= \sum_{m=0}^{\infty} (-L^{-1}R)^m \{ \Phi + L^{-1}g \}
\end{aligned}$$

Differentiating u and noting that Ru can be written RIu' , we have

$$u' = \Phi' + Ig - IRlu'$$

We now solve the u' equation by decomposition just as we did for the u equation:

$$\sum_{m=0}^{\infty} u'_m = \sum_{m=0}^{\infty} \Phi'_m + Ig - IRI \sum_{m=0}^{\infty} u'_m$$

Hence,

$$\begin{aligned}
\sum_{m=0}^{\infty} u'_m &= \sum_{m=0}^{\infty} \Phi'_m + Ig - IRI \sum_{m=0}^{\infty} u'_m \\
u'_0 &= \Phi'_0 + Ig
\end{aligned}$$

and for $m \geq 1$,

$$\begin{aligned}
u'_m &= \Phi'_m - IRI u'_{m-1} \\
&= \sum_{n=0}^m (-IRI)^n \Phi'_{m-n} + (-IRI)^m Ig
\end{aligned}$$

so that

$$\begin{aligned}
u' &= \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^m (-IRI)^n \Phi'_{m-n} + (-IRI)^m Ig \right\} \\
&= \sum_{m=0}^{\infty} \left\{ (-IRI)^m \sum_{n=0}^{\infty} \Phi'_n + (-IRI)^m Ig \right\}
\end{aligned}$$

or

$$u' = \sum_{m=0}^{\infty} (-IRI)^m \{ \Phi' + Ig \}$$

We now need to determine Φ' to determine the constants of integration c_0 and c_1 involved in Φ' or $c_{0,m}$ and $c_{1,m}$ involved in $\Phi_{0,m}$. For simplicity and clarity we let $g = 0$ and calculate the series for u . Beginning with $Lu + Ru = 0$, we have

$$u = \Phi - L^{-1}Ru$$

$$\Phi = c_0 + c_1x$$

$$\begin{aligned} u &= \sum_{m=0}^{\infty} (-L^{-1}R)^m \Phi = \sum_{m=0}^{\infty} (-L^{-1}R)^m (c_0 + c_1x) \\ &= (c_0 + c_1x) - L^{-1}R(c_0 + c_1x) + (L^{-1}R)^2(c_0 + c_1x) - \dots \end{aligned}$$

The series for du/dx is

$$\begin{aligned} u' &= (d/dx) \{ c_0 + c_1x - L^{-1}R c_0 - L^{-1}R c_1x + (L^{-1}R)^2 c_0 + (L^{-1}R)^2 c_1x - \dots \} \\ &= c_1 - IR c_0 - IR c_1x + IR^2 c_0 + IR^2 c_1x \dots \end{aligned}$$

Noting that $Ic_1 = xc_1$

$$u' = c_1 - IR c_0 - IR c_1 + IR^2 c_0 + IR^2 R c_1 - \dots$$

Rearranging and collecting terms, we have

$$u' = (c_1 - IR c_0) - IR(c_1 - IR c_0) + (IR)^2(c_1 - IR c_0) - \dots$$

$$u' = \Phi' - IR\Phi' + (IR)^2\Phi' - \dots = u'_0 - u'_1 + u'_2 - \dots$$

Thus

$$\Phi' = c_1 - IR c_0$$

$$\Phi'_m = c_{1,m} - IR c_{0,m}$$

and we note that $(d/dx)u_m \neq u'_m$ but

$$\sum_{m=0}^{\infty} (d/dx)u_m = \sum_{m=0}^{\infty} u'_m$$

i.e., the decomposition is not unique. Thus, although du_m/dx is not the same as the corresponding derivative of the m th component of u , the infinite sums are the same.

Returning to the computation of the solution in general and matching the solution to the given conditions,

$$u_0 = c_{0,0} + xc_{1,0} + L^{-1}g$$

$$u'_0 = c_{1,0} - IRc_{0,0} + Ig$$

$$\phi'_1 = u'_0$$

$$\phi'_1(b_1) = \beta_1$$

$$\phi'_1(b_2) = \beta_2$$

which determines $c_{0,0}$ and $c_{0,1}$. For $g = 0$, the matrix equation determining the integration constants is*

$$\begin{pmatrix} -\int R(b_1)db_1 & 1 \\ -\int R(b_2)db_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_{0,0} \\ c_{1,0} \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$u_m = \Phi_m - L^{-1}Ru_{m-1}$$

$$\Phi_m = c_{0,m} + xc_{1,m}$$

$$u'_m = \Phi'_m - IRI u'_{m-1}$$

$$\phi'_{m+1} = \sum_{n=0}^m u'_n$$

Matching ϕ'_{m+1} to the conditions,

$$\phi'_{m+1}(b_1) = \beta_1$$

$$\phi'_{m+1}(b_2) = \beta_2$$

determines $c_{0,m}$ and $c_{1,m}$. Thus

$$\begin{pmatrix} -\int R(b_1)db_1 & 1 \\ -\int R(b_2)db_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_{0,m} \\ c_{1,m} \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

* If R is constant, for instance $R = \rho$, the equation for $c_{0,0}$ and $c_{1,0}$ is

$$\begin{pmatrix} -\rho b_1 & 1 \\ -\rho b_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_{0,0} \\ c_{1,0} \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

where $\beta_{1,m}$ and $\beta_{2,m}$ are determined from

$$\phi'_{m+1} = u'_m + \phi'_m$$

where ϕ'_{m-1} represents the sum $\sum_{n=0}^m u'_n$. Now the constants $c_{0,m}$, $c_{1,m}$ are determined for all m . Hence all the Φ'_m and Φ_m are determined.

$$u_m = \sum_{n=0}^m (-L^{-1}R)^n \Phi_{m-n} + (-L^{-1}R)^m L^{-1}g$$

$$u = \sum_{m=0}^{\infty} \left[\sum_{n=0}^m (-L^{-1}R)^n \Phi_{m-n} + (-L^{-1}R)^m L^{-1}g \right]$$

Upon rearranging terms

$$u = \sum_{m=0}^{\infty} \left[(-L^{-1}R)^m \sum_{n=0}^{\infty} \Phi_n + (-L^{-1}R)^m L^{-1}g \right]$$

$$u = \sum_{m=0}^{\infty} (-L^{-1}R)^m \{ \Phi + L^{-1}g \}$$

where $\Phi = \sum_{n=0}^{\infty} \Phi_n$.

We have shown a technique for linear operator equations for development of an invertible matrix for the vector equation, which determines all constants of integration. This method is readily extended to the nonlinear case and also to partial differential equations.

Consider an example: let $R = 1$ and $g = 0$. (Then $IRI = I^2 = L^{-1}$.) We have $d^2u/dx^2 + u = 0$. Substituting $g = 0$ and $R = 1$ in the previously derived solution u , we have

$$u = \sum_{m=0}^{\infty} \left[(-L^{-1}R)^m \sum_{n=0}^{\infty} \Phi_n + (-L^{-1}R)^m L^{-1}g \right]$$

$$u = \sum_{m=0}^{\infty} (-L^{-1}R)^m \{ \Phi + L^{-1}g \}$$

We compute

$$u = c_0 \cos x + c_1 \sin x$$

$$u' = -c_0 \sin x + c_1 \cos x$$

Matching u' at the boundaries

$$\frac{du}{dx}(b_1) = \beta_1$$

$$\frac{du}{dx}(b_2) = \beta_2$$

Then

$$-c_0 \sin b_1 + c_1 \cos b_1 = \beta_1$$

$$-c_0 \sin b_2 + c_1 \cos b_2 = \beta_2$$

$$\begin{pmatrix} -\sin b_1 & \cos b_1 \\ -\sin b_2 & \cos b_2 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

which, for a non-zero determinant, determines c_0 and c_1 and a unique solution $u = c_0 \cos x + c_1 \sin x$ satisfying the given Neumann conditions.

SUMMARY:

We have shown the solution for Neumann conditions of linear ordinary differential equations. The procedure can be simplified considerably for linear differential equations but is a general procedure for nonlinear differential and partial differential equations for boundary-value problems. The procedure of decomposition of the initial term, as well as of the solution, yields faster convergence because when we have found an n -term approximant, the resulting composite initial term u_0 incorporates more of the solution, and hence we accelerate convergence.

INTEGRAL BOUNDARY CONDITIONS

We first consider an expository linear example:

$$d^2u/dx^2 + \gamma u = 0$$

with conditions given as:

$$u(x = \xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta_1 u(x) dx$$

$$u(x = \xi_2) = b_2 + \int_{\xi_1}^{\xi_2} \beta_2 u(x) dx$$

γ , β_1 , and β_2 are assumed constants here although they can be functions of x with minor modifications to the procedure given. In decomposition format we have $Lu + Ru = 0$ or $L^{-1}Lu = -L^{-1}Ru$ or

$$u = c_0 + c_1 x - \gamma I_x^2 \sum_{n=0}^{\infty} u_n$$

where I_x^2 is a two-fold pure integration with respect to x .

Since $c_0 + c_1 x$ is identified as u_0 , we have $u_1 = -\gamma I_x^2 u_0$, $u_2 = -\gamma I_x^2 u_1, \dots$. Consequently, we write

$$u = \sum_{v=0}^{\infty} (-\gamma)^v [c_0^v x^{2v} / (2v)! + c_1^v x^{2v+1} / (2v+1)!]$$

Since the successive approximants φ_m represent u to increasing accuracy as m increases, each φ_m must satisfy the boundary conditions for $m = 1, 2, \dots$. When $m=1$,

$$\varphi_1(\xi_1) = u_0(\xi_1) = b_1$$

$$\varphi_1(\xi_2) = u_0(\xi_2) = b_2$$

or

$$c_0 + c_1 \xi_1 = b_1$$

$$c_0 + c_1 \xi_2 = b_2$$

thus the "matching coefficients" c_0 and c_1 must satisfy

$$\begin{bmatrix} 1 & \xi_1 \\ 1 & \xi_2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The ξ_1 and ξ_2 are distinct points, i.e., $\xi_1 \neq \xi_2$, in a two-point boundary problem; hence

$$c_0 = (\xi_2 b_1 - \xi_1 b_2) / (\xi_2 - \xi_1)$$

$$c_1 = (b_2 - b_1) / (\xi_2 - \xi_1)$$

Next we determine $\varphi_2 = \varphi_1 + u_1$. Since $\varphi_1(\xi_1) = b_1$ and $\varphi_1(\xi_2) = b_2$ already, and

$$\varphi_2(\xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta_1 \varphi_1(x) dx$$

$$\varphi_2(\xi_2) = b_2 + \int_{\xi_1}^{\xi_2} \beta_2 \varphi_1(x) dx$$

Equivalently,

$$u_1(\xi_1) = \int_{\xi_1}^{\xi_2} \beta_1 u_0(x) dx$$

$$u_1(\xi_2) = \int_{\xi_1}^{\xi_2} \beta_2 u_0(x) dx$$

The next decomposition component is $u_1(x) = -\gamma[c_0 x^2/2 + c_1 x^3/3!]$ and we form $\varphi_2 = \varphi_1 + u_1$, which we match to the given conditions

$$c_0 + c_1 x - \gamma[c_0 x^2/2 + c_1 x^3/3!] = b_1 + \int_{\xi_1}^{\xi_2} \beta(c_0 + c_1 x) dx$$

or

$$c_0 + c_1 x = b_1 + \int_{\xi_1}^{\xi_2} \beta(c_0 + c_1 x) dx + \gamma[c_0 x^2/2 + c_1 x^3/3!]$$

Therefore to match the boundary conditions,

$$c_0^{(1)} + c_1^{(1)} \xi_1 = \int_{\xi_1}^{\xi_2} \beta_1 (c_0^{(0)} + c_1^{(0)} x) dx + \gamma[c_0^{(0)} \xi_1^2/2 + c_1^{(0)} \xi_1^3/3!]$$

$$c_0^{(1)} + c_1^{(1)} \xi_2 = \int_{\xi_1}^{\xi_2} \beta_2 (c_0^{(0)} + c_1^{(0)} x) dx + \gamma[c_0^{(0)} \xi_2^2/2 + c_1^{(0)} \xi_2^3/3!]$$

We have added the superscripts (1) to distinguish new values from the previously calculated values of c_0 and c_1 on the right-hand side. The right-hand sides of the two equations are symbolized as $b_1^{(1)}$ and $b_2^{(1)}$. Then

$$c_0^{(1)} + c_1^{(1)} \xi_1 = b_1^{(1)}$$

$$c_0^{(1)} + c_1^{(1)} \xi_2 = b_2^{(1)}$$

which is solvable as before for a new c_0 and c_1 . As m increases, φ_m approaches the u very closely so the error vanishes.

A MORE GENERAL PROCEDURE:

$$d^2u/dx^2 + \gamma u = 0$$

with the conditions

$$u(\xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta_1 u(x) dx$$

$$u(\xi_2) = b_2 + \int_{\xi_1}^{\xi_2} \beta_2 u(x) dx$$

We write $Lu = d^2u/dx^2$ and $L^{-1} = I_x^2$ where I_x^2 is the two-fold indefinite integration operator. Then

$$\sum_{n=0}^{\infty} u_n = \Phi - L^{-1} R \sum_{n=0}^{\infty} u_n$$

where $u_0 = \Phi = c_0 + c_1 x$ and $u_m = (-L^{-1} R)^m \Phi$ or

$$u_m = (-1)^m \gamma^m I_x^{2m} (c_0 + c_1 x)$$

so that

$$u = c_0 \sum_{m=0}^{\infty} (-1)^m \gamma^m \frac{x^{2m}}{(2m)!} + c_1 \sum_{m=0}^{\infty} (-1)^m \gamma^m \frac{x^{2m+1}}{(2m+1)!}$$

Even though the sums are recognizable, write

$$u = c_0 \mu_0(x) + c_1 \mu_1(x)$$

where μ_0 and μ_1 represent the sums. Next we can evaluate the constants c_0 and c_1 at the boundary conditions. (We call them matching coefficients.) Thus

$$c_0 \mu_0(\xi_1) + c_1 \mu_1(\xi_1) = b_1 + c_0 \beta_1 \int_{\xi_1}^{\xi_2} \mu_0(x) dx + c_1 \beta_1 \int_{\xi_1}^{\xi_2} \mu_1(x) dx$$

$$c_0 \mu_0(\xi_2) + c_1 \mu_1(\xi_2) = b_2 + c_0 \beta_2 \int_{\xi_1}^{\xi_2} \mu_0(x) dx + c_1 \beta_2 \int_{\xi_1}^{\xi_2} \mu_1(x) dx$$

which we can write as

$$\alpha_{11} c_0 + \alpha_{12} c_1 = b_1$$

$$\alpha_{21} c_0 + \alpha_{22} c_1 = b_2$$

by defining

$$\alpha_{11} = \left\{ \mu_0(\xi_1) - \beta_1 \int_{\xi_1}^{\xi_2} \mu_0(x) dx \right\}$$

$$\alpha_{12} = \left\{ \mu_1(\xi_1) - \beta_1 \int_{\xi_1}^{\xi_2} \mu_1(x) dx \right\}$$

$$\alpha_{21} = \left\{ \mu_0(\xi_2) - \beta_2 \int_{\xi_1}^{\xi_2} \mu_0(x) dx \right\}$$

$$\alpha_{22} = \left\{ \mu_1(\xi_2) - \beta_2 \int_{\xi_1}^{\xi_2} \mu_1(x) dx \right\}$$

Forming the vector equation for the matching constants,

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

If the α matrix is not singular ($\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21} \neq 0$), then we can determine its inverse

$$\alpha^{-1} = \frac{1}{|\alpha|} \begin{pmatrix} \alpha_{22} - \alpha_{12} \\ -\alpha_{21} & \alpha_{11} \end{pmatrix}$$

so that

$$\bar{c} = \alpha^{-1} \bar{b}$$

Consequently

$$c_0 = \frac{\alpha_{22}}{|\alpha|} b_1 - \frac{\alpha_{12}}{|\alpha|} b_2$$

$$c_1 = -\frac{\alpha_{21}}{|\alpha|} b_1 + \frac{\alpha_{11}}{|\alpha|} b_2$$

which we can write as

$$c_0 = \frac{\alpha_{22} b_1 - \alpha_{12} b_2}{\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}}$$

$$c_1 = \frac{\alpha_{11} b_2 - \alpha_{21} b_1}{\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}}$$

We have now computed

$$u(x) = c_0 \mu_0(x) + c_1 \mu_1(x)$$

Where of course in this case,

$$\mu_0(x) = \cos(\sqrt{\gamma} x)$$

$$\mu_1(x) = \sin \frac{(\sqrt{\gamma} x)}{\sqrt{\gamma}}$$

ANOTHER ALTERNATIVE:

Start with the exact integral boundary condition:

$$u(\xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta u(x) d\gamma x$$

Let us decompose the equation and the conditions

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \text{ and } u(\xi_1) = \sum_{n=0}^{\infty} u_n(\xi_1)$$

hence

$$\sum_{n=0}^{\infty} u_n(\xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta \sum_{n=0}^{\infty} u_n(x) dx$$

Then

$$u_0(\xi_1) = b_1$$

$$u_n(\xi_1) = \int_{\xi_1}^{\xi_2} \beta u_{n-1}(x) dx \quad (n \geq 1)$$

Now we use our earlier notation [1] where λ is a grouping parameter for collecting terms. Thus

$$u(\xi_1) = b_1 + \lambda \int_{\xi_1}^{\xi_2} \beta u(x) dx$$

$$u(\xi_1) = \sum_{n=0}^{\infty} u_n(\xi_1) \lambda^n$$

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \lambda^n$$

Substituting,

$$\sum_{n=0}^{\infty} u_n(\xi_1) \lambda^n = b_1 + \lambda \int_{\xi_1}^{\xi_2} \beta \sum_{n=0}^{\infty} u_n(x) \lambda^n dx$$

Equating like powers of λ ,

$$\begin{aligned}\lambda^0 &: u_0(\xi_1) = b_1 \\ \lambda^{n \geq 1} &: u_n(\xi_1) = \int_{\xi_1}^{\xi_2} \beta u_{n-1}(x) dx\end{aligned}$$

then we set $\lambda = 1$. Since each $u_n(x)$ has two undetermined coefficients, i.e., the $c_0^{(n)}$ and $c_1^{(n)}$, we notice that we have an algorithm to compute $c_0^{(n)}$ and $c_1^{(n)}$ without explicit reference to them. However, the earlier way of writing approximate boundary conditions $\varphi_{m+1}(\xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta \varphi_m(x) dx$ is appealing because the limit as m approaches infinity is explicitly the exact boundary condition.

EXERCISE: Carry out the computations for the matching coefficients and verify solutions.

REMARKS: The single decomposition (where we carry along the constants of integration) is applicable to all linear ordinary differential equations for either initial conditions or linear integral boundary conditions. (However, if the boundary conditions are nonlinear, we need to use double decomposition. If we try to use single decomposition with nonlinear integral boundary conditions, we must solve for the matching coefficients c_0 and c_1 as roots of algebraic or transcendental equations, then find which roots are correct. It will be more simple to use double decomposition in such cases.) We will now use double decomposition for the case already considered by single decomposition.

USING DOUBLE DECOMPOSITION:

$$\begin{aligned}u &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)} \\ c_0 &= \sum_{m=0}^{\infty} c_0^{(m)} \\ c_1 &= \sum_{m=0}^{\infty} c_1^{(m)} \\ u_n &= \sum_{m=0}^{\infty} u_n^{(m)}\end{aligned}$$

so that

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)} = \sum_{m=0}^{\infty} c_0^{(m)} + x \sum_{m=0}^{\infty} c_1^{(m)} - \gamma I_x^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)}$$

We have previously shown for boundary-value problems by application of staggered summation (see Appendix II) that

$$u_m = \sum_{n=0}^m u_n^{(m-n)}$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^m u_n^{(m-n)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)}$$

Substituting into the equation,

$$\sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} c_0^{(m)} + x \sum_{m=0}^{\infty} c_1^{(m)} - \gamma I_x^2 \sum_{m=0}^{\infty} u_m$$

we have

$$u_0 = c_0^{(0)} + x c_1^{(0)}$$

$$u_1 = c_0^{(1)} - x c_1^{(1)} - \gamma I_x^2 u_0$$

$$\vdots$$

$$u_m = c_0^{(m)} + x c_1^{(m)} - \gamma I_x^2 u_{m-1}$$

or

$$u_0 = c_0^{(0)} + x c_1^{(0)}$$

$$u_1 = c_0^{(1)} + x c_1^{(1)} - \gamma c_0^{(0)} \frac{x^2}{2!} - \gamma c_1^{(0)} \frac{x^3}{3!}$$

$$u_2 = c_0^{(2)} + x c_1^{(2)} - \gamma c_0^{(1)} \frac{x^2}{2!} - \gamma c_1^{(1)} \frac{x^3}{3!} + \gamma^2 c_0^{(0)} \frac{x^4}{4!} + \gamma^2 c_1^{(0)} \frac{x^5}{5!}$$

$$u_3 = c_0^{(3)} + x c_1^{(3)} - \gamma c_0^{(2)} \frac{x^2}{2!} - \gamma c_1^{(2)} \frac{x^3}{3!} + \gamma^2 c_0^{(1)} \frac{x^4}{4!}$$

$$+ \gamma^2 c_1^{(1)} \frac{x^5}{5!} - \gamma^3 c_0^{(0)} \frac{x^6}{6!} - \gamma^3 c_1^{(0)} \frac{x^7}{7!}$$

$$\vdots$$

$$u_m = \sum_{n=0}^m (-\gamma)^n \left\{ c_0^{(m-n)} x^{2n} / (2n)! + c_1^{(m-n)} x^{2n+1} / (2n+1)! \right\}$$

where

$$u_m = \sum_{n=0}^m u_n^{(m-n)}$$

and

$$u_n^{(m-n)} = (-\gamma)^n \left\{ c_0^{(m-n)} x^{2n} / (2n)! + c_1^{(m-n)} x^{2n+1} / (2n+1)! \right\}$$

$$u = \sum_{(p=0)}^{\infty} \sum_{n=0}^m (-\gamma)^n \left\{ c_0^{(m-n)} x^{2n} / (2n)! + c_1^{(m-n)} x^{2n+1} / (2n+1)! \right\}$$

A convenient rearrangement by staggered summation results in

$$u = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-\gamma)^m \left\{ c_0^{(n)} x^{2m} / (2m)! + c_1^{(n)} x^{2m+1} / (2m+1)! \right\}$$

or

$$u = \sum_{m=0}^{\infty} (-\gamma)^m \left\{ c_0 x^{2m} / (2m)! + c_1 x^{2m+1} / (2m+1)! \right\}$$

since

$$c_0 = \sum_{n=0}^{\infty} c_0^{(n)} \text{ and } c_1 = \sum_{n=0}^{\infty} c_1^{(n)}$$

We now form the approximants to the solution u

$$\varphi_{m+1}[u] = \sum_{n=0}^m u_n$$

and form the approximate boundary conditions. For $m = 1$,

$$\varphi_1(\xi_1) = b_1 \text{ and } \varphi_1(\xi_2) = b_2$$

and for $m > 1$,

$$\varphi_{m+1}(\xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta_1 \varphi_m(x) dx$$

$$\varphi_{m+1}(\xi_2) = b_2 + \int_{\xi_1}^{\xi_2} \beta_2 \varphi_m(x) dx$$

(noting that $\lim_{m \rightarrow \infty} \varphi_{m+1} = \lim_{m \rightarrow \infty} \varphi_m = u$ and that the approximate boundary conditions become the exact boundary conditions in the limit).

We have $\varphi_1 = u_0 = c_0^{(0)} + x c_1^{(0)}$. Since $\varphi_1(\xi_1) = b_1$ and $\varphi_1(\xi_2) = b_2$, we have $u_0(\xi_1) = b_1$ and $u_0(\xi_2) = b_2$

$$u_1 = c_0^{(1)} + xc_1^{(1)} - \gamma I_x^2 u_0$$

$$\varphi_2 = u_0 + u_1$$

$$\varphi_2(\xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta_1 \varphi_1(x) dx$$

$$\varphi_2(\xi_2) = b_2 + \int_{\xi_1}^{\xi_2} \beta_2 \varphi_1(x) dx$$

Since $\varphi_1(\xi_1) = b_1$ and $\varphi_1(\xi_2) = b_2$, then

$$u_1(\xi_1) = \int_{\xi_1}^{\xi_2} \beta_1 \varphi_1(x) dx = \int_{\xi_1}^{\xi_2} \beta_1 u_0(x) dx$$

$$u_1(\xi_2) = \int_{\xi_1}^{\xi_2} \beta_2 \varphi_1(x) dx = \int_{\xi_1}^{\xi_2} \beta_2 u_0(x) dx$$

$$u_2 = c_0^{(2)} + xc_1^{(2)} - \gamma I_x^2 u_1$$

$$\varphi_3 = \varphi_2 + u_2$$

$$\varphi_3(\xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta_1 \varphi_2(x) dx$$

$$\varphi_3(\xi_2) = b_2 + \int_{\xi_1}^{\xi_2} \beta_2 \varphi_2(x) dx$$

Since

$$\varphi_2(\xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta_1 \varphi_1(x) dx$$

$$\varphi_2(\xi_2) = b_2 + \int_{\xi_1}^{\xi_2} \beta_2 \varphi_1(x) dx$$

$$\varphi_2 = \varphi_1 + u_1$$

then

$$u_2(\xi_1) = \int_{\xi_1}^{\xi_2} \beta_1 u_1(x) dx$$

$$u_2(\xi_2) = \int_{\xi_1}^{\xi_2} \beta_2 u_1(x) dx$$

$$u_m = c_0^{(m)} + xc_1^{(m)} - \gamma I_x^2 u_{m-1}$$

$$\varphi_{m+1} = \varphi_m + u_m$$

$$\varphi_{m+1}(\xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta_1 \varphi_m(x) dx$$

$$\varphi_{m+1}(\xi_2) = b_2 + \int_{\xi_1}^{\xi_2} \beta_2 \varphi_m(x) dx$$

Since $\varphi_m = \varphi_{m-1} + u_{m-1}$

$$u_m(\xi_1) = \int_{\xi_1}^{\xi_2} \beta_1 u_{m-1}(x) dx$$

$$u_m(\xi_2) = \int_{\xi_1}^{\xi_2} \beta_2 u_{m-1}(x) dx$$

Now using

$$u_0 = c_0^{(0)} + x c_1^{(0)}$$

$$u_m = c_0^{(m)} + x c_1^{(m)} - \gamma I_x^2 u_{m-1} \quad m \geq 1$$

Thus

$$u_0 = c_0^{(0)} + x c_1^{(0)}$$

$$u_1 = c_0^{(1)} + x c_1^{(1)} + \{(-\gamma)[c_0^{(0)} x^2/2! + c_1^{(0)} x^3/3!]\}$$

$\vdots$

$$u_m = c_0^{(m)} + x c_1^{(m)} + \sum_{n=1}^m (-\gamma)^n \{c_0^{(m-n)} x^{2n}/(2n)! + c_1^{(m-n)} x^{2n+1}/(2n+1)!\}$$

We can now write

$$c_0^{(0)} + \xi_1 c_1^{(0)} = b_1^{(0)}$$

$$c_0^{(0)} + \xi_2 c_1^{(0)} = b_2^{(0)}$$

$$c_0^{(1)} + \xi_1 c_1^{(1)} = b_1^{(1)}$$

$$c_0^{(1)} + \xi_2 c_1^{(1)} = b_2^{(1)}$$

$\vdots$

$$c_0^{(m)} + \xi_1 c_1^{(m)} = b_1^{(m)}$$

$$c_0^{(m)} + \xi_2 c_1^{(m)} = b_2^{(m)}$$

For computational convenience, we define $b_1^{(0)} = b_1$ and $b_2^{(0)} = b_2$. For $m = 0$

$$b_1^{(m)} = \left[\int_{\xi_1}^{\xi_2} \beta_1 \left\{ \sum_{n=0}^{m-1} (-\gamma)^n (c_0^{(m-1-n)} x^{2n}/(2n)! + c_1^{(m-1-n)} x^{2n+1}/(2n+1)!) \right\} dx \right. \\ \left. - \sum_{n=1}^m (-\gamma)^n \{ (c_0^{(m-n)} \xi_1^{2n}/(2n)! + c_1^{(m-n)} \xi_1^{2n+1}/(2n+1)!) \} \right]$$

$$b_2^{(m)} = \left[\int_{\xi_2}^{\xi_1} \beta_2 \left\{ \sum_{n=0}^{m-1} (-\gamma)^n (c_0^{(m-1-n)} x^{2n}/(2n)! + c_1^{(m-1-n)} x^{2n+1}/(2n+1)!) \right\} dx \right. \\ \left. - \sum_{n=1}^m (-\gamma)^n \{ (c_0^{(m-n)} \xi_2^{2n}/(2n)! + c_1^{(m-n)} \xi_2^{2n+1}/(2n+1)!) \} \right]$$

Now we can write for $m \geq 0$

$$c_0^{(m)} + \xi_1 c_1^{(m)} = b_1^{(m)}$$

$$c_0^{(m)} + \xi_2 c_1^{(m)} = b_2^{(m)}$$

which are a set of simultaneous equations for the matching coefficients $c_0^{(m)}$ and $c_1^{(m)}$. Equivalently,

$$\begin{bmatrix} 1 & \xi_1 \\ 1 & \xi_2 \end{bmatrix} \cdot \begin{bmatrix} c_0^{(m)} \\ c_1^{(m)} \end{bmatrix} = \begin{bmatrix} b_1^{(m)} \\ b_2^{(m)} \end{bmatrix}$$

If ξ_1 and ξ_2 are distinct as they must be for two-point boundary conditions,

$$c_0^{(m)} = \frac{\xi_2 b_1^{(m)} - \xi_1 b_2^{(m)}}{\xi_2 - \xi_1}$$

$$c_1^{(m)} = \frac{b_2^{(m)} - b_1^{(m)}}{\xi_2 - \xi_1}$$

so that the solution is determined.

ACCELERATION OF CONVERGENCE:

We can accelerate convergence while minimizing further matching coefficients by now transposing from the boundary-value format, denoted by B.V., to an initial-value format, denoted by I.V. The procedure is to first compute a current best estimate of $u_0^{(I.V.)}$ (the first term of the decomposition series in initial-value formulation)

$$u_0^{(I.V.)} \approx \sum_{n=0}^m u_0^{(n)} = \varphi_{m+1}[u_0^{(I.V.)}]$$

then compute $u_m^{(I.V.)}$ for $m \geq 1$:

$$u_m^{(I.V.)} = (-\gamma)^m (I_x^{2m}) u_0^{(I.V.)}$$

$$u^{(I.V.)} = \sum_{m=0}^{\infty} (-\gamma)^m (I_x^{2m}) u_0^{(I.V.)}$$

$$u^{(I.V.)} = \varphi_{p+1}[u^{(I.V.)}] = \sum_{m=0}^p (-\gamma)^m (I_x^{2m}) u_0^{(I.V.)}$$

For a reasonable approximation to $u_0^{(L.V.)}$ we have

$$\varphi_{p+1}[u^{(L.V.)}] \approx \sum_{m=0}^p (-\gamma)^m (I_x^{2m}) \varphi_{n+1}[u_0^{(L.V.)}]$$

where $\varphi_{n+1}[u_0^{(L.V.)}] = \sum_{v=0}^n u_0^{(v)} = u_0^{(L.V.)}$. Substituting,

$$\varphi_{p+1}[u^{(L.V.)}] \approx \sum_{m=0}^p (-\gamma)^m (I_x^{2m}) \varphi_{n+1}[u_0^{(L.V.)}]$$

where

$$u_0^{(v)} = c_0^{(v)} + x c_1^{(v)}$$

Consequently,

$$u^{(L.V.)} \approx \sum_{m=0}^p (-\gamma)^m (I_x^{2m}) \sum_{v=0}^n [c_0^{(v)} + x c_1^{(v)}]$$

or

$$\begin{aligned} u^{(L.V.)} &\approx \sum_{m=0}^p (-\gamma)^m (I_x^{2m}) \varphi_{n+1}[c_0] \\ &\quad + \sum_{m=0}^p (-\gamma)^m (I_x^{2m}) \varphi_{n+1}[c_1] \end{aligned}$$

The limits of the series for the boundary-value solution and the initial-value solution are the same, i.e.,

$$u = u^{(B.V.)} = u^{(L.V.)}$$

$$\begin{aligned} u &\approx \varphi_{n+1}[c_0] \sum_{m=0}^p (-\gamma)^m (I_x^{2m})(1) \\ &\quad + \varphi_{n+1}[c_1] \sum_{m=0}^p (-\gamma)^m (I_x^{2m})(x) \end{aligned}$$

which we can symbolize by the algorithmic form

$$u \approx \varphi_{p+1, n+1}[u] \equiv \left\{ (\varphi_{n+1}[c_0]) \cdot (\varphi_{p+1}(\cos \sqrt{\gamma} x)) + (\varphi_{n+1}[c_1]) \cdot \left(\frac{(\sin \sqrt{\gamma} x)}{\sqrt{\gamma}} \right) \right\}$$

Then in the double limit

$$\begin{aligned}
 u &= \lim_{\substack{p \rightarrow \infty \\ n \rightarrow \infty}} \varphi_{p+1, n+1}[u] \\
 &= c_0 \cos \sqrt{\gamma} x + c_1 \frac{\sin \sqrt{\gamma} x}{\sqrt{\gamma}}
 \end{aligned}$$

which satisfies the original linear ordinary differential equation $d^2u/dx^2 + \gamma u = 0$ as well as the original linear integral two-point boundary conditions.

MATCHING COEFFICIENTS FOR NONLINEAR INTEGRAL CONDITIONS:

For linear equations, we wrote $\varphi_{m+1} = \sum_{n=0}^m u_n$ as an $(m+1)$ -term approximation to u , i.e., for a sufficiently high value of m ,

$$\varphi_{m+1}[u] \approx u \text{ or } \lim_{m \rightarrow \infty} \varphi_{m+1}[u] = u$$

(We note that in practice m does not need to be large.) For a nonlinear function of the solution such as $h(u)$, we expand in the A_n : thus

$$\varphi_{m+1}[h(u)] = \sum_{n=0}^m A_n[h(u)]$$

which we can write simply as $\sum_{n=0}^m A_n$ if the context is clear. Thus $h(u) \approx \varphi_{m+1}[h(u)]$ or $h(u) = \lim_{m \rightarrow \infty} \varphi_{m+1}[h(u)]$. The exact boundary conditions are

$$u(\xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta_1 h_1(u) dx$$

$$u(\xi_2) = b_2 + \int_{\xi_1}^{\xi_2} \beta_2 h_2(u) dx$$

The approximate boundary conditions are:

$$\varphi_{m+1}[u(\xi_1)] = b_1 + \int_{\xi_1}^{\xi_2} \beta_1 \varphi_m[h_1(u(x))] dx$$

$$\varphi_{m+1}[u(\xi_2)] = b_2 + \int_{\xi_1}^{\xi_2} \beta_2 \varphi_m[h_2(u(x))] dx$$

which yield the exact boundary conditions in the limit. For $m = 1$ we have

$$\varphi_1[u(\xi_1)] = b_1$$

$$\varphi_1[u(\xi_2)] = b_2$$

The coefficients $c_j^{(m)}$, $c_1^{(m)}$ are now calculable by decomposition of u and $h(u)$, which means that nonlinear integral boundary conditions can also be dealt with analogously to linear integral boundary conditions.

SUGGESTIONS FOR RESEARCH:

- 1) Linear equation, linear integral boundary conditions: (β_1 and β_2 can be constants or functions of x)

$$u(x = \xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta_1(x) u(x) dx$$

$$u(x = \xi_2) = b_2 + \int_{\xi_1}^{\xi_2} \beta_2(x) u(x) dx$$

- 2) Linear equation, nonlinear integral boundary conditions

$$u(x = \xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta_1(x) h_1(u(x)) dx$$

$$u(x = \xi_2) = b_2 + \int_{\xi_1}^{\xi_2} \beta_2(x) h_2(u(x)) dx$$

- 3) Nonlinear equation, linear integral boundary conditions

$$u(x = \xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta_1(x) u(x) dx$$

$$u(x = \xi_2) = b_2 + \int_{\xi_1}^{\xi_2} \beta_2(x) u(x) dx$$

- 4) Nonlinear equation, nonlinear integral boundary conditions

$$u(x = \xi_1) = b_1 + \int_{\xi_1}^{\xi_2} \beta_1(x) h_1(u(x)) dx$$

$$u(x = \xi_2) = b_2 + \int_{\xi_1}^{\xi_2} \beta_2(x) h_2(u(x)) dx$$

- 5) Linear (two-dimensional) partial differential equation, linear contour-integral boundary conditions. Note that $C(x,y) = 0$ implies $x = \xi(y)$

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 + \gamma(x,y)u = g(x,y)$$

$$u(x,y)|_{C_1(x,y)=0} = b_1 + \oint_{C_1} \beta_1(x,y)u d\ell$$

$$u(x,y)|_{C_2(x,y)=0} = b_2 + \oint_{C_2} \beta_2(x,y)u d\ell$$

- 6) Linear (three - dimensional) partial differential equation, linear surface-integral boundary conditions. $S(x,y,z)$ implies $x = \xi(y,z)$

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 + \partial^2 u / \partial z^2 + \gamma(x,y,z)u = g(x,y,z)$$

$$u(x,y,z)|_{S_1(x,y,z)=0} = b_1 + \oint_{S_1} \beta_1(x,y,z)u ds$$

$$u(x,y,z)|_{S_2(x,y,z)=0} = b_2 + \oint_{S_2} \beta_2(x,y,z)u ds$$

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SUGGESTED READING

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BOUNDARY CONDITIONS AT INFINITY

Solutions of problems with boundary conditions involving a limit at infinity can be difficult. To build some intuition, we will begin with a simple example modelled by a linear differential equation. Consider the function $u = e^{-x} = \sum_{m=0}^{\infty} (-x)^m / m!$. We obviously have $u(0) = 1$ and $u(\infty) = 0$. By the latter, we clearly mean that $\lim_{x \rightarrow \infty} u(x) = 0$. This function satisfies the differential equation $d^2u/dx^2 - u = 0$. Letting L denote d^2/dx^2 , we have $Lu - u = 0$ which is in our standard format $Lu + Ru = 0$ with $R = -1$. With L^{-1} defined as a two-fold indefinite integration operator, we have $L^{-1}Lu = L^{-1}u$ so that $u = C_0 + C_1 x + L^{-1}u$. (Since we are dealing with a linear ordinary differential equation, double decomposition is unnecessary.) We identify

$$u_0 = C_0 + C_1 x$$

$$u_1 = L^{-1}u_0$$

$$u_2 = L^{-1}u_1$$

$$\vdots$$

$$\begin{aligned} u_m &= L^{-1}u_{m-1} = (L^{-1})^m u_0 \\ &= C_0 x^{2m} / (2m)! + C_1 x^{2m+1} / (2m+1)! \end{aligned}$$

The resulting solution is $u = \sum_{m=0}^{\infty} u_m$ or

$$u = C_0 \cosh x + C_1 \sinh x$$

Even from the first-order approximation $\varphi_1 = u_0$, it is clear that $C_0 = 1$ since $u(0) = 1$. We also have the condition $u(\infty) = 0$ which might make us jump to the conclusion that $C_1 = 0$. However, we would soon see that we do not then get a verifiable solution. Since u must approach zero as $x \rightarrow \infty$, we have

$$\lim_{x \rightarrow \infty} [\cosh x + C_1 \sinh x] = 0$$

so that

$$C_1 \lim_{x \rightarrow \infty} \sinh x = - \lim_{x \rightarrow \infty} \cosh x$$

or

$$C_1 = -\frac{\lim_{x \rightarrow \infty} \cosh x}{\lim_{x \rightarrow \infty} \sinh x} = -\lim_{x \rightarrow \infty} \coth x = -1$$

and we see that $u = \cosh x - \sinh x$. Substitution of the exponential forms

$$\cosh x = (e^x + e^{-x})/2$$

$$\sinh x = (e^x - e^{-x})/2$$

shows that indeed $u = e^{-x}$ which we started with. Equivalently, since

$$\cosh x = \sum_{m=0}^{\infty} x^{2m} / (2m)!$$

$$\sinh x = \sum_{m=0}^{\infty} x^{2m+1} / (2m+1)!$$

we have

$$\begin{aligned} u &= (1 - x^2/2! + x^4/4! - \dots) - (x + x^3/3! + x^5/5! + \dots) \\ &= 1 - x + x^2/2! - x^3/3! + \dots = e^{-x} \end{aligned}$$

It is instructive to write

$$C_1 = -\lim_{x \rightarrow \infty} \xi_0(x) / \xi_1(x) = -\lim_{x \rightarrow \infty} \gamma(x)$$

where

$$\xi_0 = \cosh x \quad |x| < \infty$$

$$\xi_1 = \sinh x \quad |x| < \infty$$

$$\gamma(x) = \coth x \quad |x| < \pi \text{ for the series representing } \coth x$$

EXERCISE: Write the series for $\coth x$. Using the Padé transform, show that the limit, as $x \rightarrow \infty$, is 1. (See Appendix I.) For problems in which the decomposition series is not recognized as we did above, we complete the division, e.g.,

$$\gamma(x) = \frac{1 - x^2/2! + x^4/4! - \dots}{x + x^3/3! + x^5/5! + \dots}$$

then determine the limit with the Padé approximant.

EXERCISE: Consider the equation $d^2 u/dx^2 - \rho(x)u = 0$ with $u(0) = 1$ and $u(+\infty) = 0$ and verify the solution.

EXAMPLE: Consider a generic second-order linear homogeneous differential equation with boundary conditions specified at infinity

$$d^2 u/dx^2 + \alpha(x) du/dx + \beta(x)u = 0$$

Let $R = \alpha d/dx + \beta$

$$\alpha(x) = \sum_{m=0}^{\infty} \alpha_m x^m$$

$$\beta(x) = \sum_{m=0}^{\infty} \beta_m x^m$$

$$u(0) = 1$$

$$\lim_{x \rightarrow \infty} u(x) = 0$$

we have $u = \sum_{m=0}^{\infty} u_m$ with $u_0 = C_0 + C_1 x$. Then

$$u_m = (-L^{-1}R)^m u_0 = (-L^{-1}R)^m (1) + (-L^{-1}R)^m C_1 x$$

$$u = \sum_{m=0}^{\infty} (-L^{-1}R)^m \cdot 1 + C_1 \sum_{m=0}^{\infty} (-L^{-1}R)^m \cdot x$$

which we will write

$$u = \xi_0(x) + C_1 \xi_1(x)$$

Since $\lim_{x \rightarrow \infty} u(x) = 0$ we have

$$C_1 = -\lim_{x \rightarrow \infty} \frac{\xi_0(x)}{\xi_1(x)} = -\lim_{x \rightarrow \infty} \gamma(x)$$

$$\xi_0(x) = \sum_{m=0}^{\infty} \xi_0^{(m)} x^m \quad \left(\xi_0^{(m)} \equiv \xi_{0,m} \right)$$

$$\xi_1(x) = \sum_{m=0}^{\infty} \xi_1^{(m)} x^m$$

$$\gamma(x) = \sum_{m=0}^{\infty} \gamma_m x^m = \frac{\sum_{m=0}^{\infty} \xi_0^{(m)} x^m}{\sum_{m=0}^{\infty} \xi_1^{(m)} x^m}$$

The series for $\gamma(x)$ will usually have a finite radius of convergence. Therefore, in order to evaluate the limit at $+\infty$ to compute C_1 , we must transform the series for $\gamma(x)$ or a truncated series approximating $\gamma(x)$ to a representation suitable at infinity. The Padé approximant is useful here. Write $\gamma(x) = \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \dots$ and calculate $\Gamma(x)$ where $\Gamma(x)$ is the Padé approximant of $\gamma(x)$. (See Appendix I.) Then $C_1 = -\lim_{x \rightarrow \infty} \Gamma(x)$.

MODIFIED DECOMPOSITION AND CONDITIONS AT INFINITY:

Consider $d^2 u/dx^2 - u = 0$ with given conditions $u(0) = 1$ and $u(+\infty) = 0$ using the modified decomposition. In some cases, it is possible that the resulting recurrence relation could provide insight and illuminate the matching of the solution to the condition at infinity. A finite decomposition approximant such as $\varphi_s = \sum_{m=0}^s u_m$ with a finite radius of convergence might be transformed into a finite fraction which is accurate for large x to obtain the limit at infinity. Of course, other techniques may be applicable such as Euler transforms, analytic continuation, etc. If we recognize well-known functions as in our first example, this additional step will be eliminated.

SOLUTION: $L^{-1}Lu = L^{-1}u$ where $L^{-1}(\cdot) = C_0 + C_1 x + I_x^2(\cdot)$ where

$$I_x(\cdot) = \int (\cdot) dx$$

We have $u = C_0 + C_1 x + I_x^2 u$ where we let $u = \sum_{m=0}^{\infty} a_m x^m$ instead of $\sum_{m=0}^{\infty} u_m$. Then

$$I_x^2 u = \sum_{m=2}^{\infty} \frac{a_{m-2}}{(m-1)(m)} x^m$$

Consequently,

$$\sum_{m=0}^{\infty} a_m x^m = C_0 + C_1 x + \sum_{m=2}^{\infty} \frac{a_{m-2}}{(m)(m-1)} x^m$$

so that we have recurrence relations for the coefficients:

$$a_0 = C_0$$

$$a_1 = C_1$$

$$a_m = \frac{a_{m-2}}{m(m-1)}$$

so that

$$a_2 = C_0/2!$$

$$a_3 = C_1/3!$$

$$a_4 = C_0/4!$$

$$a_5 = C_1/5!$$

$$\vdots$$

We can now write $a_{2m} = C_0/(2m)!$ and $a_{2m+1} = C_1/(2m+1)!$ so the solution can be written as

$$u = \sum_{m=0}^{\infty} \{C_0 x^{2m}/(2m)! + C_1 x^{2m+1}/(2m+1)!\}$$

$$u = C_0 \sum_{m=0}^{\infty} x^{2m}/(2m)! + C_1 \sum_{m=0}^{\infty} x^{2m+1}/(2m+1)!$$

At this point, we recognize the summations; but let us assume that we do not see that these are series representations of the hyperbolic trigonometric functions or know their limit values at infinity, and must proceed in a way which is usable when the series are not recognized. Evaluation at the zero boundary $u(0) = 1$ requires $C_0 = 1$. To evaluate at the second boundary gives us

$$\lim_{x \rightarrow \infty} u(x) = 0 \text{ or } \sum_{m=0}^{\infty} x^{2m}/(2m)! + C_1 \lim_{x \rightarrow \infty} \sum_{m=0}^{\infty} x^{2m+1}/(2m+1)! = 0$$

hence

$$C_1 = -\lim_{x \rightarrow \infty} \left\{ \frac{\sum_{m=0}^{\infty} x^{2m}/(2m)!}{\sum_{m=0}^{\infty} x^{2m+1}/(2m+1)!} \right\}$$

$$C_1 = -\lim_{x \rightarrow \infty} \left\{ \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \frac{x^7}{4725} + \dots \right\}$$

EXERCISE: Verify this series. It can also be written

$$C_1 = -\lim_{x \rightarrow \infty} \left\{ \frac{1}{x} + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} 2^{2m}}{(2m)!} B_m x^{2m-1} \right\}$$

where the B_m are the Bernoulli numbers*. We list, for convenience of the reader,

$B_1 = 1/6$	$B_6 = 691/2730$
$B_2 = 1/30$	$B_7 = 7/6$
$B_3 = 1/42$	$B_8 = 3617/510$
$B_4 = 1/30$	$B_9 = 43,867/698$
$B_5 = 5/66$	$B_{10} = 174,611/330$

The radius of convergence of this series is π . Thus to evaluate

$$\lim_{x \rightarrow \infty} \left\{ \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \dots \right\}$$

we need to transform the series, or at least, transform the truncated series to a representation which will be valid as $x \rightarrow +\infty$.

EXERCISE: Show, using Padé approximants, that the limit of

$$\frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \dots$$

or

$$\frac{1}{x} + \sum_{m=1}^{\infty} (-1)^{m-1} \frac{2^{2m}}{(2m)!} B_m x^{2m-1}$$

is 1 as $x \rightarrow \infty$. (*Hint:* Ignore the $1/x$ which obviously does not contribute.) Write the Padé approximant $[2/2]$ and since C_1 is equal to the negative of this limit, show that C_1 is equal to -1 . Hence

* H. B. Dwight. Table of Integrals and Other Mathematical Data, 4th ed.. MacMillan and Co., N.Y. (1961).

$$u = \sum_{m=0}^{\infty} x^{2m}/(2m)! - \sum_{m=0}^{\infty} x^{2m+1}/(2m+1)! = \{1 + x^2/2! + x^4/4! + \dots\} \\ - \{x + x^3/3! + x^5/5! + \dots\} = 1 - x + x^2/2! - x^3/3! + \dots = \sum_{m=0}^{\infty} (-1)^m x^m/(m)!$$

Thus the solution is computed. Of course, it is the exponential function and it is easily verified by substitution.

We observe that it may be necessary to transform the solution series to a representation for which we can easily determine the limit as $x \rightarrow \infty$. A useful technique is that of Padé approximants which allows us to evaluate the matching coefficients C_0 and C_1 , i.e., the "constants of integration," (particularly the condition at infinity). Of course, since we do recognize the functions we can use the fact that the limit as $x \rightarrow \infty$ of $\coth x = 1$ and that the limit of e^{-x} is zero. But in general, the functions are not well-known nor will we know a priori their limits at infinity.

EXAMPLE: Consider the case $d^2u/dx^2 - \rho(x)u = 0$ with $u(0) = 1$ and $\lim_{x \rightarrow \infty} u(x) = 0$. Let $\rho(x) = \sum_{m=0}^{\infty} \rho_m x^m$. Write $Lu = \left\{ \sum_{m=0}^{\infty} \rho_m x^m \right\} u$ with $L = d^2/dx^2$. Operating on both sides with $L^{-1} = \iint (\cdot) dx dx$,

$$u = C_0 + C_1 x + L^{-1} \left\{ \sum_{m=0}^{\infty} \rho_m x^m \right\} u = \sum_{m=0}^{\infty} a_m x^m \\ \rho(x)u = \left\{ \sum_{m=0}^{\infty} \rho_m x^m \right\} \left\{ \sum_{m=0}^{\infty} a_m x^m \right\} = \sum_{m=0}^{\infty} x^m \sum_{n=0}^m \rho_n a_{m-n} \\ L^{-1} \rho(x)u = L^{-1} \sum_{m=0}^{\infty} x^m \sum_{n=0}^m \rho_n a_{m-n} = \sum_{m=0}^{\infty} \frac{x^{m+2}}{(m+1)(m+2)} \sum_{n=0}^m \rho_n a_{m-n} \\ = \sum_{m=2}^{\infty} \frac{x^m}{m(m-1)} \sum_{n=0}^{m-2} \rho_n a_{m-2-n} \\ u = C_0 + C_1 x + \sum_{m=2}^{\infty} x^m \frac{\sum_{n=0}^{m-2} \rho_n a_{m-2-n}}{m(m-1)}$$

Therefore

$$a_0 = C_0$$

$$a_1 = C_1$$

and for $m \geq 2$

$$a_m = \frac{\sum_{n=0}^{m-2} \rho_n a_{m-2-n}}{m(m-1)}$$

so that

$$a_2 = \rho_0 a_0 / 1 \cdot 2$$

$$a_3 = (\rho_0 a_1 + \rho_1 a_0) / 2 \cdot 3$$

$$a_4 = (\rho_0 a_2 + \rho_1 a_1 + \rho_2 a_0) / 3 \cdot 4$$

$$a_5 = (\rho_0 a_3 + \rho_1 a_2 + \rho_2 a_1 + \rho_3 a_0) / 4 \cdot 5$$

$\vdots$

Consequently

$$a_0 = C_0$$

$$a_1 = C_1$$

$$a_2 = \rho_0 C_0 / 2!$$

$$a_3 = \rho_0 C_1 / 3! + \rho_1 C_0 / 3!$$

$$a_4 = C_0 / 4! + \rho_1 C_1 / 3 \cdot 4 + \rho_2 C_0 / 3 \cdot 4$$

$$a_5 = \rho_0^2 C_1 / 5! + \rho_0 \rho_1 C_0 / 5!$$

$$+ \rho_1 \rho_0 C_0 / 2 \cdot 4 \cdot 5 + \rho_2 C_1 / 4 \cdot 5 + \rho_3 C_0 / 4 \cdot 5$$

$\vdots$

Thus,

$$u = C_0 \left\{ 1 + \rho_0 x^2 / 2! + \rho_1 x^3 / 3! + (\rho_0^2 + 2\rho_2) x^4 / 4! + \dots \right\} \\ + C_1 \left\{ x + \rho_0 x^3 / 3! + 2\rho_1 x^4 / 4! + (\rho_0^2 + 6\rho_2) x^5 / 5! + \dots \right\}$$

Since $u(0) = 1$, $C_0 = 1$. To evaluate C_1 we write

$$C_1 = - \lim_{x \rightarrow \infty} \frac{\{1 + \rho_0 x^2 / 2! + \rho_0^2 x^4 / 4! + \dots\} 4! + \dots}{\{x + \rho_0 x^3 / 3! + 2\rho_1 x^4 / 4! + \rho_0^2 x^5 / 5! + \dots\}} \\ C_1 = - \lim_{x \rightarrow \infty} \left\{ \frac{1}{x} + \rho_0 \frac{x}{3} - \rho_0^2 \frac{x^3}{45} + \dots \right\}$$

which is the same as our earlier result if $\rho_0 = 1$, $\rho_1 = \rho_2 = 0$. Of course, if $\rho = \sum_{m=0}^{\infty} \rho_m x^m$ we must include the neglected terms of ρ in the ratio for C_1 .

EXAMPLE: Linear Third-order Equation: The function $u = e^{-x}$ satisfies the equation $d^3u/dx^3 + u = 0$ with conditions $u(0) = 1$, $u(\infty) = 0$, and $u'(\infty) = 0$. Let us investigate how to determine this solution if only the equation and conditions are known. $L = d^3/dx^3$ and L^{-1} is now a three-fold integration. From $Lu = -u$ and $L^{-1}Lu = -L^{-1}u$,

$$u = C_0 + C_1x + C_2x^2/2 - L^{-1} \sum_{n=0}^{\infty} u_n$$

where

$$u_0 = C_0 + C_1x + C_2x^2/2$$

$$u_1 = -L^{-1}u_0$$

$$u_2 = -L^{-1}u_1 = (-L^{-1})^2 u_0$$

$\vdots$

$$u_m = (-L^{-1})^m u_0 = C_0(-1)^m x^{3m}/(3m)! + C_1(-1)^m x^{3m-1}/(3m+1)! + C_2(-1)^m x^{3m+2}/(3m+2)!$$

Since $u = \sum_{m=0}^{\infty} u_m$, we write

$$u = C_0 \xi_0(x) + C_1 \xi_1(x) + C_2 \xi_2(x)$$

with

$$\xi_0 = \sum_{m=0}^{\infty} (-1)^m x^{3m}/(3m)!$$

$$\xi_1 = \sum_{m=0}^{\infty} (-1)^m x^{3m}/(3m+1)!$$

$$\xi_2 = \sum_{m=0}^{\infty} (-1)^m x^{3m+2}/(3m+2)!$$

Thus we have

$$u = C_0 \xi_0(x) + C_1 \xi_1(x) + C_2 \xi_2(x)$$

$$u' = C_0 \xi_0'(x) + C_1 \xi_1'(x) + C_2 \xi_2'(x)$$

Since $\varphi_1 = u_0$ must satisfy $u(0) = 1$, we know $C_0 = 1$ so that

$$u = \xi_0 + C_1 \xi_1 + C_2 \xi_2$$

$$u' = \xi_0' + C_1 \xi_1' + C_2 \xi_2'$$

Since u and u' must approach zero in the limit as $x \rightarrow \infty$, we have

$$\lim_{x \rightarrow \infty} \{\xi_0 + C_1 \xi_1 + C_2 \xi_2\} = 0$$

$$\lim_{x \rightarrow \infty} \{\xi'_0 + C_1 \xi'_1 + C_2 \xi'_2\} = 0$$

or

$$\lim_{x \rightarrow \infty} \begin{pmatrix} \xi_1 & \xi_2 \\ \xi'_1 & \xi'_2 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = - \lim_{x \rightarrow \infty} \begin{pmatrix} \xi_0 \\ \xi'_0 \end{pmatrix}$$

Solving for the C_1 , C_2 and writing $D = \xi_1 \xi'_2 - \xi'_1 \xi_2$,

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = - \lim_{x \rightarrow \infty} \frac{1}{D} \begin{pmatrix} \xi'_2 & -\xi_2 \\ -\xi'_1 & \xi_1 \end{pmatrix} \cdot \begin{pmatrix} \xi_0 \\ \xi'_0 \end{pmatrix}$$

$$C_1 = - \lim_{x \rightarrow \infty} (\xi'_2 \xi_0 - \xi_2 \xi'_0) / D$$

$$C_2 = - \lim_{x \rightarrow \infty} (-\xi'_1 \xi_0 - \xi_1 \xi'_0) / D$$

Symbolizing this as $C_1 = - \lim_{x \rightarrow \infty} \rho(x)$ and $C_2 = - \lim_{x \rightarrow \infty} \sigma(x)$, we can obtain Padé approximants of ρ and σ to get the limits.

EXERCISE: Show that $C_1 = -1$ and $C_2 = 1$ so that $u = e^x$.

EXAMPLE: Nonlinear 3rd-order equation—As an example we consider the Blasius equation of boundary layer theory:

$$d^3 u / dx^3 + (1/2) u d^2 u / dx^2 = 0$$

or in our format $Lu + Nu = 0$ with $L = d^3/dx^3$, $Nu = (1/2)uu''$, L^{-1} defined as a triple integration and the A_n calculated for $(1/2)uu''$. Applying decomposition,

$$u = \alpha + \beta x + \gamma x^2/2 - L^{-1} \sum_{n=0}^{\infty} A_n$$

and

$$u_0 = \alpha + \beta x + \gamma x^2/2$$

The given conditions are $u(0) = 0$, $u'(0) = 0$, and $u' \rightarrow 1$ as $u \rightarrow +\infty$. The first two conditions require $\alpha = \beta = 0$ so that

$$u_0 = \gamma x^2/2$$

$$u_1 = -L^{-1}A_0 = -(1/2)L^{-1}(u_0 u_0'') = -(\gamma^2/2)(x^5/5!)$$

$$u_2 = -L^{-1}A_1 = -(1/2)L^{-1}(u_1 u_0'' + u_0 u_1'') = 11(\gamma^3/4)(x^8/8!)$$

$$u_3 = -L^{-1}A_2 = -(1/2)L^{-1}(u_2 u_0'' + u_1 u_1'' + u_0 u_2'') = -375(\gamma^4/8)(x^{11}/11!)$$

$$\vdots$$

Thus,

$$u = x^2/2 - (\gamma^2/2)(x^5/5!) + 11(\gamma^3/4)(x^8/8!) - 375(\gamma^4/8)(x^{11}/11!)$$

which is the Blasius series! Now,

$$u' = \gamma x - (\gamma^2/2)(x^4/4!) + 11(\gamma^3/4)(x^7/7!) - \dots$$

We note that $u'(0) = \gamma$ and we have the remaining condition $u'(x) \rightarrow 1$ as $x \rightarrow \infty$ for the evaluation of γ .

Since the $u'(x)$ series lacks the first term in $\sum_{n=0}^{\infty} c_n x^n$, we translate, using $x = z + \xi$ with ξ taken as $1/2$, so that the series for each new coefficient will converge very rapidly. We then have a numerically equal series $\sum_{n=0}^{\infty} b_n z^n$ with non-zero $b_0, b_1, b_2, \dots$ and can find the limit of the series. (See Appendix I.) Since this involves γ and we have $u'(\infty) = 1$, we can evaluate γ directly without use of numerical methods such as shooting techniques.

EXERCISE: Carry out the evaluation and verify the solution $u(x)$. Transform $f(x) = \sum_{n=0}^{\infty} c_n x^n$ where $c_0 = 0$, (using $z = x - \xi$ with $\xi < 1$ and within the radius of convergence) to $\sum_{n=0}^{\infty} b_n z^n$ with $b_0 \neq 0$. If the Blasius problem is given as an initial-value problem, we have $u''' + uu'' = 0$ with $u(0) = u'(0) = 0$ and $u''(0) = 1$. [1] The given conditions are $u(0) = u'(0) = 0$ and $u''(0) = 1$. The first two conditions require $\alpha = \beta = 0$ so that

$$u_0 = \gamma x^2/2$$

$$u_1 = -L^{-1}A_0 = -(1/2)L^{-1}(u_0 u_0'') = -\gamma^2 x^5/5!$$

$$u_2 = -L^{-1}A_1 = -(1/2)L^{-1}(u_1 u_0'' + u_0 u_1'') = 11\gamma^3 x^8/8!$$

$$u_3 = -L^{-1}A_2 = -(1/2)L^{-1}(u_2 u_0'' + u_1 u_1'' + u_0 u_2'') = -375\gamma^4 x^{11}/11!$$

A four-term approximant $\varphi_4 = \sum_{m=0}^3 u_m$ is given by

$$u = \gamma x^2/2(\gamma^2/2)x^5/5! + 11(\gamma^3/4)x^8/8! - 375(\gamma^4/8)x^{11}/11!$$

Now,

$$u'' = \gamma - \gamma^2 x^3/3! + \dots$$

We have the remaining condition $u'' = 1$ which gives us $\gamma = 1$.

EXERCISE: Substitute the resulting solution, carrying terms through x^5 to see that the differential equation and given conditions are satisfied.

EXAMPLE: Consider the nonlinear 3rd-order equation

$$d^3u/dx^3 + \mu d^2u/dx^2 - (du/dx)^2 + \alpha = 0$$

for $0 < u < \infty$ and given conditions $u(0) = 0$, $u'(0) = \beta$, and $u'(\infty) = \gamma$. We let $L = d^3/dx^3$, L^{-1} is defined as a 3-fold integration, and we write $\sum_{n=0}^{\infty} A_n\{uu''\}$ for the first nonlinear term and $\sum_{n=0}^{\infty} A_n\{(u')^2\}$ for the second. By decomposition,

$$u = u_0 - mL^{-1} \sum_{n=0}^{\infty} A_n\{uu''\} + L^{-1} \sum_{n=0}^{\infty} A_n\{u'u'\}$$

$$u_0 = \xi + \eta x + \sigma x^2/2 - \alpha x^3/3!$$

$$u_1 = mL^{-1}A_0\{uu''\} + L^{-1}A_0\{u'u'\}$$

$$u_2 = mL^{-1}A_1\{uu''\} + L^{-1}A_1\{u'u'\}$$

We can now form an n -term approximant $\varphi_n = \sum_{m=0}^{n-1} u_m$ which converges to u as n approaches. Because of the condition on $u(0)$, we have $\xi = 0$ and because of the condition $u'(0) = \beta$ we have $\eta = \beta$. Evaluating the A_n for $n = 0$

$$A_0\{uu''\} = (\beta x + \sigma x^2/2 - \alpha x^3/3!)(\sigma - \alpha x)$$

$$A_0\{u'u'\} = (\beta + \sigma x - \alpha x^2/3!)^2$$

Hence,

$$\begin{aligned}
 u_1 &= mL^{-1} \{ (\beta x + \sigma x^2/2 - \alpha x^3/3!) (\sigma - \alpha x) \} \\
 &\quad + L^{-1} \{ \beta + \sigma x - \alpha x^2/2! \} \\
 &\quad \vdots
 \end{aligned}$$

EXERCISE: Determine the Padé limit and use it to evaluate the remaining unknown constant σ . Verify the solution for u .

EXERCISE: Consider the solution of $u''' + u = 0$ using "modified decomposition". Let $u = \sum_{m=0}^{\infty} a_m x^m$ and $Lu = -u$ with $L = d^3/dx^3$. Since $u_0 = C_0 + C_1 x + C_2 x^2/2$ (where C_0, C_1, C_2 are determined from the specified conditions), show

$$a_0 = C_0$$

$$a_1 = C_1$$

$$a_2 = \frac{C_2}{2}$$

$$a_{m+3} = \frac{-a_m}{(m+1)(m+2)(m+3)}$$

for $m = 0, 1, 2, \dots$.

EXERCISE: Show that the solution is

$$\begin{aligned}
 u &= C_0 \sum_{m=0}^{\infty} (-1)^m x^{3m} / (3m)! + C_1 \sum_{m=0}^{\infty} (-1)^m x^{3m+1} / (3m+1)! \\
 &\quad + C_2 \sum_{m=0}^{\infty} (-1)^m x^{3m+2} / (3m+2)!
 \end{aligned}$$

EXERCISE: Using the same conditions as previously used for the decomposition method, i.e., $u(0) = 1$, $u(\infty) = u'(\infty) = 0$, show $u = e^{-x}$.

SUGGESTED READING

1. R. E. Meyer, *Introduction to Mathematical Fluid Dynamics*, Wiley-Interscience (1971).

INTEGRAL EQUATIONS

Integral equations of Volterra type arise quite naturally in physical applications modelled by initial-value problems. Consider the linear Volterra equation of the second kind. (Fredholm equations of the second kind which are associated with boundary value problems for a finite interval $[a, b]$, are similar except that the upper limit is b .)

$$\varphi(x) = f(x) + \lambda \int_a^x K(x, y) \varphi(y) dy$$

with $a \leq x, y \leq b$, and let $\lambda = 1$. Using decomposition $\varphi = \sum_{n=0}^{\infty} \varphi_n(x)$, we identify $\varphi_0 = f(x)$ assuming $f(x) \neq 0$, then

$$\varphi(x) = \varphi_0 + \int_a^x K(x, y) \sum_{n=0}^{\infty} \varphi_n(y) dy$$

and write $\varphi = \sum_{n=0}^{\infty} \varphi_n$ as the solution, or with an m -term approximant $\varphi = \sum_{n=0}^{m-1} \varphi_n$. In some cases, exact solutions are determinable. Consider an example:

$$\begin{aligned} K(x, y) &= |y - x| \\ f(x) &= x \end{aligned}$$

Then

$$\varphi_1 = \int_0^x |y - x| \varphi_0(y) dy = \int_0^x |y - x| y dy = -x^3 / 3!$$

Thus the two-term approximant is

$$\varphi \approx \varphi_1 + \varphi_2 = x - (x^3 / 3!)$$

or $\varphi = \sin x$ as is easily verified either by calculating more terms or by substitution.

Several illuminating further examples on convergence appear in [1].

1) Consider the equation

$$u(x) = \frac{2x}{3} + \int_0^1 x t u(t) dt$$

$$\sum_{n=0}^{\infty} u_n(x) = u_0 + \int_0^1 x t \sum_{n=0}^{\infty} u_n(t) dt$$

$$u_0 = \frac{2x}{3}$$

$$u_1 = \int_0^1 x t \cdot \frac{2t}{3} dt = x \left[\frac{2}{3} \frac{t^3}{3} \right]_{t=0}^{t=1}$$

$$u_1 = \frac{2x}{(3)^2}$$

$$u_2 = x \int_0^1 t \frac{2t}{(3)^2} dt = \frac{2x}{(3)^3}$$

⋮

$$u_n(x) = \frac{2}{(3)^{n+1}} x$$

The sequence

$$\sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^n = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$$

The partial sums are

$$S_1 = 1.000$$

$$S_2 = 1.333 \dots$$

$$S_3 = 1.449$$

$$S_4 = 1.481$$

$$S_5 = 1.493$$

and the $\lim_{n \rightarrow \infty} S_n = 1.5$. Then the solution of the problem is:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = \sum_{n=0}^{\infty} \frac{2x}{3^{n+1}} = \frac{2x}{3} \sum_{n=0}^{\infty} \frac{1}{3^n} = x$$

2) Consider the nonlinear integral equation

$$u(x) = \int_0^1 (x-t) u^3(t) dt + \frac{3x}{4} + \frac{1}{5}$$

We get

$$u_0(x) = 0.75x + 0.20$$

$$u_1(x) = A_0 = \int_0^1 (x-t) u_0^3(t) dt = 0.23x - 0.19$$

$$\begin{aligned} u_2(x) &= A_1 = 3 \int_0^1 (x-t) u_1(t) u_0^2(t) dt \\ &= (0.00354712)x - 0.004464994 \\ &\vdots \end{aligned}$$

The approximant to the solution with three terms is:

$$u(x) \approx u_0(x) + u_1(x) + u_2(x) = 0.98x + 0.003$$

which is very near to the exact solution which is x .

- 3) Consider a nonlinear biological problem: Find a real function u defined on $\mathbb{R}$ by

$$u(x) + 0.25x \int_0^1 K(x,t) g(u(t)) dt = 1$$

with $K(x,t) = 1/x+t$ and $g(u) = 1/u$. The solution by decomposition is:

$$u(x) \approx 0.25x (\log(x+1) - \log x)$$

a three-term approximant results in

x	ϕ_3
0.1	1.059947
0.2	1.089588
0.3	1.109975
0.4	1.125276
0.5	1.137327
0.6	1.147124
0.7	1.155278
0.8	1.162186
0.9	1.168123
1.0	1.173287

Of course, we can improve the approximation with more terms until the desired accuracy is achieved.

4) Find a real function u satisfying the integral equation:

$$u(x) = \lambda \int_0^1 xt |u(t)|^2 dt + x$$

where λ is a real parameter, $\lambda \in [0, 1]$. With a five-term approximation, decomposition gives the following results:

$$u_0(x) = x$$

$$u_1(x) = A_0 = \lambda \int_0^1 xt |u_0(t)|^2 dt = \frac{\lambda x}{4}$$

$$u_2(x) = A_1 = 2\lambda \int_0^1 xt u_0(t) u_1(t) dt = \frac{\lambda^2 x}{8}$$

$$u_3(x) = A_2 = \lambda \int_0^1 xt [4u_0(t)u_2(t) + 2u_1^2(t)] dt = \frac{5\lambda^3}{32} x$$

$$u_4(x) = A_3 = \lambda \int_0^1 xt [12(u_3(t)u_0(t) + u_1(t)u_2(t))] dt = \frac{9}{16} \lambda^4 x$$

$$\vdots$$

$$u(x) = \left(1 + \frac{\lambda}{4} + \frac{\lambda^2}{8} + \frac{5\lambda^3}{32} + \frac{9\lambda^4}{16} \right) x$$

Assuming a value of $\lambda = 1/10$, [1] show an error of $9.7 \times 10^{-4} \sim 10^{-3}$.

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NONLINEAR OSCILLATIONS IN PHYSICAL SYSTEMS

Nonlinear oscillating systems are generally analyzed by approximation methods which involve some sort of linearization. These replace an actual nonlinear system with a so-called "equivalent" linear system and employ averaging which is not generally valid. While the linearizations commonly used are adequate in some cases, they may be grossly inadequate in others since essentially new phenomena can occur in nonlinear systems which cannot occur in linear systems. Thus, correct solution of a nonlinear system is much more significant a matter than simply getting more accuracy when we solve the nonlinear system rather than a linearized approximation. If we want to know how a physical system behaves, it is essential to retain the nonlinearity for complete understanding of behavior despite the convenience of linearity and superposition. Physical problems are nonlinear: linearity is a special case just as a deterministic system is a special case of a stochastic system. In a linear system, cause and effect are proportional. Such a linear relation sometimes occurs but is the exception rather than the rule. The general case is nonlinear and may be stochastic as well. In such cases, it is natural to make limiting assumptions—which is not always justified. Using decomposition, these become unnecessary even for the strongly nonlinear case and the case of stochastic (large fluctuation) behavior, as well as in the cases where perturbation would be applicable or in the linear and/or deterministic limits. "Smallness" assumptions, linearized models, or assumption of sometimes physically unrealistic processes may result, of course, in mathematical simplicity but again may not be justified in all circumstances.

Here we are concerned with the study of vibrations, or equivalently with oscillatory motion and the associated forces. Vibrations can occur in any mechanical system having mass and elasticity. Consequently, they can occur in structures and machines of all kinds. In proposed large space structures containing men and machines, such vibrations will result in difficult and crucial control problems and also lifetime or duration considerations, since vibrations can lead to eventual failure.

Oscillations can be regular and periodic, or they can be random as in an earthquake. Randomness leads to stochastic differential equations. In

deterministic systems—the special case where randomness vanishes—the equations modelling the phenomena or system provide instantaneous values for any time. When random functions are involved, the instantaneous values are unpredictable and it is necessary to resort to a statistical description. Such random functions of time, or stochastic processes occur in problems, for example, such as pressure gusts encountered by aircraft, jet engine noise, or ground motion in earthquakes so that F may be a nonlinear stochastic operator in the most general case.* In such cases, we write

$$\mathcal{F}u \equiv Lu + Ru + \mathcal{R}y + Nu + \mathcal{N}u = g$$

where the script letters indicate stochasticity. Still more generally, Nu may be a function of $u, u', \dots$ as well, but this causes no difficulty. In any case Nu and $\mathcal{N}u$ can be written in terms of the A_n . Although convergence of the decomposition series for y will be most rapid when we invert the entire linear deterministic operator, computation of the integrals will, of course, be more difficult also since we will not then have simple Green's functions. We will let L denote the highest-order linear differential operator.

In an oscillator we have generally an external force or driving term $x(t)$, a restoring force $f(u)$ dependent on the displacement u , and a damping force, since energy is always dissipated in friction or resistance to motion. Usually this is dependent on velocity and we will write it as $g(u')$.

If we have a free oscillating mass m on a spring with no damping, we can write $mu'' + ku = 0$ if the spring obeys Hooke's Law, i.e., assuming displacement proportional to force. Of course no spring really behaves this way. Often the force needed for a given compression is not the same as for an extension of the same amount. Such asymmetry is represented by a quadratic force, or force proportional to u^2 rather than u . We may have a symmetric behavior but proportionality to u^3 . Then the solution is not the harmonic solution which one gets for the model equation $mu'' + ku = 0$ though it is still a periodic solution. The damping force $g(u')$ may be u'' where c is constant, or it may be more complicated such as $g(u', u'^2)$ so it depends on v^2 as well as v . By usual methods, analytic solutions then become impossible.

* When the highest derivative appears in a nonlinear term, multiple solutions or branches exist [1].

Suppose we write $-f(u)$ for the restoring force, $-h(u')$ for the damping force, and represent the driving force with g ; the resulting equation will be $u'' + f(u) + h(u') = g$. Suppose the restoring force is represented by an odd function so that $f(u) = -f(-u)$. We have this in most applications; it means simply that if we reverse the displacement then the restoring force reverses its direction. A pendulum, for example, behaves this way. We might take the first two terms of the power series for $f(u)$ and write $f(u) = \alpha u + \beta u^3$. Then we have $u'' + \alpha u + \beta u^3 = g$. If we have damping also, we have

$$u'' + cu' + \alpha u + \beta u^3 = g$$

assuming the damping force is $-cu'$. (This is Duffing's equation [2].)

The simple case of the harmonic oscillator $mu'' + ku = g$, or the case avoiding the assumption of limited motion, has been discussed completely in [3], and we will consider more realistic cases here with damping and nonlinearity. We might note, however, that if instead of $\sin u \approx u$, we go a step further and write $\sin u = u - u^3/3!$ we get the Duffing equation with ε as a small parameter, i.e., a perturbation result. It is clear then that we may well have other nonlinearities than u^3 , which we consider in the following sections.

The Duffing oscillator in a random force field modelled by $u'' + \alpha u' + \beta u + \gamma u^3 = g(t)$ can be analyzed without limiting the force $g(t)$ to a white noise and allowing α, β, γ to be stochastic processes as well. The same applies to the Van der Pol oscillator modelled by $u'' + \xi u^2 u' - \zeta u' + u = g(t)$. These equations are in our standard form $Fu = g(t)$ which can be solved by the decomposition method [2-5]. If the equation is linear and deterministic, we have simply $Lu = g$ or $Lu + Ru = g$.

THE DUFFING AND VAN DER POL OSCILLATOR EQUATIONS AND REAL-LIFE PHYSICAL PHENOMENA:

Suppose we make measurements in the laboratory and observe a function $f(u)$ in a "Duffing" experiment and find an odd function $f(u) = b_1 u + b_3 u^3 + b_5 u^5 + \dots$ or $f(u) = \sum_{n=0}^{\infty} b_n u^{2n+1}$ as shown in Figure 1

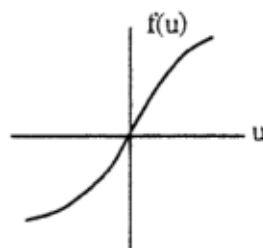


Figure 1

or, on the other hand, we observe a measurement in a "Van der Pol" experiment which yields an even function $f(u) = b_0 + b_1 u^2 + b_2 u^4 + \dots$ or $f(u) = \sum_{n=0}^{\infty} b_n u^{2n}$ as in Figure 2.

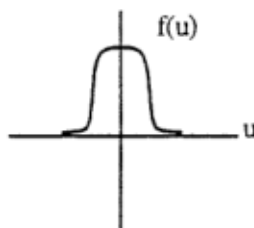


Figure 2

In the Duffing case our nonlinear oscillator equation is

$$u'' + \alpha u' + f(u) = g(t) \quad (1)$$

$$u'' + \alpha u' + \sum_{n=0}^{\infty} b_n u^{2n+1} = g(t)$$

or

$$u'' + \alpha u' + b_0 u + b_1 u^3 + b_2 u^5 + \dots = g(t)$$

and if we retain only the first term of the summation,

$$u'' + \alpha u' + \beta u + \gamma u^3 = g(t)$$

Thus equation (1) subsumes the Duffing equation.

Similarly, we can consider an oscillator equation with nonlinear damping subsuming the Van der Pol oscillator equation:

$$u'' + f(u)u' + \beta u = g(t)$$

$$u'' + \left\{ \sum_{n=0}^{\infty} b_n u^{2n} \right\} u' + \beta u = g(t)$$

$$u'' + \{b_0 + b_1 u^2 + b_2 u^4 + \dots\} u' + \beta u = g(t)$$

If we retain only the summation to $n = 1$, we have the Van der Pol equation

$$u'' + \alpha u' + \beta u + \gamma u' u^2 = g(t)$$

with $u(0) = c_0$ and $u'(0) = c_1$. We see that these equations, as commonly used, are simply first-order perturbations of the real physical models.

Mathematics has progressed considerably using linearity and linear operator theory. Nonlinear differential equations derived for physical phenomena, e.g., in electronic devices, have utilized perturbation theory or linearization of actual behavior. This is so pervasive in the training and acceptance of what is possible that models of physical phenomena may be oversimplified under the assumption that considering the true behavior will represent serious difficulties in the analysis. It is hoped that the decomposition method may contribute to the development of more sophisticated models and result in physically realistic solutions to frontier problems.

DIFFERENTIAL EQUATIONS WITH EMPIRICAL NONLINEARITIES:

Nonlinearities which are specified only through experimental measurements then require curve-fitting techniques yielding series representations. We will consider a generic anharmonic oscillator (subsuming the cases of Duffing and Van der Pol oscillators [2])

$$u'' + \alpha(u, u') + \beta(u) = g(t)$$

with $u(0) = c_0$, $u'(0) = c_1$, $g(t) = \sum_{n=0}^{\infty} g_n t^n$ with $\alpha(u, u')$ and $\beta(u)$ assumed to be given as empirical graphs (i.e., as a plotted surface for α and a plotted contour for β). The curve-fitting procedures result in:

$$\alpha(u, u') = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} u^m u'^n$$

$$\beta(u) = \sum_{m=0}^{\infty} \beta_m u^m$$

Then the equation

$$u'' + \alpha(u, u') + \beta(u) = g(t)$$

becomes

$$u'' + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} u^m u'^n + \sum_{m=0}^{\infty} \beta_m u^m = \sum_{n=0}^{\infty} g_n t^n$$

Using the decomposition and the polynomials $A_n\{f(u, v)\}$, discussed in Chapter 3, and the customary $A_n[f(u)]$ which we will now call B_n , we can write

$$\alpha(u, u') = \sum_{n=0}^{\infty} A_n(u_0, \dots, u_n; u'_0, \dots, u'_n)$$

$$\beta(u) = \sum_{n=0}^{\infty} B_n(u_0, \dots, u_n)$$

$$\begin{aligned} \beta(u) &= \sum_{\mu=0}^{\infty} B_{\mu}[\beta(u)] = \sum_{m=0}^{\infty} \beta_m u^m = \sum_{m=0}^{\infty} \beta_m \sum_{\mu=0}^{\infty} A_{\mu}[u^m] \\ &= \sum_{\mu=0}^{\infty} \left\{ \sum_{m=0}^{\infty} \beta_m A_{\mu}[u^m] \right\} \end{aligned}$$

i.e.,

$$B_{\mu}[\beta(u)] = \sum_{m=0}^{\infty} \beta_m A_{\mu}[u^m]$$

$$\begin{aligned} \alpha(u, u') &= \sum_{\mu=0}^{\infty} A_{\mu}[\alpha(u, u')] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} u^m u'^n \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} \sum_{\mu=0}^{\infty} A_{\mu}[u^m] \cdot \sum_{\nu=0}^{\infty} A_{\nu}[u'^n] \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} A_{\mu-\nu}[u^m] \cdot A_{\nu}[u'^n] \\ &= \sum_{\mu=0}^{\infty} \left\{ \sum_{\nu=0}^{\mu} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} A_{\mu-\nu}[u^m] \cdot A_{\nu}[u'^n] \right\} \end{aligned}$$

i.e.

$$A_{\mu}[\alpha(u, u')] = \sum_{\nu=0}^{\mu} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} A_{\mu-\nu}[u^m] \cdot A_{\nu}[u'^n]$$

where the A_n , B_n are now specified. We can now use the decomposition method to write

$$Lu = g(t) - \beta(u) - \alpha(u, u')$$

where $L = \partial^2/\partial t^2$ and L^{-1} is the two-fold integration from 0 to t . Operating with L^{-1} , and substituting

$$u = u_0 - L^{-1}\beta(u) - L^{-1}\alpha(u, u')$$

$$u_0 = c_0 + c_1 t + L^{-1}g(t)$$

$$u_{m+1} = -L^{-1}B_m - L^{-1}A_m \quad (m \geq 0)$$

we can write $u = \sum_{m=0}^{\infty} u_m$ and the approximant $\phi_m[u] = \phi_m = \sum_{n=0}^{m-1} u_n$.

Now if we approximate the input function $g(t) = \sum_{n=0}^{\infty} g_n t^n$ by the m th order approximant

$$\phi_m[g] = \sum_{n=0}^{m-1} g_n t^n$$

then we can compute the corresponding m th order simulant to the solution $\sigma_m[u]$ or σ_m which satisfies the equation

$$\sigma'' + \alpha(\sigma_m, \sigma'_m) + \beta(\sigma_m) = \phi_m[g]$$

$$\sigma_m(0) = c_0$$

$$\sigma'_m(0) = c_1$$

Thus the simulant to the solution σ_m is the result when $\phi_m[g]$ is used for the input.

To summarize for a given pre-set precision, we need only approximate the input and the nonlinearities to compute a convergent sequence of solution simulants which approach the solution more and more closely as the series for the approximants are carried farther. Obviously, the techniques discussed can be valuable in solid-state or vacuum tube electronics and device simulation. The outcome should be useful in getting realistic models. The Van der Pol equation, for example, is assumed to have the $u'u^2$ nonlinearity and we have used $\alpha(u, u')$. By using the "best" empirical nonlinearity, we are in a better position to refine the model.

Professor S.N. Venkatarangan (Indian Institute of Technology at Madras) and his students have prepared several papers and dissertations nearing publication using the decomposition concept. In [6] he finds a closed form

solution for a (particular) Duffing equation by applying a Laplace transform to the decomposition series, then converting the transformed series into a meromorphic function by forming its Padé approximant and finally doing the inversion. The technique was also applied to the Van der Pol equation and the Rayleigh equation.

Professors F. Jin-Quing and Y. Wei-Guang (China Institute of Atomic Energy) have developed computer programs using the decomposition method to study accuracy of the solution of the Duffing equation and for the first time to study chaotic behavior [7]. The error is only 0.0001% in four terms, which corresponds closely to our results.

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SOLUTION OF THE DUFFING EQUATION

THE DUFFING EQUATION:

Consider the Duffing equation with variable excitation and constant coefficients α, β, γ

$$\begin{aligned}u'' + \alpha u' + \beta u + \gamma u^3 &= \delta(t) \\ u(0) &= c_0 \quad u'(0) = c_1\end{aligned}$$

$\delta(t)$ will be written as a series $\delta(t) = \sum_{n=0}^{\infty} \delta_n t^n$. Let $L = d^2/dt^2$. Then L^{-1} will be the two-fold integration from 0 to t .

$$Lu = \delta(t) - \alpha u' - \beta u - \gamma u^3$$

Operating with L^{-1} ,

$$\begin{aligned}L^{-1}Lu &= L^{-1}\delta(t) - \alpha L^{-1}u' - \beta L^{-1}u - \gamma L^{-1}u^3 \\ u &= u(0) + tu'(0) + L^{-1}\delta(t) - \alpha L^{-1}u' - \beta L^{-1}u - \gamma L^{-1}u^3\end{aligned}$$

Replace u by $\sum_{n=0}^{\infty} u_n$ and the nonlinearity u^3 by $\sum_{n=0}^{\infty} A_n$. We have

$$\begin{aligned}A_0 &= u_0^3 \\ A_1 &= 3u_0^2 u_1 \\ A_2 &= 3u_0^2 u_2 + 3u_1^2 u_0 \\ A_3 &= u_1^3 + 3u_0^2 u_3 + 6u_0 u_1 u_2 \\ &\vdots\end{aligned}$$

(A_0 through A_{10} are listed in Chapter 2 for reference.) A convenient algorithm which also gives us the correct result for this specific case is

$$A_n = \sum_{\nu=0}^n \sum_{\mu=0}^{\nu} u_{n-\nu} u_{\nu-\mu} u_{\mu}$$

We identify

$$\begin{aligned}
u_0 &= u(0) + tu'(0) + L^{-1}\delta(t) \\
u_1 &= -\alpha L^{-1}(d/dt)u_0 - \beta L^{-1}u_0 - \gamma L^{-1}A_0(u_0) \\
u_2 &= -\alpha L^{-1}(d/dt)u_1 - \beta L^{-1}u_1 - \gamma L^{-1}A_1(u_0, u_1) \\
&\vdots \\
u_{m+1} &= -\alpha L^{-1}(d/dt)u_m - \beta L^{-1}u_m - \gamma L^{-1}A_m(u_0, \dots, u_m)
\end{aligned}$$

The solution is the convergent series $\sum_{m=0}^{\infty} u_m$ and $\varphi_m = \sum_{m=0}^{m-1} u_m$ is the approximant to the solution.

Although our definition of L as only the highest-ordered derivative rather than the entire linear operator avoids difficult Green's functions, we still have integrations of the function $\delta(t)$ and the A_n . If $\delta(t)$ is a function such as $\sin \omega t$, the nonlinearity results in a third power of $\sin \omega t$, and we see that a proliferation of terms and computations can occur. (see page 254). We observe, however, that we don't calculate u but a rapidly converging approximant φ_n . Since this is the case, we need not use $\delta(t)$ but an approximant of its series, or, $\varphi_m[\delta] = \sum_{n=0}^{m-1} \delta_n t^n$. The corresponding solution is called the simulant $\sigma_m[u]$. It satisfies, in this problem, the equation

$$\begin{aligned}
\sigma_m'' + \alpha \sigma_m' + \beta \sigma_m + \gamma \sigma_m^3 &= \varphi_m[\delta] \\
\sigma_m(0) = c_0 \quad \sigma_m'(0) &= c_1
\end{aligned}$$

with $\sigma_m[\delta] = \sum_{n=0}^{m-1} \delta_n t^n$. Of course, $\lim_{m \rightarrow \infty} \varphi_m[\delta] = \delta(t)$ and $\lim_{m \rightarrow \infty} \sigma_m[u] = u$.

Consider an example with $\alpha = \beta = \gamma = 1$ and $\delta(t) = e^{-t} \sin x + e^{-3t} \sin^3 x$. Solution by decomposition will converge to $u = e^{-t} \sin x$. But we choose to approximate e^{-t} by the terms up to but not including the cubic, so $e^{-t} = 1 - t + (t^2/2)$. For $\sin x$ we write x . Now $g = x - xt + (xt^2/2)$ and $L^{-1}g = xt^2/2$ since we are dropping cubic terms and beyond. Hence,

$$\begin{aligned}
u_0 &= \sin x - t \sin x + L^{-1}g = x - xt + (xt^2/2) \\
u_1 &= 0
\end{aligned}$$

so we have

$$\sigma_1 = \left(1 - t + \frac{t^2}{2}\right)(x)$$

for our "solution" or simulant for our approximant of δ which, of course, is $e^{-t} \sin x$ to the same approximation as used for the sinusoidal and exponential functions. If σ_m is the m th order simulant, $\varphi_\mu[u] = \varphi_\mu[\sigma_m]$.

If we carry more terms for $\varphi_m[\delta]$, we can identify correspondingly more terms of $\sigma_m[u]$ or, in the limit, $u = e^{-t} \sin x$. As soon as we recognize, or think we recognize, a known series, we can verify whether the equation and conditions are satisfied.

If we do not recognize the series, we can verify that computation of more terms, either for the actual solution u with the excitation δ , or for $\sigma_m[u]$ with the excitation $\varphi_m[\delta]$, yields results which have converged sufficiently for the decimal places of interest, remembering that we are interested in physical problems. We can plot results for $\varphi_m[u]$, or $\sigma_m[u]$, for $m = 1, 2, 3, \dots, N$ to show the convergence and establish a solution approximant to a sufficient accuracy.

We note that for a specified m th order approximant of the excitation δ , we obtain the m th order simulant σ_m of the solution. If we approximate to, and including, cubic terms, we get

$$u_0 = \sigma_1[u] = \left(1 - t + \frac{t^2}{2} - \frac{t^3}{3!}\right) \left(x - \frac{x^3}{3!}\right) = e^{-t} \sin x$$

i.e., to the same approximation with $u_1 = u_2 = \dots = 0$, we have $u = e^{-t} \sin x$.

If one chooses to use the function δ rather than $\varphi_m[\delta]$, and the result is not a known series, one doesn't simply keep calculating with the straightforward decomposition: a stopping rule is required, i.e., there is no point in further calculation, with the proliferation of terms which result from nonlinearity, if the results are past the necessary accuracy. If we need three decimal places and the solution has stabilized at least that far, it is sufficient. Consider now the example:

$$y'' + 2y' + y + 8y^3 = e^{-3t}$$

$$y(0) = 1/2 \quad y'(0) = -1/2$$

Suppose we first write $e^{-3t} \cong 1 - 3t$. We have

$$y_0 = (1/2)(1 - t + t^2 - t^3)$$

$$y_1 = -2L^{-1} \frac{d}{dt} y_0 - L^{-1} y_0 - 8L^{-1} y_0^3 \quad \text{since } A_0(y^3) = y_0^3$$

$$y_2 = -2L^{-1} \frac{d}{dt} y_1 - L^{-1} y_1 - 8L^{-1} A_1(y^3)$$

$$\vdots$$

$$A_0 = y_0^3$$

$$A_1 = 3y_0^2 y_1$$

$$A_2 = 3y_0^2 y_2 + 3y_0 y_1^2$$

$$A_3 = 3y_0^2 y_3 + 6y_0 y_1 y_2 + y_1^3$$

$$A_4 = 3y_0^2 y_4 + 3y_0 y_2^2 + 6y_0 y_1 y_3 + 3y_1^2 y_2$$

$$A_5 = 3y_0^2 y_5 + 6y_0 y_1 y_4 + 6y_0 y_2 y_3 + 3y_1^2 y_3 + 3y_1 y_2^2$$

$$A_6 = y_2^3 + 6y_1 y_2 y_3 + 3y_1^2 y_4 + 3y_0 y_3^2$$

$$+ 6y_0 y_2 y_4 + 6y_0 y_1 y_5 + 3y_0^2 y_6$$

$$\vdots$$

$$y_1 = -2L^{-1} \frac{d}{dt} \frac{1}{2} (1 - t + t^2 - t^3) - L^{-1} \frac{1}{2} (1 - t + t^2 - t^3)$$

$$- 8L^{-1} \frac{1}{8} (1 - 3t + 6t^2 - 10t^3 + \dots)$$

$$= \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^4}{4} - \frac{t^2}{4} + \frac{t^3}{12} - \frac{t^4}{12} + \frac{t^5}{20} - \frac{t^2}{2} + \frac{t^3}{2} - \frac{t^4}{2} + \dots$$

$$\varphi_2 = \frac{1}{2} (1 - t + t^2 - t^3) + \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^2}{4} + \frac{t^3}{12}$$

$$- \frac{t^4}{12} + \frac{t^5}{20} - \frac{t^2}{2} + \frac{t^3}{2} - \frac{t^4}{2} + \dots$$

$$= \frac{1}{2} \left(1 - t + \frac{t^2}{2} + \dots \right) \dots$$

$$= \frac{1}{2} e^{-t} + N$$

Substitution into the differential equation shows $N = 0$ so $(1/2)e^{-t}$ is the (exact) solution. Alternatively, we can compute more terms and group together terms of the same power. If we approximate e^{-t} with another term of the

series, we get

$$y_0 = \frac{1}{2} \left(1 - t + t^2 - t^3 + \frac{3t^4}{4} \right)$$

$$y_1 = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{3t^5}{20} - \frac{t^4}{4} + \frac{t^3}{12} - \frac{t^4}{24} + \frac{t^5}{40}$$

$$- \frac{t^6}{40} - \frac{t^2}{2} + \frac{t^3}{2} - \frac{t^5}{2} + \dots$$

so that the two-term approximation φ_2 is given by

$$\varphi_2 = \frac{1}{2} \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} \right) + \dots$$

so we have another term of e^{-t} .

Finally, we can use numerical results with increasing n for φ_n to show that the results are converging to the solution, i.e., there is no further change within the accuracy of our graph or table.

If we calculate the one-term approximant $\varphi_1 = y_0$ for the solution using three terms of the series for the forcing function, we get the following results:

t	$(1/2)e^{-t}$	φ_1
0	.500	.50
.1	.452	.45
.2	.409	.42
.3	.370	.38
.4	.335	.35
.5	.303	.31

Convergence to the exact solution will be best if the actual forcing function, or at least more terms of its series are used. In practice, φ_n will consist of very few terms. Another example is given by

$$u'' + u' + u + u^3 = \cos^3 t - \sin t$$

$$u(0) = 1 \qquad u'(0) = 0$$

The solution by decomposition with the above excitation is $u = \cos t$, obtained more easily if we approximate the series represented by the above excitation. (Those already familiar with the asymptotic decomposition method will see immediately that the solution for $t \rightarrow \infty$ is also $\cos t$. Hence, $u = \cos t$ is the solution for all t .)

There is no problem in carrying out solutions where any or all of the parameters α, β, γ , as well as the excitation δ , are time-varying functions.

ASYMPTOTIC DECOMPOSITION FOR THE DUFFING EQUATION:

Consider the case of unity coefficients for convenience:

$$u'' + u' + u + u^3 = \delta$$

Since we are interested in the solution as $t \rightarrow \infty$, we introduce the notation

$$\Omega[u] = \lim_{t \rightarrow \infty} u(t) = \lim_{t \rightarrow \infty} \lim_{m \rightarrow \infty} \varphi_m$$

for the asymptotic solution. Of course the φ_m must be computed by the asymptotic decomposition method for the above identity to be valid.

EXAMPLE: Constant δ

$$u^3 = \delta - u - u' - u''$$

$$u = \sum_{n=0}^{\infty} u_n$$

$$u^3 = \sum_{n=0}^{\infty} A_n$$

$$A_0 = \delta = u_0^3$$

Hence

$$u_0 = \delta^{1/3}$$

$$A_1 = -u_0 - u_0' - u_0''$$

or

$$3u_0^2 u_1 = -u_0$$

$$3\delta^{2/3} u_1 = -\delta^{1/3}$$

$$u_1 = -\delta^{1/3}/3$$

$$A_2 = -u_1 - u_1' - u_1''$$

$$3u_0^2 u_2 + 3u_0 u_1^2 = -u_1$$

$$A_3 = -u_2 - u_2' - u_2'' = 0$$

$$3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3 = 0$$

$$3\delta^{5/3} u_3 + 6\delta^{1/3} (-\delta^{1/3}/3)(0) - \delta^{1/3}/27 = 0$$

$$3\delta^{5/3} u_3 = \delta^{1/3}/27$$

$$u_3 = \delta^{-5/3}/81$$

$$\vdots$$

$$\varphi_1 = \delta^{1/3}$$

$$\varphi_2 = \delta^{1/3} - \delta^{1/3}/3 = \varphi_3$$

$$\varphi_4 = \delta^{1/3} - \delta^{1/3}/3 + \delta^{5/3}/3^4 = \delta^{1/3} - 1/3\delta^{1/3} + 1/3^4\delta^{5/3}$$

$$\vdots$$

$$\text{If } \delta = 1, \Omega[u] = 1 - \frac{1}{3} + \frac{1}{81} - \dots$$

$$\text{If } \delta \gg 1, \Omega[u] = \delta^{1/3}$$

$$\text{If } \delta < 1, \text{ the series diverges.}$$

Thus we get results for $\delta \geq 1$ in this approach. Asymptotic decomposition is applicable to linear as well as nonlinear equations; hence,

$$u = \delta - u^3 - u' - u''$$

$$u_0 = \delta$$

$$u_1 = -A_0 - u_0' - u_0''$$

$$u_1 = -u_0^3 = -\delta^3$$

$$u_2 = -A_1 - u_1' - u_1''$$

$$u_2 = -3u_0^2 u_1 = -3\delta^2(-\delta^3)$$

$$u_2 = 3\delta^5$$

$$u_3 = -A_2 - u_2' - u_2''$$

$$u_3 = -3u_0^2 u_2 - 3u_0 u_1^2$$

$$u_3 = -12\delta^7$$

Now

$$\varphi_4 = \delta - \delta^3 + 3\delta^5 - 12\delta^7$$

and we notice that the series converges for $0 < \delta < 1$.

It appears therefore that the magnitude of the excitation must be considered for a convergent result. Thus if $u'' + u' + u + u^3 = \delta$ for $0 < \delta < 1$ we solve for the u , i.e., we write $u = \delta - u^3 - u' - u''$; while if $\delta > 1$, we write $u^3 = \delta - u - u' - u''$. Thus we have a choice in asymptotic decomposition which we can use to advantage to obtain a convergent series. (We also notice that if $\delta(t) = \sin t$ which is between 0 and 1, $\Omega[u]$ behaves like a Fourier sine series.)

STAGGERED SUMMATION TECHNIQUE:

Let's first consider the harmonic oscillator with variable excitation to illustrate procedure:

$$u'' + \alpha u = \beta(t) \quad \alpha \text{ constant}$$

$$u(0) = c_0$$

$$u'(0) = c_1$$

Assume $\beta(t) = \sum_{n=0}^{\infty} \beta_n t^n$.

$$Lu + \alpha u = \beta(t)$$

$$Lu = \beta(t) - \alpha u$$

$$u = u(0) + u'(0)t + L^{-1} \beta(t) - L^{-1} \alpha u$$

$$u_0 = c_0 + c_1 t + \sum_{n=0}^{\infty} \beta_n t^{n+2} / (n+1)(n+2)$$

We now will write u_0 as a series; thus,

$$u_0 = \sum_{n=0}^{\infty} a_n^{(0)} t^n$$

where

$$a_0^{(0)} = c_0$$

$$a_1^{(0)} = c_1$$

$$a_{n+2}^{(0)} = \beta_n / (n+1)(n+2)$$

$$u = u_0 - L^{-1} \alpha u$$

With the decomposition $u = \sum_{n=0}^{\infty} u_n$,

$$\begin{aligned}
 u_1 &= -L^{-1} \alpha u_0 \\
 u_2 &= -L^{-1} \alpha u_1 \\
 u_3 &= -L^{-1} \alpha u_2 \\
 &\vdots \\
 u_m &= -L^{-1} \alpha u_{m-1} = (-L^{-1} \alpha)^m u_0
 \end{aligned}$$

for $m \geq 1$ and $u_0 = \sum_{n=0}^{\infty} a_n^{(0)} t^n$. Hence

$$u_m = (-1)^m \alpha^m \sum_{n=0}^{\infty} \frac{a_n^{(0)} t^{n+2m}}{\prod_{v=1}^{2m} (n+v)}$$

The solution of the equation we began with is $u = \sum_{m=0}^{\infty} u_m$. If we let

$$a_n^{(m)} = \frac{(-1)^m \alpha^m a_n^{(0)}}{\prod_{v=1}^{2m} (n+v)}$$

we can write $u = \sum_{m=0}^{\infty} t^{2m} \sum_{n=0}^{\infty} a_n^{(m)} t^n$. We now rearrange results by staggered summation as shown in the following tabulation.

$u_0 = a_0^{(0)} + a_1^{(0)}t$	$+ a_2^{(0)}t^2 + a_3^{(0)}t^3$	$+ a_4^{(0)}t^4 + a_5^{(0)}t^5$	$+ \dots$
$u_1 =$	$a_0^{(1)}t^2 + a_1^{(1)}t^3$	$+ a_2^{(1)}t^4 + a_3^{(1)}t^5$	$+ \dots$
$u_2 =$		$a_0^{(2)}t^4 + a_1^{(2)}t^5$	$+ \dots$
$u_3 =$			$+ \dots$
$\vdots$			
$u = a_0^{(0)} + a_1^{(0)}t$	$+ (a_2^{(0)} + a_0^{(1)})t^2$ $+ (a_3^{(0)} + a_1^{(1)})t^3$	$+ (a_4^{(0)} + a_2^{(1)} + a_0^{(2)})t^4$ $+ (a_5^{(0)} + a_3^{(1)} + a_1^{(2)})t^5$	$+ \dots$
$u = u_0$	$+ u_1$	$+ u_2$	$+ \dots$

$$u = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \alpha_n t^n$$

Alternatively, the solution components can be computed as follows:

$$u_0 = \sum_{n=0}^{\infty} a_n^{(0)} t^n$$

$$u_1 = -L^{-1} \alpha u_0 = t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n$$

where

$$a_n^{(1)} = \frac{-\alpha a_n^{(0)}}{(n+1)(n+2)}$$

$$u_2 = -L^{-1} \alpha u_1 = t^4 \sum_{n=0}^{\infty} a_n^{(2)} t^n$$

where

$$a_n^{(2)} = \frac{-\alpha a_n^{(1)}}{(n+3)(n+4)}$$

$$u_3 = t^6 \sum_{n=0}^{\infty} a_n^{(3)} t^n$$

$$a_n^{(3)} = \frac{-\alpha a_n^{(2)}}{(n+5)(n+6)}$$

Continuing, we can now write

$$u_m = t^{2m} \sum_{n=0}^{\infty} a_n^{(m)} t^n$$

$$a_n^{(m)} = \frac{-\alpha a_n^{(m-1)}}{(n+2m-1)(n+2m)}$$

Since $u = \sum_{m=0}^{\infty} u_m$,

$$u = \sum_{m=0}^{\infty} t^{2m} \sum_{n=0}^{\infty} a_n^{(m)} t^n$$

which is the previously derived result but may be more convenient for programming. By staggered summation,

$$u = \sum_{m=0}^{\infty} a_m t^m$$

For $m = 0$

$$a_0 = a_0^{(0)}$$

$$a_1 = a_1^{(0)}$$

For $m > 1$,

$$a_{2m} = \sum_{\mu=0}^m a_{2m-2\mu}^{(\mu)}$$

$$a_{2m+1} = \sum_{\mu=0}^m a_{2m-2\mu+1}^{(\mu)}$$

We note that we can write the $a_n^{(m)}$ in terms of the $a_n^{(0)}$

$$\begin{aligned} a_n^{(1)} &= \frac{-\alpha a_n^{(0)}}{(n+1)(n+2)} \\ a_n^{(2)} &= \frac{-\alpha a_n^{(1)}}{(n+3)(n+4)} = \frac{\alpha^2 a_n^{(0)}}{\prod_{v=1}^2 (n+v)} \\ a_n^{(3)} &= \frac{-\alpha a_n^{(2)}}{(n+5)(n+6)} = \frac{-\alpha^3 a_n^{(0)}}{\prod_{v=1}^3 (n+v)} \\ &\vdots \\ a_n^{(m)} &= \frac{(-1)^m \alpha^m a_n^{(0)}}{\prod_{v=1}^{2m} (n+v)} \end{aligned}$$

as expected.

SUMMARY OF FORMULAS:

$$u = \sum_{m=0}^{\infty} t^{2m} \sum_{n=0}^{\infty} a_n^{(m)} t^n$$

where for $m = 0$

$$a_0^{(0)} = c_0$$

$$a_1^{(0)} = c_1$$

$$a_{n+2}^{(0)} = \frac{\beta_n}{(n+1)(n+2)}$$

and for $m \geq 1$

$$a_n^{(m)} = \frac{-\alpha a_n^{(m-1)}}{(n+2m-1)(n+2m)}$$

Since α is constant-valued

$$a_n^{(m)} = \frac{(-1)^m \alpha^m a_n^{(0)}}{\prod_{v=1}^m (n+v)}$$

HARMONIC OSCILLATOR WITH VARIABLE EXCITATION AND SYSTEM COEFFICIENT:

$$u'' + \alpha(t)u = \beta(t)$$

$$\alpha(t) = \sum_{n=0}^{\infty} \alpha_n t^n$$

$$\beta(t) = \sum_{n=0}^{\infty} \beta_n t^n$$

$$u(0) = c_0$$

$$u'(0) = c_1$$

$$L = d^2/dt^2$$

We have

$$Lu + \alpha(t)u = \beta(t)$$

$$Lu = \beta(t) - \alpha(t)u$$

$$L^{-1} Lu = L^{-1} \beta(t) - L^{-1} \alpha(t) u(t)$$

$$u = u(0) + u'(0)t + L^{-1} \beta(t) - L^{-1} \alpha(t) u(t)$$

$$u_0 = u(0) + u'(0)t + L^{-1} \beta(t) = c_0 + c_1 t + L^{-1} \beta(t)$$

$$u_0 = c_0 + c_1 t + L^{-1} \sum_{n=0}^{\infty} \beta_n t^n$$

$$u_0 = c_0 + c_1 t + L^{-1} \sum_{n=0}^{\infty} \beta_n t^{n+2} / (n+1)(n+2)$$

which we can write as $u_0 = \sum_{n=0}^{\infty} a_n^{(0)} t^n$ where

$$a_0^{(0)} = c_0$$

$$a_1^{(0)} = c_1$$

$$a_{n+2}^{(0)} = \frac{\beta_n}{(n+1)(n+2)}$$

The following components are derived from

$$u = u_0 - L^{-1} \alpha(t) \sum_{n=0}^{\infty} u_n$$

Thus

$$u_1 = -L^{-1} \alpha(t) u_0$$

$$u_2 = -L^{-1} \alpha(t) u_1$$

$$u_3 = -L^{-1} \alpha(t) u_2$$

$$\vdots$$

$$u_m = -L^{-1} \alpha(t) u_{m-1} = (-L^{-1} \alpha(t))^m u_0$$

Since

$$\alpha(t) = \sum_{n=0}^{\infty} \alpha_n t^n$$

$$\alpha(t) u_0 = \left(\sum_{n=0}^{\infty} \alpha_n t^n \right) \left(\sum_{n=0}^{\infty} a_n^{(0)} t^n \right) = \sum_{n=0}^{\infty} t^n \left\{ \sum_{v=0}^n \alpha_{n-v} a_v^{(0)} \right\}$$

hence

$$u_1 = -L^{-1} \sum_{n=0}^{\infty} t^n \left\{ \sum_{v=0}^n \alpha_{n-v} a_v^{(0)} \right\}$$

$$u_1 = - \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} \left\{ \sum_{v=0}^n \alpha_{n-v} a_v^{(0)} \right\}$$

$$u_1 = t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n$$

where

$$a_n^{(1)} = \frac{- \sum_{v=0}^n \alpha_{n-v} a_v^{(0)}}{(n+1)(n+2)}$$

$$u_2 = -L^{-1} \alpha(t) u_1$$

$$\alpha(t)u_1 = \left\{ \sum_{n=0}^{\infty} \alpha_n t^n \right\} \left\{ t^2 \sum_{n=0}^n a_n^{(1)} t^n \right\} = t^2 \sum_{n=0}^{\infty} t^n \left\{ \sum_{v=0}^n \alpha_{n-v} a_v^{(1)} \right\}$$

Substituting, we have

$$u_2 = -L^{-1} t^2 \sum_{n=0}^{\infty} t^n \left\{ \sum_{v=0}^n \alpha_{n-v} a_v^{(1)} \right\} = - \sum_{n=0}^{\infty} \frac{t^{n+4}}{(n+3)(n+4)} \left\{ \sum_{v=0}^n \alpha_{n-v} a_v^{(1)} \right\}$$

$$u_2 = t^4 \sum_{n=0}^{\infty} a_n^{(2)} t^n$$

where

$$a_n^{(2)} = \frac{- \sum_{v=0}^n \alpha_{n-v} a_v^{(1)}}{(n+3)(n+4)}$$

$$\alpha(t)u_2 = \left\{ \sum_{n=0}^{\infty} \alpha_n t^n \right\} \left\{ t^4 \sum_{n=0}^{\infty} a_n^{(2)} t^n \right\}$$

$$= t^4 \sum_{n=0}^{\infty} t^n \left\{ \sum_{v=0}^n \alpha_{n-v} a_v^{(2)} \right\}$$

Upon substitution

$$u_3 = -L^{-1} t^4 \sum_{n=0}^{\infty} t^n \left\{ \sum_{v=0}^n \alpha_{n-v} a_v^{(2)} \right\}$$

$$= - \sum_{n=0}^{\infty} \frac{t^{n+6}}{(n+5)(n+6)} \left\{ \sum_{v=0}^n \alpha_{n-v} a_v^{(2)} \right\}$$

$$u_3 = t^6 \sum_{n=0}^{\infty} a_n^{(3)} t^n$$

where

$$a_n^{(3)} = \frac{- \sum_{v=0}^n \alpha_{n-v} a_v^{(2)}}{(n+5)(n+6)}$$

$$\vdots$$

$$u_m = t^{2m} \sum_{n=0}^{\infty} a_n^{(m)} t^n$$

where

$$a_n^{(m)} = \frac{-\sum_{v=0}^n \alpha_{n-v} a_v^{(m-1)}}{(n+2m-1)(n+2m)}$$

Finally, since

$$u = \sum_{m=0}^{\infty} u_m$$

we have the decomposition solution

$$u = \sum_{m=0}^{\infty} t^{2m} \sum_{n=0}^{\infty} a_n^{(m)} t^n$$

This result can also be rearranged by the staggered summation procedure: thus

$$u = \sum_{m=0}^{\infty} a_m t^m$$

where for $m=0$ we have

$$a_0 = a_0^{(0)}$$

$$a_1 = a_1^{(0)}$$

and for $m \geq 1$ we have

$$a_{2m} = \sum_{\mu=0}^m a_{2m-2\mu}^{(\mu)}$$

and

$$a_{2m+1} = \sum_{\mu=0}^m a_{2m-2\mu-1}^{(\mu)}$$

SUMMARY:

$$u = \sum_{m=0}^{\infty} t^{2m} \sum_{n=0}^{\infty} a_n^{(m)} t^n$$

For $m=0$, we have

$$a_0^{(0)} = c_0$$

$$a_1^{(0)} = c_1$$

$$a_{n+2}^{(0)} = \frac{\beta_n}{(n+1)(n+2)}$$

and for $m \geq 1$, we have

$$a_n^{(m)} = \frac{-\sum_{v=0}^n \alpha_{n-v} a_v^{(m-1)}}{(n+2m-1)(n+2m)}$$

HARMONIC OSCILLATOR WITH VARIABLE EXCITATION AND A DAMPING TERM:

$$u'' + \alpha u' - \beta u = \gamma(t)$$

Let a, b be numerical constants and $\gamma(t) = \sum_{n=0}^{\infty} \gamma_n t^n$. Specify

$$u(0) = c_0$$

$$u'(0) = c_1$$

$$L = d^2 / dt^2$$

to get

$$u = u(0) + u'(0)t + L^{-1} \gamma(t) - L^{-1} \alpha u' - L^{-1} \beta u$$

$$u_0 = c_1 + c_2 t + \sum_{n=0}^{\infty} \frac{\gamma_n t^{n+2}}{(n+1)(n+2)}$$

$$u_0 = \sum_{n=0}^{\infty} a_n^{(0)} t^n$$

where

$$a_0^{(0)} = c_0$$

$$a_1^{(0)} = c_1$$

$$a_{n+2}^{(0)} = \frac{\gamma_n}{(n+1)(n+2)}$$

Continuing,

$$u_1 = -L^{-1} \alpha u'_0 - L^{-1} \beta u_0$$

$$u_2 = -L^{-1} \alpha u'_1 - L^{-1} \beta u_1$$

$$\vdots$$

$$u_m = -L^{-1} \alpha u'_{m-1} - L^{-1} \beta u_{m-1}$$

Since $u'_0 = \sum_{n=0}^{\infty} a_{n+1}^{(0)} t^n (n+1) = \sum_{n=0}^{\infty} b_n^{(0)} t^n$ where $b_n^{(0)} = (n+1) a_{n+1}^{(0)}$

$$u_1 = t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n$$

$$a_n^{(1)} = \frac{-\alpha(n+1)a_{n+1}^{(0)} - \beta a_n^{(0)}}{(n+1)(n+2)} = \frac{-\alpha b_n^{(0)} - \beta a_n^{(0)}}{(n+1)(n+2)}$$

The next component u_2 is given by

$$u_2 = -L^{-1} \alpha u_1' - L^{-1} \beta u_1$$

and

$$u_1 = \sum_{n=0}^{\infty} a_n^{(1)} t^{n+2}$$

$$u_1' = \sum_{n=0}^{\infty} (n+2) a_n^{(1)} t^{n+1} = t \sum_{n=0}^{\infty} b_n^{(1)} t^n$$

where $b_n^{(1)} = (n+2)a_n^{(1)}$. Thus,

$$u_1 = t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n$$

$$u_1' = t \sum_{n=0}^{\infty} b_n^{(1)} t^n$$

$$u_2 = t^3 \sum_{n=0}^{\infty} \frac{(-\alpha) b_n^{(1)} t^n}{(n+2)(n+3)} + t^4 \sum_{n=0}^{\infty} \frac{(-\beta) a_n^{(1)} t^n}{(n+3)(n+4)}$$

We can combine these terms to write

$$u_2 = t^3 \sum_{n=0}^{\infty} a_n^{(2)} t^n$$

if

$$a_0^{(2)} = \frac{-\alpha b_0^{(1)}}{(2)(3)}$$

and

$$a_{n+1}^{(2)} = \frac{-\alpha b_{n+1}^{(1)} - \beta a_n^{(1)}}{(n+3)(n+4)}$$

Now going on to u_3 ,

$$u_3 = -L^{-1} \alpha u_2' - L^{-1} \beta u_2$$

Since

$$u_2 = \sum_{n=0}^{\infty} a_n^{(2)} t^{n+3} \text{ or } t^3 \sum_{n=0}^{\infty} a_n^{(2)} t^n$$

$$u_2' = \sum_{n=0}^{\infty} (n+3) a_n^{(2)} t^{n+2} = t^2 \sum_{n=0}^{\infty} b_n^{(2)} t^n$$

with $b_n^{(2)} = (n+3)a_n^{(2)}$,

$$u_2 = t^4 \sum_{n=0}^{\infty} \frac{-\alpha b_n^{(2)} t^n}{(n+3)(n+4)} + t^5 \sum_{n=0}^{\infty} \frac{-\beta a_n^{(2)} t^n}{(n+4)(n+5)}$$

$$u_3 = t^4 \sum_{n=0}^{\infty} a_n^{(3)} t^n$$

where $a_0^{(3)} = -\alpha b_0^{(2)} / (3)(4)$ and

$$a_{n+1}^{(3)} = \frac{-\alpha b_{n+1}^{(2)} - \beta a_n^{(2)}}{(n+4)(n+5)}$$

noting that $b_n^{(m-1)} = (n+m)a_n^{(m-1)}$. Finally, we write

$$u_m = t^{m+1} \sum_{n=0}^{\infty} a_n^{(m)} t^n$$

where $a_0^{(m)} = -\alpha b_0^{(m-1)} / m(m+1)$ and

$$a_{n+1}^{(m)} = \frac{-\alpha b_{n+1}^{(m-1)} - \beta a_n^{(m-1)}}{(n+m+1)(n+m+2)}$$

Thus, for $m=0$,

$$u_0 = \sum_{n=0}^{\infty} a_n^{(0)} t^n$$

and for $m > 0$,

$$u_1 = t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n$$

$$u_2 = t^3 \sum_{n=0}^{\infty} a_n^{(2)} t^n$$

$$u_3 = t^4 \sum_{n=0}^{\infty} a_n^{(3)} t^n$$

$$u_m = t^{m+1} \sum_{n=0}^{\infty} a_n^{(m)} t^n \quad (m > 0)$$

$$u = \sum_{m=0}^{\infty} u_m = u_0 + \sum_{m=1}^{\infty} u_m$$

$$u = \sum_{n=0}^{\infty} a_n^{(0)} t^n + \sum_{m=1}^{\infty} t^{m+1} \sum_{n=0}^{\infty} a_n^{(m)} t^n$$

which is the solution by decomposition.

EXERCISE: For a damped linear oscillator described by $d^2u/dt^2 + 2du/dt + u = 0$ with $u(0) = a$ and $u'(0) = 0$, show that the solution for decomposition is

$$u = a \left\{ a - \frac{t^2}{2!} + 2 \frac{t^3}{3!} - 3 \frac{t^4}{4!} + 4 \frac{t^5}{5!} - \frac{t^6}{6} + \dots \right\}$$

EXERCISE: For the undamped nonlinear (Duffing) oscillator described by $Lu + u + u^3 = 0$ with $u(0) = a$ and $u'(0) = 0$, show that the two-term solution is

$$\phi_2 = a \frac{t^4}{4!} - \frac{a^3 t^4}{4!}$$

EXERCISE: Consider $u'' + \alpha u' + \beta u + \gamma u^3 = g(t)$ with $u(0) = a$, $u'(0) = 0$. Assume $\alpha = 0$, $\beta = \gamma = 1$. For $g(t)$ assume $\sin t$ and approximate with the first two terms of the sine series. Show

$$u_0 = a - \sin t = a - t + t^3/3!$$

$$u_1 = -a^3 + 3a^2 t + 3at^2 - (1 + a^2/2)t^3$$

PROLIFERATION OF TERMS:

In nonlinear equations, where the initial component of the decomposition solution consists of several terms, the nonlinearity may result in a proliferation of terms and consequent increased computation unless proper steps are taken. Also it is sometimes convenient to write the inhomogeneous term as an infinite series to simplify integrations. It may well be the case that the excitation is known as a power series representation. We consider such a case here since it

is a "worst case scenario" from the point of view of proliferation of terms. If one seeks the complete solution (steady-state plus transient), the number of terms in each of the components u_m for $m \geq 1$ can increase rapidly because of the nonlinearity. A number of practical alternatives are possible to solve such apparent problems, and we will show that rapid convergence to the solution will be observed. Summations are utilized to organize the derived results.

We now consider the Duffing equation with variable excitation and constant coefficients.

$$u'' + \alpha u' + \beta u + \gamma u^3 = g(t)$$

We assume given conditions $u(0) = c_0$ and $u'(0) = c_1$ and assume that $g(t) = \sum_{n=0}^{\infty} g_n t^n$ since it is more general, will frequently simplify integrations (e.g., if $g(t) = \cos n\omega t$) and is a worst case from the point of view of computational difficulty arising from the action of the nonlinearity on the initial term ($u_0 = u(0) + tu'(0) + L^{-1}g(t)$). Let $L = d^2/dt^2$ and define L^{-1} as the two-fold definite integration from 0 to t . We have

$$Lu = g(t) - \alpha u' - \beta u - \gamma u^3$$

Operating with L^{-1}

$$u = u(0) + u'(0)t + L^{-1}g(t) - L^{-1}\alpha u' - L^{-1}\beta u - L^{-1}\gamma u^3$$

We identify the initial term

$$u_0 = u(0) + u'(0)t + L^{-1}g(t) = u(0) + u'(0)t + L^{-1} \sum_{n=0}^{\infty} g_n t^n$$

$$u_0 = u(0) + u'(0)t + \sum_{n=0}^{\infty} \frac{g_n t^{n+2}}{(n+1)(n+2)}$$

which we will write as

$$u_0 = \sum_{n=0}^{\infty} a_n^{(0)} t^n$$

where the coefficients are known:

$$a_0^{(0)} = c_0$$

$$a_1^{(0)} = c_1$$

$$a_{n+2}^{(0)} = \frac{g_n}{(n+1)(n+2)}$$

We can now write $u = u_0 - L^{-1}\alpha u' - L^{-1}\beta u - L^{-1}\gamma u^3$ and assume $u = \sum_{n=0}^{\infty} u_n$ and $u^3 = \sum_{n=0}^{\infty} A_n$ where A_n or equivalently, $A_n\{u^3\}$ is calculated for the function u^3 . (We can also write both u and u^3 as sums of the appropriate A_n ; then u becomes simply $u = \sum_{n=0}^{\infty} u_n$ and u' becomes $(d/dt)\sum_{n=0}^{\infty} u_n$. The A_n polynomials for u^3 are

$$A_0 = u_0^3$$

$$A_1 = 3u_0^2u_1$$

$$A_2 = 3u_0^2u_2 + 3u_1^2u_0$$

$$\vdots$$

$$A_m = \sum_{n=0}^m u_{m-n} \sum_{v=0}^n u_{n-v} u_v$$

Algorithms have been previously given to generate the A_n for general nonlinearities; however, the above form is also convenient for polynomials. We now have

$$u = u_0 - L^{-1}\alpha \sum_{n=0}^{\infty} u'_n - L^{-1}\beta \sum_{n=0}^{\infty} u_n - L^{-1}\gamma \sum_{n=0}^{\infty} A_n\{u^3\}$$

and can write the components

$$u_0 = \sum_{n=0}^{\infty} a_n^{(0)} t^n$$

$$u_1 = -L^{-1}\alpha u'_0 - L^{-1}\beta u_0 - L^{-1}\gamma A_0\{u^3\}$$

$$u_2 = -L^{-1}\alpha u'_1 - L^{-1}\beta u_1 - L^{-1}\gamma A_1\{u^3\}$$

$$\begin{aligned}
u_3 &= -L^{-1}\alpha u_2' - L^{-1}\beta u_2 - L^{-1}\gamma A_2\{u^3\} \\
&\vdots \\
u_m &= -L^{-1}\alpha u_{m-1}' - L^{-1}\beta u_{m-1} - L^{-1}\gamma A_{m-1}\{u^3\}
\end{aligned}$$

Since u_0 is given as a power series, we can use the following result:

If $u = \sum_{n=0}^{\infty} a_n t^n$, then $f(u) = \sum_{n=0}^{\infty} A_n t^n$ where the $A_n(a_0, \dots, a_n)$ are simply the A_n polynomials expressed in terms of the coefficients a_n instead of the components $u_0, u_1, \dots$.

Continuing, we compute u_0' and u_1 .

$$\begin{aligned}
u_0 &= \sum_{n=0}^{\infty} a_n^{(0)} t^n \\
u_0' &= a_1^{(0)} + a_2^{(0)} t + \dots = \sum_{n=0}^{\infty} (n+1) a_{n+1}^{(0)} t^n = \sum_{n=0}^{\infty} b_n^{(0)} t^n \\
b_n^{(0)} &= (n+1) a_{n+1}^{(0)} \\
A_0(u_0) &= A_0\{u^3\} = u_0^3 = \sum_{n=0}^{\infty} A_n^{(0)} t^n
\end{aligned}$$

using the above result. Thus

$$\begin{aligned}
A_0^{(0)} &= a_0^3 \\
A_1^{(0)} &= a_1 a_0^2 + a_0 a_1 a_0 + a_0^2 a_1 \\
&\vdots \\
A_n^{(0)} &= \sum_{v=0}^n a_{n-v}^{(0)} \sum_{\mu=0}^v a_{v-\mu}^{(0)} a_{\mu}^{(0)}
\end{aligned}$$

and

$$A_0 = \sum_{n=0}^{\infty} t^n \left\{ \sum_{v=0}^n a_{n-v}^{(0)} \sum_{\mu=0}^v a_{v-\mu}^{(0)} a_{\mu}^{(0)} \right\} = \sum_{n=0}^{\infty} c_n^{(0)} t^n$$

where

$$c_n^{(0)} = \left\{ \sum_{v=0}^n a_{n-v}^{(0)} \sum_{\mu=0}^v a_{v-\mu}^{(0)} a_{\mu}^{(0)} \right\}$$

We can now write

$$\begin{aligned}
u_1 &= -L^{-1}\alpha u'_0 - L^{-1}\beta u_0 - L^{-1}\gamma A_0 \\
&= -L^{-1}\alpha \sum_{n=0}^{\infty} (n+1)a_{n+1}^{(0)}t^n - L^{-1}\beta \sum_{n=0}^{\infty} a_n^{(0)}t^n - L^{-1}\gamma \sum_{n=0}^{\infty} A_n^{(0)}t^n \\
&= -\alpha \sum_{n=0}^{\infty} (n+1)a_{n+1}^{(0)}t^{n+2} / (n+1)(n+2) \\
&\quad -\beta \sum_{n=0}^{\infty} a_n^{(0)}t^{n+2} / (n+1)(n+2) \\
&\quad -\gamma \sum_{n=0}^{\infty} A_n^{(0)}t^{n+2} / (n+1)(n+2) \\
&= -\sum_{n=0}^{\infty} t^{n+2} \left\{ \alpha(n+1)a_{n+1}^{(0)} + \beta a_n^{(0)} + \gamma A_n^{(0)} \right\} / (n+1)(n+2)
\end{aligned}$$

We see that u_1 is known in terms of the $a_n^{(0)}$ and that we can write the equation for u_1 in the convenient form:

$$u_1 = t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n \quad \text{or} \quad \sum_{n=0}^{\infty} a_n^{(1)} t^{n+2}$$

where

$$\begin{aligned}
a_n^{(1)} &= \frac{-\alpha b_n^{(0)} - \beta a_n^{(0)} - \gamma c_n^{(0)}}{(n+1)(n+2)} \\
u'_1 &= \sum_{n=0}^{\infty} (n+2)a_n^{(1)}t^{n+1} = t \sum_{n=0}^{\infty} b_n^{(1)}t^n
\end{aligned}$$

where

$$\begin{aligned}
b_n^{(1)} &= (n+2)a_n^{(1)} \\
A_1 &= A_1(u_0, u_1) = 3u_1u_0^2
\end{aligned}$$

where

$$\begin{aligned}
u_0 &= \sum_{n=0}^{\infty} a_n^{(0)}t^n \\
u_1 &= t^2 \sum_{n=0}^{\infty} a_n^{(1)}t^n \\
A_1 &= 3t^2 \left[\sum_{n=0}^{\infty} a_n^{(1)} \sum_{n=0}^{\infty} a_n^{(0)} \sum_{n=0}^{\infty} a_n^{(0)} \right] t^n
\end{aligned}$$

where

$$c_n^{(1)} = 3 \left\{ \sum_{v=0}^n a_{n-v}^{(1)} \sum_{\mu=0}^v a_{v-\mu}^{(0)} a_{\mu}^{(0)} \right\}$$

We now have

$$u_1 = t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n$$

$$u_1' = t \sum_{n=0}^{\infty} b_n^{(1)} t^n$$

$$A_1 = t^2 \sum_{n=0}^{\infty} c_n^{(1)} t^n$$

and

$$\begin{aligned} u_2 = & -L^{-1} \alpha u_1' - L^{-1} \beta u_1 - L^{-1} \gamma A_1 = t^3 \sum_{n=0}^{\infty} \frac{(-\alpha) b_n^{(1)} t^n}{(n+2)(n+3)} \\ & + t^4 \sum_{n=0}^{\infty} \frac{(-\beta) a_n^{(1)} t^n}{(n+3)(n+4)} + t^4 \sum_{n=0}^{\infty} \frac{(-\gamma) c_n^{(1)} t^n}{(n+3)(n+4)} \end{aligned}$$

Thus,

$$\begin{aligned} u_2 = & t^3 \left\{ \frac{-\alpha b_0^{(1)}}{2 \cdot 3} - \frac{\alpha b_1^{(1)} t}{3 \cdot 4} - \frac{\alpha b_2^{(1)} t^2}{4 \cdot 5} - \dots \right\} \\ & + t^4 \left\{ \frac{-\beta a_0^{(1)}}{3 \cdot 4} - \frac{\beta a_1^{(1)} t}{4 \cdot 5} - \frac{\beta a_2^{(1)} t^2}{5 \cdot 6} - \dots \right\} \\ & + t^4 \left\{ \frac{-\gamma c_0^{(1)}}{3 \cdot 4} - \frac{\gamma c_1^{(1)} t}{4 \cdot 5} - \frac{\gamma c_2^{(1)} t^2}{5 \cdot 6} - \dots \right\} \\ u_2 = & t^3 \left\{ \frac{-\alpha b_0^{(1)}}{2 \cdot 3} - \frac{\alpha b_1^{(1)} t}{3 \cdot 4} - \frac{\alpha b_2^{(1)} t^2}{4 \cdot 5} - \dots \right\} \\ u_2 = & t^3 \left\{ \frac{-\alpha b_0^{(1)}}{2 \cdot 3} \right\} + t^3 \left\{ -\frac{\beta a_0^{(1)} t}{3 \cdot 4} - \frac{\gamma c_0^{(1)} t}{3 \cdot 4} \right\} \\ & + t^3 \left\{ \frac{-\beta a_1^{(1)} t^2}{4 \cdot 5} - \frac{\gamma c_1^{(1)} t^2}{4 \cdot 5} \right\} + t^3 \left\{ \frac{\beta a_2^{(1)} t^3}{5 \cdot 6} - \frac{\gamma c_2^{(1)} t^3}{5 \cdot 6} \right\} \\ u_2 = & t^3 \sum_{n=0}^{\infty} a_n^{(2)} t^n \end{aligned}$$

where

$$a_n^{(2)} = \frac{-\alpha b_{n+1}^{(1)} - \beta a_n^{(1)} - \gamma c_n^{(1)}}{(n+3)(n+4)}$$

Going on to u_3

$$u_3 = -L^{-1}\alpha u_2' - L^{-1}\beta u_2 - L^{-1}\gamma A_2$$

Since $u_2 = \sum_{n=0}^{\infty} a_n^{(2)} t^{n+3}$, we have $u_2' = \sum_{n=0}^{\infty} (n+3)a_n^{(2)} t^{n+2}$ or

$$u_2' = t^2 \sum_{n=0}^{\infty} b_n^{(2)} t^n$$

where

$$b_n^{(2)} = (n+3)a_n^{(2)}$$

For A_2 we have

$$A_2 = 3u_2 u_0^2 + 3u_0 u_1^2$$

where

$$u_0 = \sum_{n=0}^{\infty} a_n^{(0)} t^n$$

$$u_1 = t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n$$

$$u_2 = t^3 \sum_{n=0}^{\infty} a_n^{(2)} t^n$$

Hence

$$\begin{aligned} A_2 = & 3t^3 \sum_{n=0}^{\infty} t^n \left\{ \sum_{\nu=0}^n a_{n-\nu}^{(2)} \sum_{\mu=0}^{\nu} a_{\nu-\mu}^{(0)} a_{\mu}^{(0)} \right\} \\ & + 3t^4 \sum_{n=0}^{\infty} t^n \left\{ \sum_{\nu=0}^n a_{n-\nu}^{(0)} \sum_{\mu=0}^{\nu} a_{\nu-\mu}^{(1)} a_{\mu}^{(1)} \right\} \end{aligned}$$

which we write as

$$A_2 = 3t^3 \sum_{n=0}^{\infty} \omega_n t^n + 3t^4 \sum_{n=0}^{\infty} \psi_n t^n$$

and finally

$$A_2 = t^3 \sum_{n=0}^{\infty} c_n^{(2)} t^n$$

where

$$c_0^{(2)} = 3\omega_0 = 3a_0^{(2)} (a_0^{(0)})^2$$

and

$$c_{n+1}^{(2)} = 3\{\omega_{n+1} + \psi_n\}$$

Thus,

$$\begin{aligned} u_2 &= t^3 \sum_{n=0}^{\infty} a_n^{(2)} t^n \\ u_2' &= t^2 \sum_{n=0}^{\infty} a_n^{(2)} t^n \\ A_2 &= t^3 \sum_{n=0}^{\infty} c_n^{(2)} t^n \\ u_3 &= -L^{-1} \alpha u_2' - L^{-1} \beta u_2 - L^{-1} \gamma A_2 \\ u_3 &= t^4 \sum_{n=0}^{\infty} \frac{(-\alpha) a_n^{(2)} t^n}{(n+3)(n+4)} \\ &\quad + t^5 \sum_{n=0}^{\infty} \frac{(-\beta) a_n^{(2)} t^n}{(n+4)(n+5)} \\ &\quad + t^5 \sum_{n=0}^{\infty} \frac{(-\gamma) c_n^{(2)} t^n}{(n+4)(n+5)} \end{aligned}$$

or more simply, as before,

$$u_3 = t^4 \sum_{n=0}^{\infty} a_n^{(3)} t^n$$

where

$$\begin{aligned} a_0^{(3)} &= \frac{-\alpha b_0^{(2)}}{3 \cdot 4} \\ a_{n+1}^{(3)} &= \frac{-\alpha b_{n+1}^{(2)} - \beta a_n^{(2)} - \gamma c_n^{(2)}}{(n+4)(n+5)} \end{aligned}$$

Continuing to $u_4, u_5, \dots$, we can write the m th term

$$\begin{aligned} u_m &= t^{m+1} \sum_{n=0}^{\infty} a_n^{(m)} t^n \\ a_0^{(m)} &= \frac{-\alpha b_0^{(m-1)}}{m(m+1)} \\ a_{n+1}^{(m)} &= \frac{-\alpha b_{n+1}^{(m-1)} - \beta a_n^{(m-1)} - \gamma c_n^{(m-1)}}{(n+m+1)(n+m+2)} \end{aligned}$$

Summarizing, for $m = 0$

$$u_0 = \sum_{n=0}^{\infty} a_n^{(0)} t^n$$

$$u_1 = t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n$$

$$u_2 = t^3 \sum_{n=0}^{\infty} a_n^{(2)} t^n$$

$$u_3 = t^4 \sum_{n=0}^{\infty} a_n^{(3)} t^n$$

$$\vdots$$

$$u_m = t^{m+1} \sum_{n=0}^{\infty} a_n^{(m)} t^n$$

The decomposition solution is $u = \sum_{m=0}^{\infty} u_m$ or $u = u_0 + \sum_{m=1}^{\infty} u_m$ or

$$u = \sum_{n=0}^{\infty} a_n^{(0)} t^n + \sum_{m=1}^{\infty} t^{m+1} \sum_{n=0}^{\infty} a_n^{(m)} t^n$$

We can also write the solution by staggered summation in the form

$$u = \sum_{n=0}^{\infty} a_n t^n$$

where

$$a_0 = a_0^{(0)}$$

$$a_1 = a_1^{(0)}$$

and for $m \geq 2$

$$a_m = a_m^{(0)} + \sum_{\mu=1}^{m-1} a_{m-\mu-1}^{(\mu)}$$

A SUMMARY OF FORMULAS USED:

$$\sum_{n=0}^{\infty} a_n^{(0)} t^n + \sum_{m=1}^{\infty} t^{m+1} \sum_{n=0}^{\infty} a_n^{(m)} t^n$$

$$\begin{aligned}
(m=0) \quad & a_0^{(0)} = c_0 \\
& a_1^{(0)} = c_1 \\
& a_{n+2}^{(0)} = \frac{g_n}{(n+1)(n+2)} \\
(m=1) \quad & a_n^{(1)} = \frac{-\alpha b_n^{(0)} - \beta a_n^{(0)} - \gamma c_n^{(0)}}{(n+1)(n+2)} \\
(m \geq 2) \quad & a_n^{(m)} = \frac{-\alpha b_n^{(m-1)}}{m(m+1)} \\
& a_{n+1}^{(m)} = \frac{-\alpha b_{n+1}^{(m-1)} - \beta a_{n+1}^{(m-1)} - \gamma c_{n+1}^{(m-1)}}{(n+m+1)(n+m+2)}
\end{aligned}$$

where for $m=1$, $b_n^{(0)} = (n+1)a_{n+1}^{(0)}$ and for $m>1$, $b_n^{(m-1)} = (n-m)a_{n+1}^{(m-1)}$. We also have

$$\begin{aligned}
u &= \sum_{m=0}^{\infty} u_m \\
u^3 &= \sum_{m=0}^{\infty} A_m \\
A_m \{u^3\} &= \sum_{n=0}^m u_{m-n} \sum_{v=0}^n u_{n-v} u_v
\end{aligned}$$

ALGORITHMS FOR THE DUFFING EQUATION:

We consider now explicit solution of the Duffing equation

$$u'' + \alpha u' + \beta u + \gamma u^3 = \delta(t)$$

given initial conditions $u(0) = c_0$ and $u'(0) = c_1$. We assume $\delta(t) = \sum_{n=0}^{\infty} \delta_n t^n$.

Our objective is an efficient procedure for calculation of the decomposition series. We have

$$u = u_0 - L^{-1} \alpha u' - L^{-1} \beta u - L^{-1} \gamma u^3$$

where

$$\begin{aligned}
u_0 &= u(0) + t u'(0) + L^{-1} \delta(t) \\
&= c_0 + c_1 t + \int_0^t \int_0^t \sum_{n=0}^{\infty} \delta_n t^n dt dt
\end{aligned}$$

We note that if the input is a sinusoidal function, integrations will soon become difficult because of the effect of the nonlinearity. The result for $\delta(t) = \sin t$ is found by setting $\delta_{2n+1} = (-1)^n / (2n-1)!$ and $\delta_{2n} = 0$.

For $\cos t$, let $\delta_{2n} = (-1)^n / (2n)!$ and $\delta_{2n+1} = 0$. For $\delta(t) = c_0 \cos t + c_1 \sin t$, we can let $\delta_{2n} = (-1)^n c_0 / (2n)!$ and $\delta_{2n+1} = (-1)^n c_1 / (2n+1)!$. Finally, in many cases, a finite series will be sufficient for the required accuracy and we can write $\zeta(t) = \sum_{n=0}^N \zeta_n t^n$, i.e., $\delta_n = \zeta_n$ for $0 \leq n \leq N$ and $\delta_n = 0$ for $n > N$. Conveniently, $\delta(t)$, assumed to be uniformly convergent, can be programmed as an N -component vector with the series truncated to the precision required. Terms after u_0 are given by

$$u_m = -L^{-1} \alpha u'_{m-1} - L^{-1} \beta u_{m-1} - L^{-1} \gamma A_{m-1}$$

where the $A_{m-1} = A_{m-1}\{u^3\} = \sum_{n=0}^m \sum_{v=0}^n u_{m-n} u_{n-v} u_v$, $\varphi_m = \sum_{n=0}^{m-1} u_n$ for $m \geq 1$ is the approximant to the solution. $\varphi_{m-1} = \varphi_m + u_m$, and $\lim_{m \rightarrow \infty} \varphi_m = u$. Now write the initial term u_0 in the convenient form:

$$u_0 = \sum_{n=0}^{\infty} a_n^{(0)} t^n = \varphi_1$$

where

$$a_0^{(0)} = c_0, \quad a_1^{(0)} = c_1, \quad a_{n-2}^{(0)} = \delta_n / (n+1)(n+2)$$

The approximant φ_1 will be written

$$\varphi_1 = \sum_{n=0}^{\infty} b_n^{(1)} t^n$$

where $b_n^{(1)} = a_n^{(0)}$. The u_1 component can now be calculated and therefore the φ_2 approximant, since $\varphi_2 = \varphi_1 + u_1$. Since

$$u_0 = \sum_{n=0}^{\infty} a_n^{(0)} t^n$$

we have

$$u'_0 = \sum_{n=0}^{\infty} (n+1) a_{n+1}^{(0)} t^n$$

Computing A_0 using the given algorithm,

$$A_0 = \sum_{n=0}^{\infty} \sum_{v=0}^n u_{n-v} u_v$$

$$A_0 = u_0^3$$

which we know from our general algorithms for $f(u)$ as well. Using our result for u_0 , we can find A_0 as the Cauchy product:

$$A_0 = \left\{ \sum_{n=0}^{\infty} a_n^{(0)} t^n \right\} \cdot \left\{ \sum_{v=0}^{\infty} a_v^{(0)} t^v \right\} \cdot \left\{ \sum_{\mu=0}^{\infty} a_{\mu}^{(0)} t^{\mu} \right\}$$

or

$$A_0 = \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \sum_{\mu=0}^v a_{n-v}^{(0)} a_{v-\mu}^{(0)} a_{\mu}^{(0)}$$

which we will write as

$$A_0 = \sum_{n=0}^{\infty} A_n^{(0)} t^n$$

where

$$A_n^{(0)} = \sum_{v=0}^n \sum_{\mu=0}^v a_{n-v}^{(0)} a_{v-\mu}^{(0)} a_{\mu}^{(0)}$$

Since $u_1 = -L^{-1} \alpha u_0' - L^{-1} \beta u_0 - L^{-1} \gamma A_0$, we can now write

$$u_1 = -\alpha t^2 \sum_{n=0}^{\infty} \frac{a_{n+1}^{(0)} t^n}{(n+2)} - \beta t^2 \sum_{n=0}^{\infty} \frac{a_n^{(0)} t^n}{(n+1)(n+2)} - \gamma t^2 \sum_{n=0}^{\infty} \frac{A_n^{(0)} t^n}{(n+1)(n+2)}$$

In succinct form, $u_1 = t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n$ where

$$a_n^{(1)} = \frac{-\alpha(n+1)a_{n+1}^{(0)} - \beta a_n^{(0)} - \gamma A_n^{(0)}}{(n+1)(n+2)}$$

We now have $\varphi_2 = \varphi_1 + u_1$ where $\varphi_1 = \sum_{n=0}^{\infty} b_n^{(1)} t^n$ and $u_1 = t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n$; hence

$$\varphi_2 = \sum_{n=0}^{\infty} b_n^{(2)} t^n$$

where

$$\begin{aligned}b_0^{(2)} &= b_0^{(1)} \\ b_1^{(2)} &= b_1^{(1)} \\ b_{n+2}^{(2)} &= b_{n+2}^{(1)} + a_n^{(1)}\end{aligned}$$

Proceeding to the u_2 component and therefore the φ_3 approximant,

$$u_2 = -L^{-1}\alpha u_1' - L^{-1}\beta u_1 - L^{-1}\gamma A_1$$

so that from $u_1 = t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n$ we write

$$u_1' = t \sum_{n=0}^{\infty} (n+2) a_n^{(1)} t^n$$

and

$$A_1 = \sum_{n=0}^{m-1} \sum_{\nu=0}^n u_{m-n} u_{n-\nu} u_1 = 3u_0^2 u_1$$

or the Cauchy product

$$\begin{aligned}A_1 &= 3 \left\{ \sum_{n=0}^{\infty} a_n^{(0)} t^n \right\} \cdot \left\{ \sum_{\nu=0}^{\infty} a_{\nu}^{(0)} t^{\nu} \right\} \cdot \left\{ t^2 \sum_{\mu=0}^{\infty} a_{\mu}^{(1)} t^{\mu} \right\} \\ A_1 &= t^2 \sum_{n=0}^{\infty} A_n^{(1)} t^n\end{aligned}$$

where

$$A_n^{(1)} = 3 \sum_{\nu=0}^n \sum_{\mu=0}^{\nu} a_{n-\nu}^{(0)} a_{\nu-\mu}^{(0)} a_{\mu}^{(1)}$$

Consequently

$$u_2 = -\alpha t^3 \sum_{n=0}^{\infty} \frac{a_n^{(1)} t^n}{(n+3)} - \beta t^4 \sum_{n=0}^{\infty} \frac{a_n^{(1)} t^n}{(n+3)(n+4)} - \gamma t^4 \sum_{n=0}^{\infty} \frac{A_n^{(1)} t^n}{(n+3)(n+4)}$$

or

$$u_2 = t^3 \sum_{n=0}^{\infty} a_n^{(2)} t^n$$

where

$$a_0^{(2)} = -\alpha a_0^{(1)} / 3$$

$$a_{n+1}^{(2)} = \frac{-\alpha(n+3)a_{n+1}^{(1)} - \beta a_n^{(1)} - \gamma A_n^{(1)}}{(n+3)(n+4)}$$

Now we have the approximant $\varphi_3 = \varphi_2 + u_2$ with

$$\varphi_2 = \sum_{n=0}^{\infty} b_n^{(2)} t^n$$

$$u_2 = t^3 \sum_{n=0}^{\infty} a_n^{(2)} t^n$$

$$\varphi_3 = \sum_{n=0}^{\infty} b_n^{(3)} t^n$$

where

$$b_0^{(3)} = b_0^{(2)}$$

$$b_1^{(3)} = b_1^{(2)}$$

$$b_2^{(3)} = b_2^{(2)}$$

$$b_{n+3}^{(3)} = b_{n+3}^{(2)} + a_n^{(2)}$$

Proceeding to the u_3 component and consequently the φ_4 approximant, we have $u_3 = -L^{-1}\alpha u_2' - L^{-1}\beta u_2 - L^{-1}\gamma A_2$ using

$$u_2 = t^3 \sum_{n=0}^{\infty} a_n^{(2)} t^n$$

$$u_2' = t^2 \sum_{n=0}^{\infty} (n+3) a_n^{(2)} t^n$$

$$A_2 = \sum_{n=0}^{m=2} \sum_{v=0}^n u_{m-n} u_{n-v} u_v = 3u_0^2 u_2 + 3u_1^2 u_0$$

$$A_2 = 3 \left\{ \sum_{n=0}^{\infty} a_n^{(0)} t^n \right\} \cdot \left\{ \sum_{v=0}^{\infty} a_v^{(0)} t^v \right\} \cdot \left\{ t^3 \sum_{\mu=0}^{\infty} a_{\mu}^{(2)} t^{\mu} \right\}$$

$$+ 3 \left\{ t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n \right\} \cdot \left\{ t^2 \sum_{v=0}^{\infty} a_v^{(1)} t^v \right\} \cdot \left\{ \sum_{\mu=0}^{\infty} a_{\mu}^{(0)} t^{\mu} \right\}$$

$$= t^3 \sum_{n=0}^{\infty} t^n \cdot 3 \sum_{v=0}^n \sum_{\mu=0}^v a_{n-v}^{(0)} a_{v-\mu}^{(0)} a_{\mu}^{(2)} + t^4 \sum_{n=0}^{\infty} t^n \cdot 3 \sum_{v=0}^n \sum_{\mu=0}^v a_{n-v}^{(1)} a_{v-\mu}^{(1)} a_{\mu}^{(0)}$$

which we will write as

$$A_2 = t^3 \sum_{n=0}^{\infty} A_n^{(2)} t^n$$

where

$$\begin{aligned} A_0^{(2)} &= 3 \sum_{v=0}^{n=0} \sum_{\mu=0}^v a_{n-v}^{(0)} a_{v-\mu}^{(0)} a_{\mu}^{(2)} \\ A_{n+1}^{(2)} &= 3 \sum_{v=0}^{n+1} \sum_{\mu=0}^v a_{n+1-v}^{(0)} a_{v-\mu}^{(0)} a_{\mu}^{(2)} \\ &\quad + 3 \sum_{v=0}^n \sum_{\mu=0}^v a_{n-v}^{(1)} a_{v-\mu}^{(1)} a_{\mu}^{(0)} \end{aligned}$$

Thus

$$\begin{aligned} u_3 &= -\alpha t^4 \sum_{n=0}^{\infty} \frac{a_n^{(2)} t^n}{(n+4)} - \beta t^5 \sum_{n=0}^{\infty} \frac{a_n^{(2)} t^n}{(n+4)(n+5)} \\ &\quad - \gamma t^5 \sum_{n=0}^{\infty} \frac{A_n^{(2)} t^n}{(n+4)(n+5)} \end{aligned}$$

or

$$u_3 = t^4 \sum_{n=0}^{\infty} a_n^{(3)} t^n$$

where

$$\begin{aligned} a_0^{(3)} &= -\alpha a_0^{(2)} / 4 \\ a_{n+1}^{(3)} &= \frac{-\alpha(n+4)a_{n+1}^{(2)} - \beta a_n^{(2)} - \gamma A_n^{(2)}}{(n+4)(n+5)} \end{aligned}$$

The next approximant $\varphi_4 = \varphi_3 + u_3$ using

$$\varphi_3 = \sum_{n=0}^{\infty} b_n^{(3)} t^n$$

$$u_3 = t^4 \sum_{n=0}^{\infty} a_n^{(3)} t^n$$

so that

$$\varphi_4 = \sum_{n=0}^{\infty} b_n^{(4)} t^n$$

with $b_0^{(4)} = b_0^{(3)}$, $b_1^{(4)} = b_1^{(3)}$, $b_2^{(4)} = b_2^{(3)}$, $b_3^{(4)} = b_3^{(3)}$, and $b_{n+4}^{(4)} = b_{n+4}^{(3)} + a_n^{(3)}$

Continuing to the u_4 component and the φ_5 approximant,

$$u_4 = -L^{-1}\alpha u_3' - L^{-1}\beta u_3 - L^{-1}\gamma A_3$$

for which we need

$$u_3 = t^4 \sum_{n=0}^{\infty} a_n^{(3)} t^n$$

$$u_3' = t^3 \sum_{n=0}^{\infty} (n+4) a_n^{(3)} t^n$$

$$A_3 = \sum_{n=0}^{\infty} \sum_{v=0}^n u_{n-n} u_{n-v} u_v$$

$$A_3 = u_1^3 + 3u_0^2 u_3 + 6u_0 u_1 u_2$$

$$\begin{aligned} A_3 = & \left\{ t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n \right\} \cdot \left\{ t^2 \sum_{v=0}^{\infty} a_v^{(1)} t^v \right\} \cdot \left\{ t^2 \sum_{\mu=0}^{\infty} a_{\mu}^{(1)} t^{\mu} \right\} \\ & + 3 \left\{ \sum_{n=0}^{\infty} a_n^{(0)} t^n \right\} \cdot \left\{ \sum_{v=0}^{\infty} a_v^{(0)} t^v \right\} \cdot \left\{ t^4 \sum_{\mu=0}^{\infty} a_{\mu}^{(3)} t^{\mu} \right\} \\ & + 6 \left\{ \sum_{n=0}^{\infty} a_n^{(c)} t^n \right\} \cdot \left\{ t^2 \sum_{v=0}^{\infty} a_v^{(1)} t^v \right\} \cdot \left\{ t^3 \sum_{\mu=0}^{\infty} a_{\mu}^{(2)} t^{\mu} \right\} \end{aligned}$$

$$\begin{aligned} A_3 = & t^6 \sum_{n=0}^{\infty} t^n \sum_{v=0}^n \sum_{\mu=0}^v a_{n-v}^{(1)} a_{n-v-\mu}^{(1)} a_{\mu}^{(1)} \\ & + t^4 \sum_{n=0}^{\infty} t^n \cdot 3 \sum_{v=0}^n \sum_{\mu=0}^v a_{n-v}^{(0)} a_{v-\mu}^{(0)} a_{\mu}^{(3)} \\ & + t^5 \sum_{n=0}^{\infty} t^n \cdot 6 \sum_{v=0}^n \sum_{\mu=0}^v a_{n-v}^{(0)} a_{v-\mu}^{(1)} a_{\mu}^{(2)} \end{aligned}$$

which we write finally as

$$A_3 = t^4 \sum_{n=0}^{\infty} A_n^{(3)} t^n$$

where

$$\begin{aligned}
A_0^{(3)} &= 3 \sum_{\nu=0}^n \sum_{\mu=0}^{\nu} a_{\nu-\mu}^{(0)} a_{\mu}^{(3)} \\
A_1^{(3)} &= 3 \sum_{\nu=0}^{n+1} \sum_{\mu=0}^{\nu} a_{n-\nu}^{(0)} a_{\nu-\mu}^{(0)} a_{\mu}^{(3)} \\
&\quad + 6 \sum_{\nu=0}^n \sum_{\mu=0}^{\nu} a_{n-\nu}^{(0)} a_{\nu-\mu}^{(1)} a_{\mu}^{(2)} \\
A_{n+2}^{(3)} &= 3 \sum_{\nu=0}^{n+2} \sum_{\mu=0}^{\nu} a_{n-2-\nu}^{(0)} a_{\nu-\mu}^{(0)} a_{\mu}^{(3)} \\
&\quad + 6 \sum_{\nu=0}^{n+1} \sum_{\mu=0}^{\nu} a_{n+1-\nu}^{(0)} a_{\nu-\mu}^{(1)} a_{\mu}^{(2)} \\
&\quad + \sum_{\nu=0}^n \sum_{\mu=0}^{\nu} a_{n-\nu}^{(1)} a_{\nu-\mu}^{(1)} a_{\mu}^{(1)}
\end{aligned}$$

Thus the u_4 component is given by

$$\begin{aligned}
u_4 &= -\alpha t^5 \sum_{n=0}^{\infty} \frac{a_n^{(3)} t^n}{(n+5)} - \beta t^6 \sum_{n=0}^{\infty} \frac{a_n^{(3)} t^n}{(n+5)(n+6)} \\
&\quad - \gamma t^6 \sum_{n=0}^{\infty} \frac{A_n^{(3)} t^n}{(n+5)(n+6)}
\end{aligned}$$

which we will write as

$$u_4 = t^5 \sum_{n=0}^{\infty} a_n^{(4)} t^n$$

where

$$\begin{aligned}
a_0^{(4)} &= \frac{-\alpha a_0^{(3)}}{5} \\
a_{n+1}^{(4)} &= \frac{-\alpha(n+5)a_n^{(3)} - \beta a_n^{(3)} - \gamma A_n^{(3)}}{(n+5)(n+6)}
\end{aligned}$$

and we have the next approximant $\varphi_5 = \varphi_4 - u_4$ with

$$\begin{aligned}
\varphi_4 &= \sum_{n=0}^{\infty} b_n^{(4)} t^n \\
u_4 &= t^5 \sum_{n=0}^{\infty} a_n^{(4)} t^n
\end{aligned}$$

We write now

$$\varphi_5 = \sum_{n=0}^{\infty} b_n^{(5)} t^n$$

with

$$\begin{aligned} b_0^{(5)} &= b_0^{(4)} & b_3^{(5)} &= b_3^{(4)} \\ b_1^{(5)} &= b_1^{(4)} & b_4^{(5)} &= b_4^{(4)} \\ b_2^{(5)} &= b_2^{(4)} & b_{n+5}^{(5)} &= b_{n+5}^{(4)} + a_n^{(4)} \end{aligned}$$

Computing the u_5 component and φ_5 approximation.

$$u_5 = -L^{-1} \alpha u_4' - L^{-1} \beta u_4 - L^{-1} \gamma A_4$$

for which we need

$$u_4 = t^5 \sum_{n=0}^{\infty} a_n^{(4)} t^n$$

$$u_4' = t^4 \sum_{n=0}^{\infty} (n+5) a_n^{(4)} t^n$$

$$A_4 = \sum_{n=0}^{m=4} \sum_{v=0}^n u_{m-n} u_{n-v} u_v$$

$$A_4 = 3u_0^2 u_4 + 3u_1^2 u_2 + 3u_2^2 u_0 + 6u_0 u_1 u_3$$

$$\begin{aligned} A_4 &= 3 \left\{ \sum_{n=0}^{\infty} a_n^{(0)} t^n \right\} \cdot \left\{ \sum_{v=0}^{\infty} a_v^{(0)} t^v \right\} \cdot \left\{ t^5 \sum_{\mu=0}^{\infty} a_{\mu}^{(4)} t^{\mu} \right\} \\ &+ 3 \left\{ t^2 \sum_{n=0}^{\infty} a_n^{(1)} t^n \right\} \cdot \left\{ t^2 \sum_{v=0}^{\infty} a_v^{(1)} t^v \right\} \cdot \left\{ t^3 \sum_{\mu=0}^{\infty} a_{\mu}^{(2)} t^{\mu} \right\} \\ &+ 3 \left\{ t^3 \sum_{n=0}^{\infty} a_n^{(2)} t^n \right\} \cdot \left\{ t^3 \sum_{v=0}^{\infty} a_v^{(2)} t^v \right\} \cdot \left\{ \sum_{\mu=0}^{\infty} a_{\mu}^{(0)} t^{\mu} \right\} \\ &+ 6 \left\{ \sum_{n=0}^{\infty} a_n^{(0)} t^n \right\} \cdot \left\{ t^2 \sum_{v=0}^{\infty} a_v^{(1)} t^v \right\} \cdot \left\{ t^4 \sum_{\mu=0}^{\infty} a_{\mu}^{(3)} t^{\mu} \right\} \\ A_4 &= t^5 \sum_{n=0}^{\infty} t^n \cdot 3 \sum_{v=0}^n \sum_{\mu=0}^v a_{n-v}^{(0)} a_{v-\mu}^{(0)} a_{\mu}^{(4)} + t^7 \sum_{n=0}^{\infty} t^n \cdot 3 \sum_{v=0}^n \sum_{\mu=0}^v a_{n-v}^{(1)} a_{v-\mu}^{(1)} a_{\mu}^{(2)} \\ &+ t^6 \sum_{n=0}^{\infty} t^n \cdot 3 \sum_{v=0}^n \sum_{\mu=0}^v a_{n-v}^{(2)} a_{v-\mu}^{(2)} a_{\mu}^{(1)} + t^6 \sum_{n=0}^{\infty} t^n \cdot 6 \sum_{v=0}^n \sum_{\mu=0}^v a_{n-v}^{(0)} a_{v-\mu}^{(1)} a_{\mu}^{(3)} \end{aligned}$$

which we now write as

$$A_4 = t^5 \sum_{n=0}^{\infty} A_n^{(4)} t^n$$

where

$$A_0^{(4)} = 3 \sum_{v=0}^{n=0} \sum_{\mu=0}^v a_{n-v}^{(0)} a_{v-\mu}^{(0)} a_{\mu}^{(4)}$$

$$A_1^{(4)} = 3 \sum_{v=0}^{n=1} \sum_{\mu=0}^v a_{n-v}^{(0)} a_{v-\mu}^{(0)} a_{\mu}^{(4)} + 3 \sum_{v=0}^{n=0} \sum_{\mu=0}^v a_{n-v}^{(2)} a_{v-\mu}^{(2)} a_{\mu}^{(1)} + 6 \sum_{v=0}^{n=0} \sum_{\mu=0}^v a_{n-v}^{(0)} a_{v-\mu}^{(1)} a_{\mu}^{(3)}$$

$$A_{n+2}^{(4)} = 3 \sum_{v=0}^{n+2} \sum_{\mu=0}^v a_{n+2-v}^{(0)} a_{v-\mu}^{(0)} a_{\mu}^{(4)} + 3 \sum_{v=0}^{n+1} \sum_{\mu=0}^v a_{n+1-v}^{(2)} a_{v-\mu}^{(2)} a_{\mu}^{(1)} \\ + 6 \sum_{v=0}^{n+1} \sum_{\mu=0}^v a_{n+1-v}^{(0)} a_{v-\mu}^{(1)} a_{\mu}^{(3)} + 3 \sum_{v=0}^n \sum_{\mu=0}^v a_{n-v}^{(1)} a_{v-\mu}^{(1)} a_{\mu}^{(2)}$$

u_5 can now be written

$$u_5 = -\alpha t^6 \sum_{n=0}^{\infty} \frac{a_n^{(4)} t^n}{(n+6)} - \beta t^7 \sum_{n=0}^{\infty} \frac{a_n^{(4)} t^n}{(n+6)(n+7)} - \gamma t^7 \sum_{n=0}^{\infty} \frac{A_n^{(4)} t^n}{(n+6)(n+7)}$$

and finally

$$u_5 = t^6 \sum_{n=0}^{\infty} a_n^{(5)} t^n$$

where

$$a_0^{(5)} = \frac{\alpha a_0^{(4)}}{6}$$

$$a_{n+1}^{(5)} = \frac{-\alpha(n+6)a_{n+1}^{(4)} - \beta a_n^{(4)} - \gamma A_n^{(4)}}{(n+6)(n+7)}$$

Thus, the six-term approximant $\varphi_6 = \varphi_5 + u_5$ where $\sum_{n=0}^{\infty} b_n^{(5)} t^n$ and

$u_5 = t^6 \sum_{n=0}^{\infty} a_n^{(5)} t^n$ and we conveniently write

$$\varphi_6 = \sum_{n=0}^{\infty} b_n^{(6)} t^n$$

with

$$\begin{aligned}
 b_0^{(6)} &= b_0^{(5)} & b_4^{(6)} &= b_4^{(5)} \\
 b_1^{(6)} &= b_1^{(5)} & b_5^{(6)} &= b_5^{(5)} \\
 b_2^{(6)} &= b_2^{(5)} & b_3^{(6)} &= b_3^{(5)} \\
 b_{n+6}^{(6)} &= b_{n+6}^{(5)} + a_n^{(5)}
 \end{aligned}$$

We now have a six-term approximant to the solution and, if necessary, can continue in the same manner. (The algorithm can easily be programmed for machine computation.)

SUMMARY:

Decomposition components are given by:

$$\begin{aligned}
 u_0 &= \sum_{n=0}^{\infty} a_n^{(0)} t^n \\
 u_m &= t^{m+1} \sum_{n=0}^{\infty} a_n^{(m)} t^n \quad m \geq 1 \\
 \sum_{m=0}^{\infty} u_m &= u
 \end{aligned}$$

The coefficient-generating algorithms are:

$$\begin{aligned}
 a_0^{(0)} &= c_0 \\
 a_1^{(0)} &= c_1 \\
 a_{n+2}^{(0)} &= \frac{\delta_n}{(n+1)(n+2)} \\
 a_n^{(1)} &= \frac{-\alpha(n+1)a_{n+1}^{(0)} - \beta a_n^{(0)} - \gamma A_n^{(0)}}{(n+1)(n+2)} \\
 a_0^{(2)} &= \frac{-\alpha a_0^{(1)}}{3} \\
 a_{n+1}^{(2)} &= \frac{-\alpha(n+3)a_{n+1}^{(1)} - \beta a_n^{(1)} - \gamma A_n^{(1)}}{(n+3)(n+4)} \\
 &\vdots \\
 a_0^{(m)} &= \frac{-\alpha a_0^{(m-1)}}{(m+1)} \quad m \geq 2
 \end{aligned}$$

$$A_1 = t^2 \sum_{n=0}^{\infty} A_n^{(1)} t^n$$

$$A_2 = t^3 \sum_{n=0}^{\infty} A_n^{(2)} t^n$$

$$\vdots$$

$$A_m = t^{m+1} \sum_{n=0}^{\infty} A_n^{(m)} t^n$$

$$\sum_{m=0}^{\infty} A_m = u^3$$

$$\varphi_m = \sum_{n=0}^{\infty} b_n^{(m)} t^n$$

$$\lim_{m \rightarrow \infty} \varphi_m = u$$

REMARKS: A method of approximation does not need infinite precision. We need to know that we have uniform convergence of input and coefficient series and generally $\delta(t)$ can be truncated after a few terms for sufficient precision. Computation will investigate the effect on increased accuracy of adding terms to the $\delta(t)$ series to develop stopping rules. An example is instructive. Consider

$$u'' + u' + u + u^3 = g(t)$$

$$u(0) = 0$$

$$u'(0) = 1$$

$$g(t) = 1 - (1/2)t^2 + (5/6)t^3$$

The solution is $\sin t$. If we calculate $u(t)$ by decomposition, we find $\varphi_2 \approx u(t) = t - (1/3!)t^3$ or $\sin t$ to the same approximation as the series for g . If we specify g further, we can get more terms of u . Practically, this means that if we are interested in accuracy to n decimal places, but further calculation results in no further change in the n places, the solution is considered known and will be verified to satisfy the equation and the given conditions. Because of the generally rapid convergence, which we have seen in all work on the subject, we would expect that most of the solution is in the early terms of the decomposition series.

If we have the φ_6 approximant to three decimal places and further change exists only in the fourth or fifth place, our result may be sufficiently accurate so that convergence to a required precision can lead to a stopping rule. As we gain insight from our computation, we can develop stopping rules, investigate

convergence rate and error, acceleration techniques (Padé, Euler, Shanks), checks on the solution by modified decomposition, asymptotic decomposition, and verification that the solution satisfies the given equation and conditions, determination of the region of convergence in initial-value problems and possible use of analytic continuation, our one-step procedure, and other special techniques. Because of our choice of L and R with R a lower-order operator than L , we always have convergence, as we saw in evaluation of matrix inverses with $|L^{-1}R| < 1$, and since our integrations are trivial, i.e., we have Green's functions of unity. Adomian and Rach [1] have previously discussed an initial-value problem with a simple diffusion equation with $u(x,0) = f(x)$ where the operator in the solution series $u(x,t) = \sum_{m=0}^{\infty} (L_x^{-1}L_x)^m f(x)$ annihilates $f(x)$ at a finite m ; the result then does not satisfy the equation and conditions.

No closed form solutions exist for the Duffing equation and solutions have always been made using perturbation, discretized methods, etc. The results have shown multiple oscillations—not only at the excitation frequency ω but also at subharmonics ω/n , or superharmonic $n\omega$ with $n = 1, 2, \dots$.

If the excitation contains several terms we get combination frequencies also. The nonlinearity causes complex interactions among the input terms; however, the series captures all of the actual result. One could plot the result and do a Fourier analysis to see components in the output. With non-conservative systems where the u' term is non-zero, one would not in general expect sinusoidal outputs unless $g(t)$ compensated appropriately for the damping. We must be certain that we have a correctly modelled problem whose solution is consistent with the given conditions, and then solve it to see if the harmonics do exist in fact. Finally, solutions are unstable against small changes which can result from round-off error or linearization between grid points in the usual computer methods.

EXAMPLES:

- 1) To show that the decomposition method yields correct solutions to the degree of approximation that we use for the excitation, we will begin with $u = \sin t = t - t^3/3!$ and substitute into the Duffing equation. $u'' + u' + u + u^3 = g(t)$. Dropping terms greater than t^3 , we have

$$g = 1 - \frac{t^2}{2} + \frac{5t^3}{6}$$

$$L^{-1}g = \frac{t^2}{2} \text{ (+higher terms > } t^3 \text{)}$$

$$u_0 = u(0) + tu'(0) + L^{-1}g$$

$$u_0 = t + \frac{t^2}{2}$$

$$u_1 = -L^{-1}(1 - \frac{t^2}{2}) - L^{-1}(t - \frac{t^3}{6}) - L^{-1}t^3$$

$$u_1 = -\frac{t^2}{2} - \frac{t^3}{6} \text{ to order of } g$$

$$\varphi_2 = t + \frac{t^2}{2} - \frac{t^2}{2} - \frac{t^3}{3!}$$

$$\varphi_2 = t - \frac{t^3}{3!} \text{ or } \sin t \text{ to order of } g$$

We now have the solution to the same degree of approximation as g . Thus, given the problem $u'' + u' + u + u^3 = g(t)$ where $g(t) = 1 - t^2/2 + 5t^3/6$ with $u(0) = 0$ and $u'(0) = 1$, we get $u = \sin t$ to the approximation $\varphi_2 = t - t^3/3!$. As the forcing function is approximated more closely, our series for u approaches the series for $\sin t$ more closely. Substitution of the approximant φ_2 into the equation must satisfy the equation to that degree of approximation and satisfy the given conditions as well.

2) Let

$$u = e^{-t} + \sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

$u' = -t^2/2$ and $u'' = -t$, substituting in the Duffing equation approximate u with the first two terms of each series. Thus $u = 1 - t^3/3!$. Then

$$u'' + u' + u + u^3 = g(t)$$

We get $g = 2 - t$, since other terms are greater than t^3 when operated on by L^{-1} . Thus $L^{-1}g = t^2 - t^3/3!$. Since $u(0) = 1$ and $u'(0) = 0$,

$$u_0 = 1 + t^2 - \frac{t^3}{3!}$$

To get $u_1 = -L^{-1}u'_0 - L^{-1}u_0 - L^{-1}u_0^3$,

$$u_1 = -L^{-1}(2t) - L^{-1}(1) - L^{-1}(1) = -\frac{t^3}{3} - \frac{t^2}{2} - \frac{t^2}{2}$$

$$u_1 = -t^2 - \frac{t^3}{3}$$

$$u'_1 = -2t - t^2$$

$$u_1^3 = 0 \quad (\text{i.e., } > t^3)$$

$$\varphi_2 = 1 + t^2 - \frac{t^3}{3!} - \frac{t^3}{3} - t^2$$

$$\varphi_2 = 1 - \frac{t^3}{2}$$

u_2 is zero to same approximation so we have $\varphi_2 = u$.

$$\varphi_2 = (1-t) + \left(t - \frac{t^3}{3!}\right)$$

Thus, given the problem

$$u'' + u' + u + u^3 = 2 - t$$

$$u(0) = 1$$

$$u'(0) = 0$$

we get $\varphi_2 = 1 - t^3/2$ which is correct even if we do not see that this is $(1-t) + (t - t^3/3!)$ and recognize or guess $e^{-t} + \sin t$.

A CONVENIENT FORM OF THE SOLUTION OF THE DUFFING EQUATION IN ASCENDING POWERS OF T:

Considering the Duffing equation $u'' + \alpha u' + \beta u + \gamma u^3 = g(t)$ where α, β, γ are constants, $u(0) = c_0$, $u'(0) = c_1$, and $g = \sum_{n=0}^{\infty} g_n t^n$, we calculate the solution through the t^6 term which should generally be sufficient. Letting $L = d^2/dt^2$, $R = \alpha(d/dt) + \beta$, and $Nu = \gamma u^3$, we have in operator format

$Lu + Ru + Nu = g$. Decomposition results in $u = \sum_{n=0}^{\infty} u_n$ with

$$u_0 = c_0 + c_1 t + L^{-1}g$$

or

$$u_0 = c_0 + c_1 t + \sum_{n=0}^{\infty} g_n t^{n+1} / (n+1)(n+2)$$

$$u_0 = c_0 + c_1 t + g_0 t^2 / 2 \cdot 1 + g_1 t^3 / 3 \cdot 2 \\ + g_2 t^4 / 4 \cdot 3 + g_3 t^5 / 5 \cdot 4 + \dots$$

The following components are:

$$u_{n+1} = -L^{-1}\alpha(d/dt)u_n - L^{-1}\beta u_n - \gamma L^{-1}u A_n$$

where the A_n for u^3 have been given by convenient algorithms. Thus,

$$u_1 = -\alpha c_1 t^2 / 2 - \beta c_0 t^2 / 2 - \alpha g_0 t^3 / 6 - \beta c_1 t^3 / 6 \\ - \alpha g_1 t^4 / 24 - \beta g_0 t^4 / 24 - \alpha g_2 t^5 / 60 \\ - \beta g_1 t^5 / 120 - \alpha g_3 t^6 / 120 - \beta g_0 t^6 / 360 - \dots \\ - \gamma c_0^3 t^2 / 2 - \gamma c_0^2 c_1 t^3 / 2 - \left(\frac{\gamma c_0^2 g_0}{8} + \frac{\gamma c_0 c_0^2}{4} \right) t^4 \\ - \left(\frac{\gamma c_0^3}{20} + \frac{\gamma c_0^2 g_1}{40} + \frac{3\gamma c_0 c_1 g_0}{20} \right) t^5 - \dots$$

We can continue to obtain $u_2, u_3, \dots$. However, it is most convenient to write the solution in ascending powers of t as

$$u = \sum_{n=0}^{\infty} c_n t^n$$

where c_0 and c_1 are given and

$$c_2 = -\frac{\alpha c_0}{2} - \frac{\beta c_0}{2} - \frac{\gamma c_0^3}{2} + \frac{g_0}{2}$$

$$c_3 = \frac{\alpha^2 c_1}{6} + \frac{\alpha \beta c_0}{6} + \frac{\alpha \gamma c_0^3}{6} - \frac{\alpha g_0}{6} - \frac{\beta c_1}{6} - \frac{\gamma c_0^2 c_1}{2} + \frac{g_1}{6}$$

$$\begin{aligned}
c_4 = & -\frac{\alpha^3 c_1}{24} - \frac{\alpha^2 \beta c_0}{24} - \frac{\alpha^2 \gamma c_0^3}{24} + \frac{\alpha^2 g_0}{24} + \frac{2\beta c_1}{12} + \frac{\alpha \gamma c_0^2 c_1}{4} - \frac{\alpha g_1}{24} \\
& + \frac{\beta^2 c_0}{24} + \frac{\beta \gamma c_0^3}{6} - \frac{\beta g_0}{24} + \frac{\gamma^2 c_0^5}{8} - \frac{\gamma c_0^2 g_0}{8} - \frac{\gamma c_0 c_1^2}{4} + \frac{g_2}{12} \\
c_5 = & -\frac{\alpha^4 c_1}{120} - \frac{\alpha^3 \beta c_0}{120} - \frac{\alpha^3 \gamma c_0^3}{120} + \frac{\alpha^3 g_0}{120} + \frac{\alpha^2 \beta c_1}{40} \\
& - \frac{3\alpha^2 \gamma c_0^2 c_1}{40} + \frac{\alpha^2 g_1}{120} - \frac{\alpha \beta^2 c_0}{60} - \frac{\alpha \beta \gamma c_0^3}{15} + \frac{\alpha \beta g_0}{60} - \frac{\alpha \gamma^2 c_0^5}{20} \\
& + \frac{\alpha \gamma c_0^2 g_0}{20} + \frac{\alpha \gamma c_0 c_1^2}{5} - \frac{\alpha g_2}{60} + \frac{\beta^2 c_1}{120} + \frac{\beta \gamma c_0^2 c_1}{5} - \frac{\beta g_1}{120} + \frac{g \gamma^2 c_0^4 c_1}{40} \\
& - \frac{\gamma c_0^2 g_1}{40} - \frac{3\gamma c_0 c_1 g_0}{20} - \frac{\gamma c_1^2}{20} + \frac{g_3}{20} \\
c_6 = & -\frac{\alpha^5 c_1}{720} - \frac{\alpha^4 \beta c_0}{720} - \frac{\alpha^4 \gamma c_0^3}{720} + \frac{\alpha^4 g_0}{720} + \frac{\alpha^3 \beta c_1}{180} \\
& + \frac{\alpha^2 \gamma c_0^2 c_1}{60} - \frac{\alpha^3 g_1}{720} + \frac{\alpha^2 \beta^2 c_0}{240} + \frac{\alpha^2 \beta \gamma c_0^3}{60} - \frac{\alpha^2 \beta g_0}{240} + \frac{\alpha \gamma^2 c_0^5}{80} \\
& - \frac{\alpha^2 \gamma c_0^2 g_0}{80} - \frac{11\alpha^2 \gamma c_0 c_1^2}{120} + \frac{\alpha^2 g_2}{360} - \frac{\alpha \beta^2 c_1}{240} - \frac{2\alpha \beta \gamma c_0^2 c_1}{15} + \frac{\alpha \beta g_1}{360} \\
& - \frac{7\alpha \gamma^2 c_0^4 c_1}{48} + \frac{\alpha \gamma c_0^2 g_1}{120} - \frac{13\alpha \gamma c_0 c_1 g_0}{120} + \frac{7\alpha \gamma c_1^2}{120} - \frac{\alpha g_3}{120} - \frac{\beta^3 c_0}{720} - \frac{5\beta^2 \gamma c_0^3}{144} \\
& + \frac{\beta^2 g_0}{720} - \frac{17\beta \gamma^2 c_0^5}{240} + \frac{7\beta \gamma c_0^2 g_0}{120} + \frac{11\beta \gamma c_0 c_1^2}{120} - \frac{\beta g_2}{360} - \frac{3\gamma^3 c_0^7}{80} + \frac{\gamma^2 c_0^4 g_0}{16} \\
& + \frac{7\gamma^2 c_0^3 c_1^2}{40} - \frac{\gamma c_0^2 g_2}{120} - \frac{\gamma c_0 c_1 g_2}{30} - \frac{\gamma^2 c_0 g_0^2}{40} - \frac{\gamma c_1^2 g_0}{20} + \frac{g_4}{30}
\end{aligned}$$

The sum $\sum_{n=0}^{\infty} c_n t^n$ is the solution through the t^6 term and if we have c_0 or c_1 equal to zero, the result becomes quite simple.

RESPONSE OF NONLINEAR STOCHASTIC OPERATORS:

Random vibrations arise, e.g., in space structures and buildings subjected to seismic events. Our objective will be consideration of randomness in physical systems, which are generally nonlinear, without the use of perturbation or linearization which may prevent our seeing real possibilities of catastrophic failure. We will also consider parameters and excitations without the usual

restrictive assumptions which are customary but do not necessarily conform to physical reality.

For the present, assume the equation $u'' + \alpha u' + \beta u + \gamma f(u) = g$ or $Lu + Ru + Nu = g$ with $L = d^2/dt^2$, $R = \alpha(d/dt) + \beta$, and $Nu = \gamma f(u)$. We can consider cases in which one or more of the α, β, γ, g may be stochastic processes, without restriction to only g being stochastic, and without further assuming a white noise excitation. Decomposition yields $u = \sum_{n=0}^{\infty} u_n$ with

$$\begin{aligned} u_0 &= u(0) + tu'(0) + L^{-1}g \\ u_{n>0} &= -L^{-1}\alpha(d/dt)u_{n-1} - L^{-1}\beta u_{n-1} - L^{-1}\gamma A_n \end{aligned}$$

where the A_n are determined for $f(u)$. The approximant $\phi_n = \sum_{n=0}^{n-1} u_n$ serves as the solution. Since the procedure converges rapidly so that a few terms are sufficient in practical cases and because the terms depend on preceding terms rather than following terms, avoiding closure problems, ensemble averages can be taken term by term to determine $\langle u \rangle$. We do make the natural assumption that the excitation and parameter processes are uncorrelated. Then taking the ensemble average of the product $\phi_m(t) \phi_m(t')$, we can also get a two-point correlation. A review of the necessary knowledge of stochastic processes for application to solution of physical problems by decomposition appears in [1].

DUFFING'S EQUATION WITHOUT PERTURBATION TO GIVEN ACCURACY:

Quantitative general solutions of Duffing's equation are easily found using the decomposition method. The motion depends on the initial conditions $u(0)$, $u'(0)$, the parameters, and the inputs. The method of solution makes no assumption on the nature of the output or on smallness of certain parameters, and is not restricted to a single input or closeness of the excitation frequency and the natural (unforced) frequency. This section will show that by seeking solutions to only the necessary accuracy, considerable computation and difficult integrations are avoidable. The appearance of harmonics and subharmonics will be demonstrated. Finally, we will demonstrate, using the Duffing equation, that decomposition subsumes perturbation.

$$u'' + \alpha u' + \omega_0^2 u + \beta u^3 = g(t)$$

Let $L = d^2/dt^2$ and $L^{-1} = \int_0^t \int_0^t (\cdot) dt dt$ and solve for Lu . Thus,

$$Lu = g(t) - \alpha u' - \omega_0^2 u - \beta u^3$$

$$L^{-1}Lu = L^{-1}g - L^{-1}\alpha u' - L^{-1}\omega_0^2 u - L^{-1}\beta u^3$$

$$u = u(0) + tu'(0) + L^{-1}g - L^{-1}\alpha u' - L^{-1}\omega_0^2 u - L^{-1}\beta u^3$$

Let u be decomposed into components $\sum_{n=0}^{\infty} u_n$ with u_0 identified as

$$u_0 = u(0) + tu'(0) + L^{-1}g$$

with other components to be determined. (The nonlinear term is written as

$\sum_{n=0}^{\infty} A_n \{u^3\}$ or briefly as $\sum_{n=0}^{\infty} A_n$.) The terms after u_0 are:

$$u_{m+1} = -L^{-1}u'_m - L^{-1}\omega_0^2 u_m - L^{-1}\beta A_m$$

for $m \geq 0$. Then $\phi_m = \sum_{n=0}^{m-1} u_n$ approximates the solution $u = \sum_{n=0}^{\infty} u_n$ in a rapidly converging series.

Although this provides general solutions, there are difficulties with trigonometric inputs, for example, and the u^3 term; we can still get difficult integrations despite defining L so that a difficult Green's function can be avoided. Also we can get a proliferation of terms, causing unnecessary computation. Hence, we will assume $g = \sum_{n=0}^{\infty} g_n t^n$ which might appear counter-intuitive because of the proliferation problem. However we only need to compute to a necessary accuracy in a physical problem which we demonstrate with some illuminating examples.

EXAMPLE: Consider $u'' + u' + u + u^3 = g(t)$ where $u(0) = 2$ and $u'(0) = -1$ and

$$g = \cos^3 t + 3e^{-t} \cos^2 t + 3e^{-2t} \cos t - \sin t + e^{-3t} + e^{-t}$$

If we approximate each function in g with the terms of its series through t^2 and calculate $L^{-1}g$, we get $9t^2/2$ to the above approximation. Hence

$u_0 = 2 - t + (9/2)t^2$ and $u_1 = -(9/2)t^2$. Therefore the two-term approximation, correct through the t^2 term, is $u \approx 2 - t$ which we can write as $u \approx (1 - t + t^2/2) + (1 - t^2/2) = e^{-t} + \cos t$, $u_0^3 = g = \cos^3 t + \dots$ and $u \approx \cos t$ for large L . Note that this is the exact solution.

EXAMPLE: Show superharmonics are possible in a Duffing equation.

$$\begin{aligned} u'' + u' + u + u^3 &= g = t + 10t + \dots \\ u(0) &= 1 & u'(0) &= 2 \end{aligned}$$

Where g is a trigonometric function as before. To avoid difficult integrations, we consider the Maclaurin approximation to three terms:

$$g \approx 3 + 7t + 10t^2$$

Since

$$u_0 = u(0) + tu'(0) + L^{-1}g$$

we have

$$u_0 = 1 + 2t + \frac{3t^2}{2}$$

if we drop terms greater than t^2 . Then

$$u_1 = -L^{-1}u_0 - L^{-1}u_0' - L^{-1}u_0^3 = -2t^2$$

The two-term approximant ϕ_2 is given by

$$\phi_2 = 1 + 2t - \frac{t^2}{2}$$

which we see can be arranged as $\phi_2 = (1 - t^2/2) + 2t$ and immediately guess

$$u = \cos t + \sin 2t$$

showing existence of a superharmonic. Needless to say, we verify the solution obtained by direct substitution. (Also we can consider another term of the Maclaurin expansion of g to get the cubic term in our approximant.)

EXAMPLE: Show that subharmonics can arise in a Duffing equation.

$$u'' + u' + u + u^3 = g$$

with $u(0) = 2$ and $u'(0) = 0$. g is again a trigonometric form whose Maclaurin expansion is given by

$$g = \frac{79}{9} - \frac{11t}{9} - \frac{11t^2}{18} \dots$$

Then

$$L^{-1}g = \frac{79t^2}{18}$$

through quadratic terms. The first term of the decomposition series is

$$u_0 = u(0) + tu'(0) + L^{-1}g = 2 + \frac{79t^2}{18}$$

Solving the Duffing equation by decomposition, we have

$$u = u_0 - L^{-1}u' - L^{-1}u - L^{-1}u^3$$

Since $\sum_{n=0}^{\infty} u_n$, the next term u_1 is

$$u_1 = -L^{-1}u'_0 - L^{-1}u_0 - L^{-1}A_0\{u^3\}$$

The polynomial $A_0\{u^3\}$ is simply u_0^3 ; therefore, $u_1 = 5t^2$ plus, of course, higher terms. The two-term approximant ϕ_2 to the solution is $u_0 + u_1$; hence

$$\phi_2 = 2 + \frac{79}{18}t^2 - 5t^2 = 2 - \frac{11}{18}t^2$$

which we recognize as

$$u = \left(1 - \frac{t^2}{2} + \dots\right) + \left(1 - \frac{t^2}{9} + \dots\right)$$

or

$$u \approx \cos t + \cos \frac{1}{3}t$$

Decomposition yields the solution—it depends on the parameters, given conditions, and the excitation.

The fact that we recognize closed forms here is perhaps interesting but of no real significance. The series can be carried far enough for computation as necessary. The traditional emphasis on closed form solutions has generally led to replacement of actual problems with more tractable but less realistic models.

PERTURBATION VS. DECOMPOSITION:

Consider the homogeneous Duffing equation with no damping and assuming a "small" nonlinear term:

$$u'' + u + \varepsilon u^3 = 0$$

$$u(0) = a$$

$$u'(0) = 0$$

Using perturbation define $u = u_0(t) + \varepsilon u_1(t) + \dots$. Hence, substituting to $O(\varepsilon)$,

$$u_0'' + \varepsilon u_1'' + u_0 \varepsilon u_1 + (u_0 + \varepsilon u_1)^3 = 0$$

Equating powers of ε ,

$$u_0'' + u_0 = 0$$

The linear solution ($\varepsilon = 0$) satisfying the given conditions is $u_0 = a \cos t$. The ε^1 terms give us

$$u_1'' + u_1 = -u_0^3 - a^3 \cos^3 t = -a^3(3 \cos t + \cos 3t)$$

with $u_1(0) = u_1'(0) = 0$. Hence we can solve for u_1 and write $u_0 + \varepsilon u_1$. We observe that the u_0 is the solution of $u_0'' + u_0 = 0$. In decomposition it is simply $u_0 = u(0) + tu'(0)$. Also the perturbative term is harder to obtain than the decomposition component $u_1 = -L^{-1}u_0 - \varepsilon L^{-1}A_0$ where the L^{-1} is merely a double integration. The perturbation case involves integration using a Green's function and more difficult integration. Further, the perturbation u_1 involves the secular term $t \sin t$, and we do not get a uniformly valid expansion which would allow a bounded u for a finite number of terms. Thus $u_1/u_0 \rightarrow \infty$ as $t \rightarrow \infty$. The results converge slowly while decomposition converges rapidly so few terms are required. The totals should be the same but not term by term.

Applying decomposition to the equivalent linear system $u'' + u = 0$ with

$u(0) = a$ and $u'(0) = 0$, the decomposition terms are

$$u_0 = a$$

$$u_1 = -L^{-1}u_0 = at^2/2$$

$$u_2 = -L^{-1}u_1 = at^4/2$$

Therefore, the three-term approximant obtained by decomposition is

$$\phi_3 = a \left(1 - \frac{t^2}{2} + \frac{t^4}{4!} \right)$$

which is the approximant to $u = a \cos t$. Now consider $u'' + u + \beta u^3 = 0$ by decomposition without assumption of smallness (and use of perturbation). We get

$$u_0 = a$$

$$u_1 = -L^{-1}u_0 - \beta L^{-1}u_0^3 = -a^2/2 - \beta a^3 t^2/2$$

$$u_2 = -L^{-1}u_1 - \beta L^{-1}u_1^3$$

$$\vdots$$

We have added $-\beta L^{-1}u_0^3$ or $-\beta a^3 t^2/2$ as a first approximation to the previous linear result. When the system is close to linear (weakly nonlinear), we get $\beta = \varepsilon$.

$$u'' + u + \varepsilon Nu = 0$$

$$u(0) = a \text{ and } u'(0) = 0$$

$$u_0 = a$$

$$u_1 = -L^{-1}u_0 - \varepsilon L^{-1}Nu$$

Now the $\varepsilon L^{-1}Nu$ approaches $\varepsilon L^{-1}u$ or $-at^2/2$. For this weakly nonlinear (or small ε) case the first approximation or ε term $u_1'' + u_1 = -a \cos t$ so $u_1 = -at^2/2$.

Thus the first decomposition addition is equal to the first-order perturbation result if and only if the nonlinearity Nu is sufficiently small. *In decomposition there is not a restriction to "close to linear"; it applies generally to nonlinear systems so "weakly nonlinear" or "linear" become special cases.*

$$u_1 = -L^{-1}u_0 = -at^2/2$$

$$\phi_2 = a \left(1 - \frac{t^2}{2} \right) = a \cos t \text{ as the number of terms increases}$$

which is easily checked by finding more components. In the equation $u'' + u + \epsilon u^3 = 0$, we had

$$u_0 = a$$

$$u_1 = -L^{-1}u_0 - \epsilon L^{-1}u_0^3$$

$$\vdots$$

For small enough ϵ , we have added $-L^{-1}u_0^3$ or $-a^3t^2/2$ as a first approximation to the linear result $u = a \cos t$. When $Nu = u$, i.e., we have a weak nonlinearity, we get

$$u'' + u + \epsilon Nu = 0$$

$$u(0) = a \text{ and } u'(0) = 0$$

$$u_0 = a$$

$$u_1 = -L^{-1}u_0 - \epsilon L^{-1}u_0$$

The $\epsilon L^{-1}Nu$ or $\epsilon L^{-1}A_0$ approaches $\epsilon L^{-1}u_0$ or $-at^2/2$. In the perturbation case the first approximation or ϵ term now satisfies $u_1'' + u_1 = -a \cos t$ so $u_1 = -a t^2/2$. Thus the first decomposition addition is equal to the first-order perturbation result. Perturbation is effective if and only if the nonlinearity Nu is almost linear. Decomposition is effective for general nonlinearities and includes perturbation as a special case.

Discontinuities in frequency response will occur as a result of varying excitation frequency, since the nonlinearity acting on the difference between excitation frequency and natural frequency causes new frequencies to appear and new multiple possible responses. With small damping, the oscillatory motion can suddenly change from slow to fast or vice-versa. In phase space we can have changes from one orbit to another and may find separated regions dependent on initial conditions, parameters, and excitation. If we change excitation frequency to approach the natural frequency, the behavior can change significantly. Decomposition yields the actual quantitative results for

real physical behavior for any given parameters, conditions, and inputs whether constants or time-varying. However, conditions must be specified. Further, decomposition provides solutions for real oscillators with any nonlinearity as determined from laboratory measurement, not only those with a simple nonlinearity u^3 which might be, in actuality, u^π .

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BOUNDARY-VALUE PROBLEMS WITH CLOSED IRREGULAR CONTOURS OR SURFACES

The simulant concept can now be used in an extremely valuable application, that of boundary-value problems for differential or partial differential equations modelling physical problems between two closed irregular contours (or surfaces). These are considered using decomposition of the boundary shape and simulation of the solution for each boundary approximant.

Our objective is to solve "two-limit" boundary-value problems analogous to two-point boundary-value problems for a second-order ordinary differential equation with Dirichlet conditions. In two dimensions, the corresponding situation is a two-contour second-order partial differential equation. In three dimensions, the analogue is a three-dimensional second-order partial differential equation solved between two surfaces. We can continue to an n -dimensional second-order partial differential equation and n -dimensional manifolds. Our special interest is in solving partial differential equations in regions bounded by contours or surfaces.

TWO-CONTOUR CASE:

Thus for the two-point, i.e., two-limit problem for second-order ordinary differential equations, which we can think of as a one-dimensional partial differential equation, we have two-point boundary conditions $u(x)|_{x=\xi_1} = b_1$ and $u(x)|_{x=\xi_2} = b_2$ where $x = \xi_1$ and $x = \xi_2$ are embedded in a line and we are considering equations such as $d^2u/dx^2 + f(u, u') = 0$. In the two-dimensional case we consider equations such as

$$u_{xx} + u_{yy} + p(x)f(u, u_x, u_y) = 0$$

with conditions such as

$$u(x, y)|_{c_1(x, y)=0} = b_1$$

$$u(x, y)|_{c_2(x, y)=0} = b_2$$

Thus the limits are smooth closed curves or contours.

TWO-SURFACE BOUNDARY-VALUE PROBLEM:

Here, we consider equations such as

$$u_{xx} + u_{yy} + u_{zz} + p(x, y, z)f(u, u_x, u_z) = 0$$

in a three-dimensional region bounded by surfaces $S_1(x, y, z) = 0$ and $S_2(x, y, z) = 0$ embedded in the region. Our boundary conditions are

$$u(x, y, z)|_{S_1=0} = b_1$$

$$u(x, y, z)|_{S_2=0} = b_2$$

The surfaces are smooth closed surfaces representing the limits. Obviously the concept can be extended to equations such as

$$\frac{\sum_{\mu=1}^n \partial^2 u}{\partial x_\mu + p(x_1, \dots, x_n)f(u, u_1, \dots, u_n)} = 0$$

with

$$u(\bar{x})|_{M_1(\bar{x})=0} = b_1$$

$$u(\bar{x})|_{M_2(\bar{x})=0} = b_2$$

where $\bar{x} = (x_1, \dots, x_n)$. M_1 and M_2 are smooth closed manifolds representing the limits in n dimensions.

Let us begin with the merely illustrative one-dimensional example for comparison, using decomposition [1]. Consider a two-point simple one-dimensional boundary-value problem $d^2u/dx^2 + \nu u = 0$ with ν a numerical constant and Dirichlet conditions $u(x = \xi_1) = b_1$ and $u(x = \xi_2) = b_2$. The first example does not require the technique of analytic simulation but serves as an introduction to the following multidimensional cases. The equation can be written in the decomposition form as $Lu + Ru = 0$. Solving by the decomposition technique yields

$$\sum_{n=0}^{\infty} u_n = c_0 + c_1 x - \nu I_x^2 \sum_{n=0}^{\infty} u_n$$

where I_x^2 is a two-fold pure integration with respect to x . By double decomposition,

$$c_0 = \sum_{n=0}^{\infty} c_0^{(n)} \quad c_1 = \sum_{m=0}^{\infty} c_1^{(m)} \quad u_n = \sum_{m=0}^{\infty} u_n^{(m)}$$

Hence

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)} = \sum_{m=0}^{\infty} c_0^{(m)} + x \sum_{m=0}^{\infty} c_1^{(m)} - \nu I_x^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)}$$

We now have

$$\begin{aligned} u_0 &= c_0^{(0)} + x c_1^{(0)} \\ u_1 &= c_0^{(1)} + x c_1^{(1)} - \nu I_x^2 u_0 \\ u_2 &= c_0^{(2)} + x c_1^{(2)} - \nu I_x^2 u_1 \\ &\vdots \\ u_m &= c_0^{(m)} + x c_1^{(m)} - \nu I_x^2 u_{m-1} \end{aligned}$$

We can write

$$\begin{aligned} u_m &= \sum_{n=0}^m u_n^{(m-n)} \\ u &= \sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} \sum_{n=0}^m u_n^{(m-n)} \\ u_m &= \sum_{n=0}^m (-\nu)^n \left\{ c_0^{(m-n)} x^{2n} / (2n)! + c_1^{(m-n)} x^{2n+1} / (2n+1)! \right\} \\ u_m &= c_0^{(m)} + x c_1^{(m)} + \sum_{n=1}^m (-\nu)^n \left\{ c_0^{(m-n)} x^{2n} / (2n)! + c_1^{(m-n)} x^{2n+1} / (2n+1)! \right\} \end{aligned}$$

In the decomposition method the $(m+1)$ -term approximant to the solution u is symbolized by $\phi_{m+1} = \sum_{n=0}^m u_n$. Thus

$$\begin{aligned} \phi_1 &= u_0 \\ \phi_2 &= \phi_1 + u_1 \\ &\vdots \\ \phi_{m+1} &= \phi_m + u_m \end{aligned}$$

The exact boundary conditions $u(x = \xi_1) = b_1$ and $u(x = \xi_2) = b_2$ can be approximated successively by the approximate boundary conditions

$$\begin{array}{ll} \phi_1(x = \xi_1) = b_1 & \phi_1(x = \xi_2) = b_2 \\ \vdots & \vdots \\ \phi_{m-1}(x = \xi_1) = b_1 & \phi_{m-1}(x = \xi_2) = b_2 \end{array}$$

But since $\phi_{m-1} = \phi_m + u_m$, then

$$\begin{array}{ll} u_0(x = \xi_1) = b_1 & u_0(x = \xi_2) = b_2 \\ u_1(x = \xi_1) = 0 & u_1(x = \xi_2) = 0 \\ \vdots & \vdots \\ u_m(x = \xi_1) = 0 & u_m(x = \xi_2) = 0 \end{array}$$

Define $b_1^{(0)} = b_1$ and $b_2^{(0)} = b_2$. Then,

$$b_1^{(m)} = \sum_{n=1}^m (-v)^n \{ c_0^{(m-n)} \xi_1^{2n} / (2n)! + c_1^{(m-n)} \xi_1^{2n+1} / (2n+1)! \}$$

$$b_2^{(m)} = \sum_{n=1}^m (-v)^n \{ c_0^{(m-n)} \xi_2^{2n} / (2n)! + c_1^{(m-n)} \xi_2^{2n+1} / (2n+1)! \}$$

$$c_0^{(m)} + \xi_1 c_1^{(m)} = b_1^{(m)}$$

$$c_0^{(m)} + \xi_2 c_1^{(m)} = b_2^{(m)}$$

or

$$\begin{pmatrix} 1 & \xi_1 \\ 1 & \xi_2 \end{pmatrix} \begin{pmatrix} c_0^{(m)} \\ c_1^{(m)} \end{pmatrix} = \begin{pmatrix} b_1^{(m)} \\ b_2^{(m)} \end{pmatrix}$$

Thus for $\xi_2 \neq \xi_1$

$$c_0^{(m)} = \frac{\xi_2 b_1^{(m)} - \xi_1 b_2^{(m)}}{\xi_2 - \xi_1}$$

$$c_1^{(m)} = \frac{b_2^{(m)} - b_1^{(m)}}{\xi_2 - \xi_1}$$

Using staggered summation

$$\begin{aligned}
 u &= \sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} \sum_{n=0}^m u_n^{(m-n)} \\
 u &= \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)} \\
 \sum_{m=0}^{\infty} \sum_{n=0}^m u_n^{(m-n)} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)}
 \end{aligned}$$

Recasting this example in the format of higher dimensional cases,

$$d^2 u / dx^2 + v u = 0$$

$$u(x) \Big|_{P_1(x)=0} = b_1$$

$$u(x) \Big|_{P_2(x)=0} = b_2$$

where $P_1(x) = x - \xi_1$ and $P_2(x) = x - \xi_2$. We consider the simulants σ_m to the approximations of the boundaries and denote them by $\sigma_m[u]$ which becomes u in the limit. Thus

$$(d^2 / dx^2) \sigma_m + v \sigma_m = 0$$

$$\sigma_m(x) \Big|_{x=\xi_1^{(m)}} = b_1$$

$$\sigma_m(x) \Big|_{x=\xi_2^{(m)}} = b_2$$

Successive simulants are $\sigma_1, \sigma_2, \dots, \sigma_m$. The $\lim_{m \rightarrow \infty} \sigma_m(x) = u(x)$. In the one-dimensional case it is $x = \xi = \rho$, a "radius". (We may develop a point sequence where $\lim_{m \rightarrow \infty} \xi_1^{(m)} = \xi_1$ and similarly for $\xi_2^{(m)}$. Thus $\lim_{m \rightarrow \infty} \sigma_m = u$.) In the two-dimensional case, the limits are not points but closed contours in R^2 described by $C(x, y) = 0$ and we can have a contour sequence $C^{(m)}(x, y) = 0$. If the contours are not smooth but consist, for example, of piecewise differentiable functions, we can represent them by smooth continuous functions as accurately as we wish, and without Gibbs phenomena, by a recent combination of techniques for decomposition of algebraic and differential equations [2]. Thus we can assume that the contours (or surfaces) are smooth though irregular in shape.

TWO-DIMENSIONAL CASE:

Now we consider a two-dimensional case with the model equation on R^2

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0$$

which we view as a two-dimensional analogue of the first example with $v(y) = \partial^2 / \partial y^2$. Analogous boundary conditions are

$$u(x, y) \Big|_{C_1(x, y)=0} = b_1$$

$$u(x, y) \Big|_{C_2(x, y)=0} = b_2$$

where C_1 and C_2 are closed contours representing the boundaries in R^2 . (We can, if we wish, let the outer contour $\rightarrow \infty$ or the inner contour approach the origin.) The model equation is written as $L_x u + L_y u = 0$ where $L_x = \partial^2 / \partial x^2$ and $L_y = \partial^2 / \partial y^2$. Operating with L_x^{-1} ,

$$L_x^{-1} L_x u = -L_x^{-1} L_y u$$

$$u = c_0(y) + x c_1(y) - L_y I_x^2 u$$

Decomposing u into $\sum_{n=0}^{\infty} u_n$ we have

$$\sum_{n=0}^{\infty} u_n = c_0(y) + x c_1(y) - L_y I_x^2 \sum_{n=0}^{\infty} u_n$$

Using double decomposition $u = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)}$. Also,

$$c_0(y) = \sum_{m=0}^{\infty} C_0^{(m)}(y)$$

$$c_1(y) = \sum_{m=0}^{\infty} C_1^{(m)}(y)$$

$$u_n = \sum_{m=0}^{\infty} u_n^{(m)}$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)} = \sum_{m=0}^{\infty} c_0^{(m)}(y) + x \sum_{m=0}^{\infty} c_1^{(m)}(y) - L_y I_x^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)}$$

$$\begin{aligned}
u_0 &= c_0^{(0)}(y) + x c_1^{(0)}(y) \\
u_1 &= c_0^{(1)}(y) + x c_1^{(1)}(y) - L_y I_x^2 u_0 \\
u_2 &= c_0^{(2)}(y) + x c_1^{(2)}(y) - L_y I_x^2 u_1 \\
&\vdots \\
u_m &= c_0^{(m)}(y) + x c_1^{(m)}(y) - L_y I_x^2 u_{m-1}
\end{aligned}$$

We can write

$$\begin{aligned}
u_m &= \sum_{n=0}^m u_n^{(m-n)} \\
u &= \sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} \sum_{n=0}^m u_n^{(m-n)} \\
u_m &= \sum_{n=0}^m (-L_y)^n \left\{ c_0^{(m-n)} x^{2n} / (2n)! + c_1^{(m-n)} x^{2n+1} / (2n+1)! \right\}
\end{aligned}$$

or

$$u_m = c_0^{(m)} + x c_1^{(m)} + \sum_{n=1}^m (-L_y)^n \left\{ c_0^{(m-n)} x^{2n} / (2n)! + c_1^{(m-n)} x^{2n+1} / (2n+1)! \right\}$$

The approximant to the solution is $\phi_{m+1} = \sum_{n=0}^m u_n$. The exact boundary conditions are:

$$u(x, y) \Big|_{C_1(x, y)=0} = b_1$$

$$u(x, y) \Big|_{C_2(x, y)=0} = b_2$$

are approximated by

$$\phi_{m+1}(x, y) \Big|_{C_1(x, y)=0} = b_1$$

$$\phi_{m+1}(x, y) \Big|_{C_2(x, y)=0} = b_2$$

for $m = 0, 1, 2, \dots$. Since $\phi_{m+1} = \phi_m + u_m$, we have

$$\begin{array}{ll}
u_0(x, y) \Big|_{C_1(x, y)=0} = b_1 & u_0(x, y) \Big|_{C_2(x, y)=0} = b_2 \\
u_1(x, y) \Big|_{C_1(x, y)=0} = 0 & u_1(x, y) \Big|_{C_2(x, y)=0} = 0 \\
\vdots & \vdots \\
u_m(x, y) \Big|_{C_1(x, y)=0} = 0 & u_m(x, y) \Big|_{C_2(x, y)=0} = 0
\end{array}$$

The interior contour or boundary is given as $C_1(x, y) = 0$ so that $x = \xi_1(y)$. The exterior boundary $C_2(x, y) = 0$ so that $x = \xi_2(y)$.^{*} These can be approximated by $\xi_1^{(m)}(y)$ and $\xi_2^{(m)}(y)$. If we wish for a particular model, we can consider $\lim_{m \rightarrow \infty} \xi_2^{(m)}(y) \rightarrow \infty$ so the exterior boundary $\rightarrow \infty$. Define $b_1^{(0)} = b_1$ and $b_2^{(0)} = b_2$ and

$$b_1^{(m)} = - \sum_{n=1}^m \left\{ \frac{\xi_1^{2n}(y)}{(2n)!} (-L_y)^n C_0^{(m-n)}(y) + \frac{\xi_1^{2n+1}(y)}{(2n+1)!} (-L_y)^n C_1^{(m-n)}(y) \right\}$$

$$b_2^{(m)} = - \sum_{n=1}^m \left\{ \frac{\xi_2^{2n}(y)}{(2n)!} (-L_y)^n C_0^{(m-n)}(y) + \frac{\xi_2^{2n+1}(y)}{(2n+1)!} (-L_y)^n C_1^{(m-n)}(y) \right\}$$

so we can write

$$C_0^{(m)} + \xi_1 C_1^{(m)} = b_1^{(m)}$$

$$C_0^{(m)} + \xi_2 C_1^{(m)} = b_2^{(m)}$$

or

$$\begin{pmatrix} 1 & \xi_1 \\ 1 & \xi_2 \end{pmatrix} \cdot \begin{pmatrix} C_0^{(m)} \\ C_1^{(m)} \end{pmatrix} = \begin{pmatrix} b_1^{(m)} \\ b_2^{(m)} \end{pmatrix}$$

Thus

$$C_0^{(m)} = \frac{\xi_2 b_1^{(m)} - \xi_1 b_2^{(m)}}{\xi_2 - \xi_1}$$

$$C_1^{(m)} = \frac{b_2^{(m)} - b_1^{(m)}}{\xi_2 - \xi_1}$$

THREE-DIMENSIONAL CASE:

We can generalize to R^3 with two closed surfaces $S_1(x, y, z) = 0$ and $S_2(x, y, z) = 0$ (or even to manifolds in R^n) and simulate the solution $u(x, y, z)$ representing the model phenomena. The Dirichlet conditions are

$$u(x, y, z) \Big|_{S_i(x, y, z)=0} = b_i \quad \text{for } i = 1, 2$$

The simulants are $\sigma_m[u] \rightarrow u$ for sufficiently high-order m , i.e., $\lim_{m \rightarrow \infty} \sigma_m = u$.

The boundary shape is subjected to decomposition and simulants are found for

^{*} See implicit function theorem. Also see chapter on decomposition of algebraic equations [1].

successive boundary approximations to higher and higher order. Thus σ_m satisfies the model equation and the boundary condition

$$\sigma_m(x, y, z) \Big|_{S_i(x, y, z)=0} = b_i$$

for $i = 1, 2$ for $m = 1, 2, 3, \dots$. As we increase the dimensionality, we can use the previous results by writing

$$L_x u + L_{y'} u = 0$$

where $L_x = \partial^2 / \partial x^2$ and $L_{y'} = \partial^2 / \partial y^2 + \partial^2 / \partial z^2$. Then

$$L_x u = -L_{y'} u$$

$$L_x^{-1} L_x u = L_x^{-1} L_{y'} u$$

where $y' = (y, z)$. Decomposition $u = \sum_{n=0}^{\infty} u_n$ yields

$$\sum_{n=0}^{\infty} u_n = C_0(y') + x C_1(y') - L_{y'} I_x^{-2} \sum_{n=0}^{\infty} u_n$$

With double decomposition $u = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)}$ and

$$C_0(y') = \sum_{m=0}^{\infty} C_0^{(m)}(y')$$

$$C_1(y') = \sum_{m=0}^{\infty} C_1^{(m)}(y')$$

$$u_n = \sum_{m=0}^{\infty} u_n^{(m)}$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)} = \sum_{m=0}^{\infty} C_0^{(m)}(y') + x C_1^{(m)}(y') - L_{y'} I_x^{-2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)}$$

so that

$$u_0 = C_0^{(0)}(y') + x C_1^{(0)}(y')$$

$$u_1 = C_0^{(1)}(y') + x C_1^{(1)}(y') - L_{y'} I_x^{-2} u_0$$

$$\begin{aligned}
 u_2 &= C_0^{(2)}(y') + x C_1^{(2)}(y') - L_{y'} I_x^2 u_1 \\
 &\vdots \\
 u_m &= C_0^{(m)}(y') + x C_1^{(m)}(y') - L_{y'} I_x^2 u_{m-1}
 \end{aligned}$$

We can write

$$\begin{aligned}
 u_m &= \sum_{n=0}^m u_n^{(m-n)} \\
 u &= \sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} \sum_{n=0}^m u_n^{(m-n)}
 \end{aligned}$$

Then

$$\begin{aligned}
 u_m &= \sum_{n=0}^m (-L_{y'})^n \{ C_0^{(m-n)} x^{2n} / (2n)! + C_1^{(m-n)} x^{2n+1} / (2n+1)! \} \\
 u_m &= C_0^{(m)} + x C_1^{(m)} + \sum_{n=1}^m (-L_{y'})^n \{ C_0^{(m-n)} x^{2n} / (2n)! + C_1^{(m-n)} x^{2n+1} / (2n+1)! \}
 \end{aligned}$$

Our $(m+1)$ term approximants to the solution are

$$\begin{aligned}
 \phi_{m+1} &= \sum_{n=0}^m u_n \\
 \phi_1 &= u_0 \\
 \phi_2 &= \phi_1 + u_1 \\
 &\vdots \\
 \phi_{m+1} &= \phi_m + u_m
 \end{aligned}$$

The exact boundary conditions are

$$\begin{aligned}
 u(x, y') \Big|_{s_1(x, y')=0} &= b_1 \\
 u(x, y') \Big|_{s_2(x, y')=0} &= b_2
 \end{aligned}$$

The approximate conditions are

$$\begin{aligned}
 \phi_m(x, y') \Big|_{s_1(x, y')=0} &= b_1 \\
 \phi_m(x, y') \Big|_{s_2(x, y')=0} &= b_2
 \end{aligned}$$

for successive m but since $\phi_{m+1} = \phi_m + u_m$,

$$u_m(x, y')|_{S_1=0} = b_1$$

$$u_m(x, y')|_{S_2=0} = b_2$$

for $m = 0, 1, 2, 3, \dots$. $S_1(x, y') = 0$ implies $x = \xi_1(y')$ and $S_2(x, y') = 0$ implies $x = \xi_2(y')$. Hence $x = \xi_1^{(m)}(y')$ and $x = \xi_2^{(m)}(y')$ are the approximate explicit functions. Defining $b_1^{(0)} = b_1$ and $b_2^{(0)} = b_2$.

$$b_1^{(m)} = - \sum_{n=1}^m \left\{ \frac{\xi_1^{2n}}{(2n)!} (-L_{y'})^n C_0^{(m-n)} + \frac{\xi_1^{2n+1}}{(2n+1)!} (-L_{y'})^n C_1^{(m-n)} \right\}$$

$$b_2^{(m)} = - \sum_{n=1}^m \left\{ \frac{\xi_2^{2n}}{(2n)!} (-L_{y'})^n C_0^{(m-n)} + \frac{\xi_2^{2n+1}}{(2n+1)!} (-L_{y'})^n C_1^{(m-n)} \right\}$$

so that

$$C_0^{(m)} + \xi_1 C_1^{(m)} = b_1^{(m)}$$

$$C_0^{(m)} + \xi_2 C_1^{(m)} = b_2^{(m)}$$

yielding

$$C_0^{(m)} = \frac{\xi_2 b_1^{(m)} - \xi_1 b_2^{(m)}}{\xi_2 - \xi_1}$$

$$C_1^{(m)} = \frac{b_2^{(m)} - b_1^{(m)}}{\xi_2 - \xi_1}$$

if we exclude the trivial case $\xi_2 \neq \xi_1$. If $\lim_{m \rightarrow \infty} S^{(m)}(x, y, z) = S(x, y, z)$, then

$$\lim_{m \rightarrow \infty} \sigma_m[u] = u.$$

The ideas used for the solution by decomposition of algebraic equations can be used to obtain smooth expansions of piecewise-differentiable functions and do so without Gibbs phenomena. Thus in our consideration of irregular contours and surfaces, we can if necessary go farther and consider non-smooth contours and surfaces as well. Physically, this means we can approximate shapes of artificial devices. Mathematically, it means broadening of the class of nonlinearities.

Consider, for example, the function formed on the domain $0 \leq x \leq 2$ by two simple parabolas, one with vertex at $x = 0, y = 0$ and one with vertex at $x = 2, y = 0$. e.g., $y = x^2$ and $y = (x-2)^2$. Write $y(x) = P_1(x)$ for $0 \leq x \leq 1$ and

$y(x) = P_2(x)$ for $1 \leq x \leq 2$. Now $y(x)$ is non-differentiable at $(1,1)$. We will show that an A_n expansion can be carried out, i.e., the analyticity requirement can be weakened. Write $[y - P_1(x)][y - P_2(x)] = 0$ considering $P_1(x)$ and $P_2(x)$ as roots of a quadratic equation to be solved by decomposition. Of course, the functions need not both be parabolas, or for that matter, other functions can be used for either one with a discontinuity in the derivative at the intersection.

Consider the example

$$y(x) = x^2 \quad 0 \leq x \leq 1$$

$$y(x) = x \quad 1 \leq x \leq \infty$$

or

$$(y - x^2)(y - x) = 0$$

The decomposition form of this quadratic equation in y is given as

$$y(x) = \frac{x^2}{1+x} + \frac{1}{x(1+x)} \sum_{n=0}^{\infty} A_n \{y^2\}$$

where the $A_n \{y^2\}$ are the A_n polynomials for the function y^2 . The decomposition of y into components results in

$$y_0 = \frac{x^2}{(1+x)}$$

$$y_{n+1} = \frac{1}{x(1+x)} A_n \{y^2\} \quad n \geq 0$$

determining $y = \sum_{n=0}^{\infty} y_n$. An n -term approximation $\varphi_n = \sum_{i=0}^{n-1} y_i$ is

$$\varphi_n = \left(x^2/(1+x) \right) \left(1 + x/(1+x)^2 + 2x^2/(1+x)^4 + 5x/(1+x)^6 + \dots \right)$$

The result can be written

$$y(x) = \left(x^2/(1+x) \right) \sum_{m=0}^{\infty} k_m \left(x^m/(1+x)^{2m} \right)$$

if we define $k_0 = 1$ and $k_{m \geq 1} = \sum_{n=0}^{m-1} k_n k_{m-1-n}$. ($k_0 = 1$, $k_1 = 1$, $k_2 = 2$, $k_3 = 5$, $k_4 = 14$, $k_5 = 42, \dots$) This result now represents the function of interest; thus, it is analogous to a Fourier expansion. Note that the limit of y as

$x \rightarrow \infty$ is x , i.e., in the region $1 \leq x < \infty$; the limit of y as $x \rightarrow 0$ is x^2 . In [1] it is shown that the smallest root of a quadratic (or higher degree polynomial) is obtained first and we see the smallest root values in $0 < x < 1$ are from the $y = x^2$ section, while in the $1 \leq x < \infty$ region, the smallest root values are from $y = x$. The farther apart the roots are, the faster the convergence will be.

The second root of $y^2 - (x + x^2)y + x^3 = 0$ is obtained by dividing this by the root already obtained, e.g., if we consider only ϕ_1 a one-term approximant of the first root or $\phi_1 = x^2/(1+x)$, we have

$$\{y^2 - (x + x^2)y + x^3\} / \{y - x^2/(1+x)\} = 0$$

Thus, neglecting the remainder, the second root is $y = (x + x^2) - x^2/(1+x)$ whose limit for small x is x and whose limit for large x is x^2 . If the first root is $f_1(x)$, the second root is $[y^2 - (x + x^2)y + x^3] / [y - f_1(x)]$ or

$$y = (x^2 + x - f_1) + (x^3 - f_1(x^2 + x - f_1)) / (y - f_1)$$

which is solvable by decomposition. We can now do numerical calculations to see how well the result approximates the function of interest. We choose one value of x in each region and one at the discontinuity. At $x = 1/2$, we must have $y = 1/4$, i.e., $\lim_{n \rightarrow \infty} \phi_n = \sum_{i=0}^{\infty} y_i$ gives exactly those values.

$x = 1/2$	$x = 2$	$x = 1$
$\phi_1 = .1666667$	$\phi_1 = 1.3333333$	$\phi_1 = .5000000$
$\phi_2 = .2037037$	$\phi_2 = 1.6296296$	$\phi_2 = .6250000$
$\phi_3 = .2201646$	$\phi_3 = 1.7613169$	$\phi_3 = .6875000$
$\phi_4 = .2293096$	$\phi_4 = 1.8344764$	$\phi_4 = .7265625$
$\phi_5 = .2349998$	$\phi_5 = 1.879998$	$\phi_5 = .7539063$
$\phi_6 = .2387932$	$\phi_6 = 1.9103456$	$\phi_6 = .7744141$
$\phi_7 = .2414426$	$\phi_7 = 1.9315408$	$\phi_7 = .79052174$
$\phi_8 = .2433561$	$\phi_8 = 1.94682485$	$\phi_8 = .8036194$
$\phi_9 = .2447734$	$\phi_9 = 1.0591975$	$\phi_9 = .9145204$
$\phi_{10} = .2458444$	$\phi_{10} = 1.9667548$	$\phi_{10} = .8238030$
$\lim_{m \rightarrow \infty} \phi_m = 1/4$	$\lim_{m \rightarrow \infty} \phi_m = 2$	$\lim_{m \rightarrow \infty} \phi_m = 1$
1.6% error by ϕ_{10}	1.6% error by ϕ_{10}	

We note from observation of the values at $x = 1$ that the expansion in the A_n polynomials does not display the Gibbs phenomenon which we see in Fourier series [2]. Instead, we get a blending effect at the point of discontinuity in the derivative—an interesting application of the decomposition method.

REFERENCES

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2. G. Adomian and R. Rach, Smooth Polynomial Expansions of Piecewise-differentiable Functions, *Appl. Math. Lett.*, **2**, (377-79) (1989).

APPLICATIONS IN PHYSICS

Real problems of physics are generally nonlinear and often stochastic as well. Linearity and determinism should be viewed as special cases only. The general practices of linearization, perturbation, white noise, and quasi-monochromatic approximations necessarily change the problems whose solutions are desired, to be tractable by convenient mathematics. They are not then identical to the physical solutions which we seek. The alternative of using the decomposition method will be explored here as we consider examples of problems of physics. These problems are often quite difficult because nonlinearity and stochasticity are involved. Decomposition makes unnecessary procedures such as closure approximations [1] and perturbation and white noise processes in differential equations which involve stochastic process parameters, inputs, or initial/boundary conditions. Decomposition also avoids discretization and consequent intensive computer calculation and yields analytic expressions rather than tables of numbers. Thus quantitative solutions are obtained for dynamical systems. (When the systems are stochastic as well, the decomposition series involves stochastic terms from which statistics can be calculated.) The method applies to linear or nonlinear, ordinary or partial differential equations and is useful for many algebraic, integral, and delay-differential equations [2]. This chapter will outline procedures for typical applications.

ANALYTICAL SOLUTION OF THE NAVIER-STOKES EQUATIONS:

The Navier-Stokes model* [1, 2] for an incompressible fluid of kinematic viscosity ν , and constant density ρ is given as :

$$\begin{aligned} (\partial \bar{u} / \partial t) + (\bar{u} \cdot \nabla) \bar{u} - \nu \nabla^2 \bar{u} + (1/\rho) \nabla p &= \bar{F} & u \in \Omega \times (0, T) \\ \nabla \cdot \bar{u} &= 0 & \Omega \times (0, T) \\ \bar{u} &= 0 & d\Omega \times (0, T) \end{aligned}$$

where $\bar{u}$ is a vector with components u, v, w .

* This treatment differs from the earlier work presented in [2] in that pressure is dynamic allowing for large velocity and turbulence.

We assume that the velocity $\bar{u}(x, y, z, t, \omega)$, the pressure, $p(x, y, z, t, \omega)$, and the external force, $\bar{F}$ are stochastic processes. In terms of velocity components u, v, w , we write

$$\begin{aligned} & (\partial u / \partial t) + u(\partial u / \partial x) + v(\partial u / \partial y) + w(\partial u / \partial z) \\ & - v\{(\partial^2 u / \partial x^2) + (\partial^2 u / \partial y^2) + (\partial^2 u / \partial z^2)\} \\ & + (1/\rho)(\partial p / \partial x) = F_x \end{aligned} \quad (1)$$

with similar equations for v (replacing F_x by F_y and $\partial p / \partial x$ by $\partial p / \partial y$) and for w (replacing F_y by F_z and $\partial p / \partial y$ by $\partial p / \partial z$). We define an initial-boundary problem by specifying initial conditions for u, v, w and for $t \geq 0$, specifying u, v, w , on the boundary.

Let's rewrite the system (1) in the equation above in the decomposition form.

$$Lu + N_1(u, v, w) = g_1$$

$$Lv + N_2(u, v, w) = g_2$$

$$Lw + N_3(u, v, w) = g_3$$

We have some choice on the definition of nonlinear terms. Let's consider

$$L = (\partial / \partial t) - v(\partial^2 / \partial x^2) - v(\partial^2 / \partial y^2) - v(\partial^2 / \partial z^2)$$

$$= L_t + L_x + L_y + L_z$$

$$N_1 = u(\partial u / \partial x) + v(\partial u / \partial y) + w(\partial u / \partial z)$$

$$N_2 = v(\partial v / \partial y) + w(\partial v / \partial z) + u(\partial v / \partial x)$$

$$N_3 = w(\partial w / \partial z) + u(\partial w / \partial x) + v(\partial w / \partial y)$$

$$g_1 = F_x - (1/\rho)(\partial p / \partial x)$$

$$g_2 = F_y - (1/\rho)(\partial p / \partial y)$$

$$g_3 = F_z - (1/\rho)(\partial p / \partial z)$$

To complete the specification of g_1, g_2, g_3 we must know the pressure function. We can assume an initial pressure which will, of course, become a function of x, y, z, t as any disturbance occurs. However, we must determine the functional dependence of pressure on the velocities u, v, w so that the g_1, g_2, g_3 are calculable.

If we find the divergence of each term in the Navier-Stokes equation, the ∇p becomes $\nabla^2 p$ or $1/\rho \nabla^2 p$ depending on the definition used for p . The first and third terms vanish from the divergence condition. The second term gives us $\nabla(\bar{u} \cdot \nabla) \cdot \bar{u}$. Thus

$$\nabla^2 p = \nabla \cdot \bar{F} - \nabla(\bar{u} \cdot \nabla) \cdot \bar{u}$$

Thus,

$$\begin{aligned} [L_x + L_y + L_z]p &= \nabla \cdot \bar{F} - \left(\frac{\partial u}{\partial x} \right)^2 - \left(\frac{\partial v}{\partial y} \right)^2 - \left(\frac{\partial w}{\partial z} \right)^2 \\ &\quad - 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} - 2 \frac{\partial w}{\partial x} \frac{\partial u}{\partial z} - 2 \frac{\partial w}{\partial y} \frac{\partial v}{\partial z} \end{aligned}$$

Symbolizing the right side by f , solving for $L_x p$ and inverting the operator L_x , we have

$$p = A + Bx + L_x^{-1}f - L_x^{-1}L_y p - L_x^{-1}L_z p$$

Writing $p = \sum_{n=0}^{\infty} p_n$ and identifying

$$p_0 = A + Bx + L_x^{-1}f$$

we have for $n \geq 0$

$$p_{n+1} = -L_x^{-1}L_y p_n - L_x^{-1}L_z p_n$$

and we can write an n -term approximation for p by $\phi_n = \sum_{i=0}^{n-1} p_i$ which converges to $\sum_{n=0}^{\infty} p_n$ or p . (Similar equations can be written for $L_y p$ and $L_z p$.)

We assumed an initial pressure which gives us the A in our equation for p_0 . The coefficient B is zero since the disturbance vanishes as $x \rightarrow \infty$. We use this p to find u, v, w . The resulting velocities are used in our equation for p , as a function of velocities, to yield an improved $p = p_0 + p_1$ (which re-calculates p_0 because of the change in f). This is used to improve results for velocities u, v, w . These calculations can proceed until we have sufficiently accurate results for u, v, w, p . We have

$$\begin{aligned} L_t u + L_x u + L_y u + L_z u &= g_1 - N_1 \\ L_t v + L_x v + L_y v + L_z v &= g_2 - N_2 \\ L_t w + L_x w + L_y w + L_z w &= g_3 - N_3 \end{aligned}$$

from which

$$\begin{aligned}
L_t u &= g_1 - N_1 - L_x u - L_y u - L_z u \\
L_x u &= g_1 - N_1 - L_t u - L_y u - L_z u \\
L_y u &= g_1 - N_1 - L_x u - L_t u - L_z u \\
L_z u &= g_1 - N_1 - L_x u - L_y u - L_t u
\end{aligned}$$

Four similar equations can be written for v with g_1 and N_1 replaced by g_2 and N_2 and also four equations for w using g_3 and N_3 .

It has been shown in Chapter 3 that when the boundary conditions are general (when conditions on any one variable depend upon all the others) that to solve for u, v, w , we can use any of the four operator equations depending on the given conditions and integrations required. If we know initial conditions, the equations involving the operator L_t on the left side will be simplest since only a single integration will be required. We can also solve the system as a boundary-value problem using any of the equations involving L_x, L_y , or L_z on the left side as discussed in Chapter 4. Hence, using the first equation of each set above and operating with L_t^{-1} , we have

$$\begin{aligned}
u &= u(0) + L_t^{-1} g_1 - L_t^{-1} N_1 - L_t^{-1} (L_x + L_y + L_z) u \\
v &= v(0) + L_t^{-1} g_2 - L_t^{-1} N_2 - L_t^{-1} (L_x + L_y + L_z) v \\
w &= w(0) + L_t^{-1} g_3 - L_t^{-1} N_3 - L_t^{-1} (L_x + L_y + L_z) w
\end{aligned}$$

Now write the decompositions $u = \sum_{n=0}^{\infty} u_n, v = \sum_{n=0}^{\infty} v_n, w = \sum_{n=0}^{\infty} w_n$. Also, write N_1, N_2, N_3 in terms of the A_n polynomials and finally identify:

$$\begin{aligned}
u_0 &= u(0) + L_t^{-1} g_1 \\
v_0 &= v(0) + L_t^{-1} g_2 \\
w_0 &= w(0) + L_t^{-1} g_3
\end{aligned}$$

The remaining components of u, v, w for $n \geq 0$ can now be determined

$$\begin{aligned}
u_{n+1} &= -L_t^{-1} A_n \{N_1\} - L_t^{-1} (L_x + L_y + L_z) u_n \\
v_{n+1} &= -L_t^{-1} A_n \{N_2\} - L_t^{-1} (L_x + L_y + L_z) v_n \\
w_{n+1} &= -L_t^{-1} A_n \{N_3\} - L_t^{-1} (L_x + L_y + L_z) w_n
\end{aligned}$$

where the notation $A_n \{\cdot\}$ refers to the A_n for the quantity in brackets. We now

have a completely calculable system which, if we ignore stochasticity for the moment, and approximate by

$$\begin{aligned}\phi_n^{(u)} &= \sum_{i=0}^{n-1} u_i \\ \phi_n^{(v)} &= \sum_{i=0}^{n-1} v_i \\ \phi_n^{(w)} &= \sum_{i=0}^{n-1} w_i\end{aligned}$$

we have found u , v , w to n -term approximations.

In the stochastic case, the expressions for u , v , w are stochastic series, i.e., series containing stochastic processes which we must solve for first- and second-order statistics, where each velocity component is replaced by a sum of a deterministic component velocity and a stochastic component. We do this now in spite of the fact that the equation obtained by replacing velocities with stochastic processes may not be the correct stochastic model since the equation was derived deterministically. An example of this is the problem of wave propagation in a random medium where it is incorrect to simply replace the velocity in the d'Alembertian operator with a stochastic quantity; i.e., a stochastic model must be derived which has the deterministic model as a limit rather than using the deterministic model to obtain a stochastic model [3]. Thus, we must obtain

$$\begin{aligned}\langle u \rangle &= \langle u_0 \rangle + \langle u_1 \rangle + \langle u_2 \rangle + \dots \\ \langle v \rangle &= \langle v_0 \rangle + \langle v_1 \rangle + \langle v_2 \rangle + \dots \\ \langle w \rangle &= \langle w_0 \rangle + \langle w_1 \rangle + \langle w_2 \rangle + \dots\end{aligned}$$

remembering that g_1 , g_2 , g_3 are stochastic, since F and p are stochastic and the A_n are stochastic.

The two-point correlation for each velocity component u , v , w is obtained by averaging the product of series for the velocity component at two space-time points. If we consider, for example, fixed space position and time scales such that stationarity can be assumed, the ergodic hypothesis may hold so that ensemble averages can be replaced with time averages of observations.

Since nonlinear terms can contain both functions of a single variable, such as $f(u)$, and also functions of variables, such as $f(u,v)$ and $f(u,v,w)$, we have listed these generalized A_n in Chapter 3. We can use the $A_n\{f(u,v)\}$ since N_1 , N_2 , N_3 can be considered term by term. Now the coefficients in the

generalized algorithm for $f(u,v)$ will involve three quantities, the derivative is $f^{(\mu,v)}$ and the summation is over μ, v . Using the resulting generalized A_n , we obtain a general solution. Smooth solutions (to the incompressible problem under consideration) do exist for short times and are continuously dependent on the initial data.

A basic question is whether the Navier-Stokes equations are an adequate model for real turbulent fluids. The linear constitutive law used in the derivation means that derivatives of the velocity components u, v, w are necessarily small. Secondly, stochasticity cannot be considered as an afterthought; it must be considered in the initial modelling. A more general model due to Ladyzhenskaya has partially addressed this issue by allowing nonlinearity in the constitutive law which leads to a global uniqueness for nonstationary three-dimensional flow. A truly nonlinear stochastic model coupled with the decomposition method of solution may resolve remaining difficulties.

SOME THOUGHTS ON THE ONSET OF TURBULENCE:

Consider first a very simple equation whose solution is trivial. Thus consider $dy/dx = (y-1)^2$ which obviously is satisfied by $y = 1$. Now consider the effect of a 1% change in a parameter by writing $dy/dx = (y-1)^2 + .01$.** This now yields a periodic solution $y = 1 + 0.1 \tan(x/10)$ which has vertical asymptotes at $(2k+1)5\pi$, $k = 0, \pm 1, \pm 2, \dots$

Now, let's make a 1% change in the initial condition, or $y(0) = 1.01$. We now have a hyperbola $y = 1 - 1/(x-100)$ *** and only one vertical asymptote at $x = 100$. Thus the effect in a *nonlinear* equation of even very small changes in inputs or parameters can result in large effects on the solution.

Suppose now that very small fluctuations are present in the input and parameter because of small inherent randomness. Then the solution could change randomly between the possibilities above and appears very complex indeed.

Now considering the Navier-Stokes system with its nonlinear terms where there could be small fluctuations in density, pressure, viscosity, and velocities, it is clear that we can expect similar effects and a "chaotic-looking" or turbulent case.

The nonlinear terms cause small fluctuations to become large fluctuations

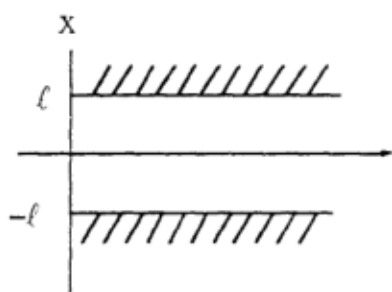
** We have considered $y' = (y-1) + a$ with $a > 0$. If $a < 0$, solution varies between two horizontal asymptotes with inflection point at $(0,1)$. The asymptotes coincide if $a = 0$. The solution $y = 1$ is a singular solution not derivable from the general solution.

*** If $y = 0.99$, the asymptote moves to $x = -100$.

while friction terms tend to remove differences in velocities. The Reynolds number is a measure of the ratio of nonlinear terms to frictional terms, so it is reasonable that if the number becomes large, the tendency to turbulence increases. However, factors such as smoothness of boundaries and the magnitude of initial fluctuations also influence the resulting flow.

In the simple deterministic case, consider one nonlinear term $u \partial u / \partial x$ divided by a molecular friction term $\nu \partial^2 u / \partial x^2$. If u and $\partial u / \partial x$ are assumed to be of the order U , and L is a typical distance over which the velocity varies by U , the ratio is of the order $(U^2/L)/(\nu \cdot U^2/L) = U \cdot L/\nu$ or the Reynolds number. In the general case, if we have a fluctuation in ν or ν we can see that large changes can occur in the tendency to turbulent behavior.

The best way, apparently, to determine when turbulence starts is to solve the stochastic Navier-Stokes system as we have outlined and study the behavior as a function of the parameters of the flow. A comparison of a deterministic solution and a stochastic solution with varying conditions should illuminate the problem of the onset of turbulence. Suppose we consider flow in a flat channel as an idealization of a pipe in a plane. We have



Replace x by x/ℓ to make the half-width unity and assume $\partial \rho / \partial x = 0$. Write

$$\begin{aligned} L_t u &= \nu (\partial^2 / \partial x^2) u - u (\partial / \partial x) u \\ u &= L_t^{-1} \nu (\partial^2 / \partial x^2) \sum_{n=0}^{\infty} u_n - L_t^{-1} \sum_{n=0}^{\infty} A_n \end{aligned}$$

where the A_n polynomials are generated for the nonlinear term. Then

$$\begin{aligned} u_0 &= u(0) + t u'(0) \\ u_{n+1} &= L_t^{-1} \nu (\partial^2 / \partial x^2) u_n - L_t^{-1} A_n \end{aligned}$$

for $n \geq 0$. If v is constant and u_0 is deterministic, u is deterministic. If u_0 has a random component, this component will cause new terms to keep appearing because of the expressions on the right side of the equation for u_{n+1} for any $n \geq 0$, especially from the term involving A_n . This is obvious by inspection of the A_n for increasing n . The effect of physically unrealistic change in the solution by a linearization is also clear. Consequently, as a result of any randomness and the nonlinearity, the flow is radically altered—the effect increasing as the fluctuation becomes larger. Random boundary conditions resulting from roughness in the walls will have the same effect.

The general problem may have random initial/boundary conditions. ρ is generally taken as a constant and set equal to unity; however, compressibility becomes a factor with increasing depth and ρ may not only be a function of z but random in turbulent conditions.

THE VAN DER POL EQUATION:

$$u'' + \alpha u' + \beta u + \gamma u'u^2 = g(t)$$

$$u(0) = c_0$$

$$u'(0) = c_1$$

Let

$$L = d^2/dt^2$$

$$L^{-1}(\cdot) = \int_0^t \int_0^t (\cdot) dt dt$$

$$g(t) = \sum_{n=0}^{\infty} g_n t^n$$

$$Lu = g(t) - \alpha(d/dt)u - \beta u - \gamma u'u^2$$

$$u = u_0 - L^{-1}\alpha(d/dt)u - L^{-1}\beta u - L^{-1}\gamma u'u^2$$

with

$$u_0 = u(0) + tu'(0) + L^{-1}g(t) = u(0) + tu'(0) + \sum_{n=0}^{\infty} g_n t^{n+2} / (n+1)(n+2)$$

Let $u = \sum_{m=0}^{\infty} u_m$, $u'u^2 = \sum_{m=0}^{\infty} A_m$. The A_m can be found from

$$A_m = \sum_{n=0}^m \sum_{v=0}^n u_{m-n} u_{n-v} u'_v$$

Since we have u_0 , all the following components are determined from

$$u_m = -L^{-1}\alpha u'_{m-1} - L^{-1}\beta u_{m-1} - L^{-1}\gamma A_{m-1}$$

The approximants to the solution are given by

$$\begin{aligned}\phi_1 &= u_0 \\ \phi_{m+1} &= \phi_m + u_m \\ \lim_{m \rightarrow \infty} \phi_m &= u\end{aligned}$$

The A_m are

$$\begin{aligned}A_0 &= u'_0 u_0^2 \\ A_1 &= u'_1 u_0^2 + 2u'_0 u_0 u_1 \\ A_2 &= u'_2 u_0^2 + 2u'_1 u_0 u_1 + u'_0 u_1^2 + 2u'_0 u_0 u_2 \\ A_3 &= u'_3 u_0^2 + 2u'_2 u_0 u_1 + u'_1 u_1^2 + 2u'_1 u_0 u_2 \\ &\quad + 2u'_0 u_1 u_2 + 2u'_0 u_0 u_3 \\ A_4 &= u'_4 u_0^2 + 2u'_3 u_0 u_1 + u'_2 u_1^2 + 2u'_2 u_0 u_2 \\ &\quad + 2u'_1 u_1 u_2 + 2u'_1 u_0 u_3 + u'_0 u_2^2 + 2u'_0 u_1 u_3 \\ &\quad + 2u'_0 u_0 u_4 \\ A_5 &= u'_5 u_0^2 + 2u'_4 u_0 u_1 + u'_3 u_1^2 + 2u'_3 u_0 u_2 \\ &\quad + 2u'_2 u_1 u_2 + 2u'_2 u_0 u_3 + u'_1 u_2^2 + 2u'_1 u_1 u_3 \\ &\quad + 2u'_1 u_0 u_4 + 2u'_0 u_2 u_3 + 2u'_0 u_1 u_4 + 2u'_0 u_0 u_5 \\ A_6 &= u'_6 u_0^2 + 2u'_5 u_0 u_1 + u'_4 u_1^2 + 2u'_4 u_0 u_2 \\ &\quad + 2u'_3 u_1 u_2 + 2u'_3 u_0 u_3 + u'_2 u_2^2 + 2u'_2 u_1 u_3 \\ &\quad + 2u'_2 u_0 u_4 + 2u'_1 u_2 u_3 + 2u'_1 u_1 u_4 + 2u'_1 u_0 u_5 \\ &\quad + u'_0 u_3^2 + 2u'_0 u_2 u_4 + 2u'_0 u_1 u_5 + 2u'_0 u_0 u_6 \\ &\quad \vdots\end{aligned}$$

NOTE: This is done by writing $u' u^2 = Nu = N_1 \cdot N_2$, writing $\sum B_n$ for u' and $\sum C_n$ for u^2 and considering the possible products, e.g., $A_5 = B_1 C_5 + B_1 C_4 + \dots + B_4 C_1 + B_5 C_0$. (See Chapter 2.)

Given the A_m , the u_m can be calculated and rearranged in ascending powers of t (see Appendix II) to get solutions to any required answer. The same procedure as with the Duffing equation can be used to write $u = \sum_{n=0}^{\infty} c_n t^n$ and calculate the c_n . However, we can get a quick approximation as in the following example.

EXAMPLE: $u'' + u' + u + u'u^2 = -\sin t - (\sin t)(\cos^2 t)$

with $u(0) = 1$ and $u'(0) = 0$. Approximating $\sin t$ by t and $\cos t$ by $1 - t^2/2$, we find $L^{-1}g$ doesn't contribute, $u_0 = 1$, $u_1 = -t^2/2$, so $\phi_2 = 1 - t^2/2$ which we recognize as a two-term approximant of $u = \cos t$ and we can verify by showing that it satisfies the equation and the given conditions.

BURGER'S EQUATION:

The equation is $u_t + uu_x = v u_{xx}$ for $x \geq 0$ and $t \geq 0$ where necessary conditions must, of course, be given. We write

$$\begin{aligned} L_t u &= v L_{xx} u - uu_x \\ u(t \neq 0) &= f(x) \end{aligned}$$

with $L_t = \partial/\partial t$ and $L_{xx} = \partial^2/\partial x^2$. Operating with $L_t^{-1} = \int_0^t (\cdot) dt$, we have

$$u = u(t=0) + L_t^{-1} v L_{xx} u - L_t^{-1} uu_x$$

We identify $u(t \neq 0) = f(x)$ as the u_0 term of the decomposition $u = \sum_{n=0}^{\infty} u_n$ and write the nonlinearity uu_x as $\sum_{n=0}^{\infty} A_n$ where

$$\begin{aligned} A_0 &= u_0 u'_0 \\ A_1 &= u_0 u'_1 + u_1 u'_0 \\ A_2 &= u_2 u'_0 + u_1 u'_1 + u_0 u'_2 \\ A_3 &= u_3 u'_0 + u_2 u'_1 + u_1 u'_2 + u_0 u'_3 \\ &\vdots \\ A_n &= u_n u'_0 + u_{n-1} u'_1 + \dots + u_1 u'_{n-1} + u_0 u'_n \end{aligned}$$

Now the components after u_0 are given by

$$u_m = L_t^{-1} v L_{xx} u_{m-1} - L_t^{-1} A_m$$

and we can write the m -term approximant which converges rapidly to the correct solution $\phi_m = \sum_{n=0}^{m-1} u_n$. Since either of the possible operator equations for $L_t u$ and $L_{xx} u$ can yield the solution in the general case where the conditions for $t = 0$ depend on x and the conditions on x depend on t , it is no longer nec-

essary to use both operator equations as in earlier work. (When the conditions are not general in this way, we have asymptotic equality.) Integrations for a difficult $f(x)$ can be made trivial by writing $f(x)$ in series form and carrying a limited number of terms. If we use the L_{xx} equation, $u_0 = A + Bx$ where the A, B are evaluated from the boundary conditions and we note that L_{xx}^{-1} represents a two-fold indefinite integration. If we have a non-zero $u(x, t = 0) = f(x)$, the problem is simply solved. If $f(x) = 0$, we must use the L_{xxx} equation.

KURAMOTO-SIVASHINSKY EQUATION:

The K-S equation is given as

$$u_t + uu_x + \mu u_{xx} + \nu u_{xxx} = 0$$

In the operator notation of the decomposition method, this is

$$L_t u + L_x u + Ru + Nu = 0$$

where

$$L_t = \partial / \partial t$$

$$L_x = \nu \partial^4 / \partial x^4$$

$$Ru = \mu \partial^2 u / \partial x^2$$

$$Nu = u(\partial / \partial x)u$$

This equation describes problems in fluid motion, fluctuations in the position of a flame front and oscillating chemical reactions. There are a number of possibilities dependent on the stated conditions. Suppose we know that $u(x, 0) = f(x)$ explicitly. Then we write

$$L_t u = -Ru - L_x u - Nu$$

and define $L_t^{-1} = \int_0^t (\cdot) dt$. Now

$$L_t^{-1} L_t u = -L_t^{-1} \mu (\partial^2 / \partial x^2) u - L_t^{-1} \nu (\partial^4 / \partial x^4) u - L_t^{-1} u (\partial / \partial x) u$$

Substituting $u = \sum_{n=0}^{\infty} u_n$, $Nu = \sum_{n=0}^{\infty} A_n$ where the A_n are generated for $u(\partial / \partial x)u$ or uu' . These are found as

$$\begin{aligned}
 A_0 &= u_0 u'_0 \\
 A_1 &= u_0 u'_1 + u_1 u'_0 \\
 A_2 &= u_0 u'_2 + u_1 u'_1 + u_2 u'_0 \\
 &\vdots \\
 A_n &= u_0 u'_n + u_1 u'_{n-1} + \cdots + u_n u'_0
 \end{aligned}$$

In the decomposition of u into $\sum_{n=0}^{\infty} u_n$, we identify $u_0 = f(x)$. Then from

$$u = f(x) - L_t^{-1} \mu (\partial^2 / \partial x^2) \sum_{n=0}^{\infty} u_n - L_t^{-1} \nu (\partial^4 / \partial x^4) \sum_{n=0}^{\infty} u_n - L_t^{-1} \sum_{n=0}^{\infty} A_n$$

we have

$$\begin{aligned}
 u_0 &= f(x) \\
 u_{n+1} &= -L_t^{-1} \mu (\partial^2 / \partial x^2) u_n - L_t^{-1} \nu (\partial^4 / \partial x^4) u_n - L_t^{-1} A_n
 \end{aligned}$$

for $n \geq 0$ so all components are calculable. We compute $\phi_n = \sum_{i=0}^{n-1} u_i$ as an n -term approximant to the solution $u = \sum_{n=0}^{\infty} u_n$. The results are sufficient for a complete solution if $f(x)$ is continuous and n -times differentiable or may be appropriately transformed by Fourier series. Now ϕ_n must satisfy the equation to n th approximation and exactly as $n \rightarrow \infty$. Of course, a numerical result depends on an explicit $f(x)$.

Given boundary conditions on x , such as:

$$\begin{aligned}
 u(x = \xi_1, t) &= b_1 \\
 u(x = \xi_2, t) &= b_2 \\
 u(x = \xi_3, t) &= b_3 \\
 u(x = \xi_4, t) &= b_4
 \end{aligned}$$

we can also solve $L_x u$ which will require four-fold (indefinite) integrations. Then

$$L_x^{-1}(\cdot) = \sum_{n=0}^3 c_n(t) x^n / n! + (1/\nu) \iiint \iiint (\cdot) dx \, dx \, dx \, dx$$

(If ν is a function of x , it must, of course, be inside the integration.) We write

$$u = c_0(t) + c_1(t)x + c_2(t)x^2/2 + c_3(t)x^3/6 \\ - I_x^4(1/v)(\partial/\partial t)u - I_x^4(1/v)Ru - I_x^4(1/v)Nu$$

where $I_x(\cdot) = \int(\cdot)dx$. By decomposition

$$u = u_0 - I_x^4(1/v)(\partial/\partial t) \sum_{n=0}^{\infty} u_n \\ - I_x^4 \left\{ (1/v) \left(\mu \partial^2 / \partial x^2 \sum_{n=0}^{\infty} u_n \right) \right\} - I_x^4(1/v) \sum_{n=0}^{\infty} A_n$$

Now

$$u_0 = c_0 + c_1 x + c_2 x^2/2 + c_3 x^3/6$$

$$u_{n+1} = -I_x^4(1/v)(\partial/\partial t)u_n - I_x^4 \left\{ (1/v) (\mu \partial^2 / \partial x^2) u_n \right\} - I_x^4(1/v)A_n$$

Since this is a nonlinear equation, we must evaluate the coefficients for each approximant ϕ_m for each $m = 1, 2, \dots$. Alternatively, we can use double decomposition. In this case we write

$$u_n = \sum_{m=0}^{\infty} u_n^{(m)} \\ u_0 = \sum_{m=0}^{\infty} u_0^{(m)} \\ A_n = \sum_{m=0}^{\infty} A_n^{(m)} \\ c_i(t) = \sum_{m=0}^{\infty} c_i^{(m)}(t) \quad i = 0, 1, 2, 3$$

Then

$$u_i = u_0^{(0)} = \sum_{k=0}^3 c_k^{(0)}(t) x^k / k! \\ \vdots \\ u_m = \sum_{k=0}^3 c_k^{(m)}(t) x^k / k! - I_x^4(1/v)(\partial/\partial t)u_{m-1} \\ - I_x^4(1/v)(\mu \partial^2 / \partial x^2)u_{m-1} - I_x^4(1/v)A_{m-1}$$

Matching the solution approximants at the boundaries determines the components of the integration constants as discussed in Chapters 3 and 4.

THE LANE-EMDEN EQUATION:

This is one of the basic equations in the theory of stellar structure in astrophysics and was recently solved by N.T. Shawagfeh [4] using the decomposition method. It is given by

$$r^{-2} \frac{d^2}{dr^2} [r^2 T] + \lambda T^m = 0$$

where m is particularly of interest in the range from 0 to 5 with the conditions

$$T(0) = 1 \quad \text{and} \quad [dT/dr]_{r=0} = 0$$

What was needed was an approximation which did not require λ to be small. Such a non-perturbative solution follows. First, the dependent and independent variables are transformed* using $\theta = \xi T$ and $\xi = \lambda^{1/2} r$ to obtain

$$\begin{aligned} \frac{\partial^2 \theta}{\partial \xi^2} &= -\xi^{1-m} \theta^m \\ \theta(0) &= 1 \\ \frac{d\theta}{d\xi} \Big|_{\xi=0} &= 0 \end{aligned}$$

which is written as

$$L\theta = -\xi^{1-m} \theta^m$$

where $L = d^2/d\xi^2$ and L^{-1} is a two-fold integration with respect to ξ . Now

$$\theta = \theta_0 - L^{-1} \xi^{1-m} \theta^m$$

where $\theta_0 = \xi(0) + \xi d\theta/d\xi|_{\xi=0} = \xi$. The nonlinearity is $f(\theta) = \theta^m = \sum_{n=0}^{\infty} A_n$ where

* This step is eliminated and results generalized in work to be published.

$$\begin{aligned}
A_0 &= f(\theta_0) \\
A_1 &= \theta_1 f'(\theta_0) \\
A_2 &= \theta_2 f'(\theta_0) + \theta_1^2 / 2 f''(\theta_0) \\
A_3 &= \theta_3 f'(\theta_0) + \theta_1 \theta_2 f''(\theta_0) + \theta_1^3 / 3! f^{(3)}(\theta_0) \\
&\vdots
\end{aligned}$$

Now $\theta = \sum_{n=0}^{\infty} \theta_n$ where

$$\begin{aligned}
\theta_0 &= \xi \\
\theta_{n+1} &= -L^{-1} \xi^{1-n} A_n
\end{aligned}$$

for $n \geq 0$. θ can now be written as a series in the form

$$\theta = \xi + c_1 \xi^3 + c_2 \xi^5 + \dots$$

where the c 's are determined as follows:

$$\begin{aligned}
c_1 &= -1/3! \\
c_2 &= m/5! \\
c_3 &= -(1/7!) \left(\frac{8m^2 - 5m}{3} \right) \\
&\vdots
\end{aligned}$$

Finally, since $\theta = \xi T$, $T = c_1 \xi^2 + c_2 \xi^4 + c_3 \xi^6 + \dots$. An accurate and easily computed solution is obtained with seven terms.

NONLINEAR TRANSPORT IN MOVING FLUIDS:

A new approach to time-dependent spread of contaminants in moving fluids is provided by decomposition which is easily extended to nonlinear and stochastic partial differential equations as well. First we consider

The one-dimensional advection equation:

$$\begin{aligned}
\partial u / \partial t + \alpha \partial u / \partial x &= 0 & 0 < t \leq T, \quad 0 \leq x \leq 1, \quad \alpha > 0 \\
u(x, 0) &= f(x) \\
u(0, t) &= g(t)
\end{aligned}$$

By decomposition and using the partial solution for t , we have

$$u = u(x, 0) - L_t^{-1} \alpha (\partial/\partial x) \sum_{n=0}^{\infty} u_n$$

where

$$L_t = \partial/\partial t \quad L_t^{-1} = \int_0^t (\cdot) dt$$

$$u = \sum_{n=0}^{\infty} u_n \quad u(x, 0) = f(x)$$

is identified as u_0 , and $f(x)$ is assumed differentiable as necessary. Then

$$u_0 = f(x)$$

$$u_1 = -\alpha L_t^{-1} (\partial/\partial x) f(x) = -\alpha t f'(x)$$

$$u_2 = (\alpha^2 t^2 / 2!) f''(x)$$

$$\vdots$$

$$u_n = (-1)^n (\alpha^n t^n / n!) f^{(n)}(x)$$

so that

$$u = \sum_{n=0}^{\infty} (-1)^n (\alpha^n t^n / n!) f^{(n)}(x)$$

and

$$\phi_{m+1} = \sum_{n=0}^m (-1)^n (\alpha^n t^n / n!) f^{(n)}(x) \quad (2)$$

is an $(m+1)$ -term approximation to u , satisfying the equation and the condition at $t=0$ using the t -dimension "partial solution".

The x -dimension partial solution is derived by

$$\alpha \partial u / \partial x = -\partial u / \partial t$$

$$L_x u = -\alpha^{-1} (\partial/\partial t) \sum_{n=0}^{\infty} u_n$$

$$u = u(0, t) - \alpha^{-1} L_x^{-1} (\partial/\partial t) \sum_{n=0}^{\infty} u_n$$

Consequently,

$$\begin{aligned}
u_0 &= g(t) \\
u_1 &= -\alpha^{-1} L_x^{-1}(\partial/\partial t)u_0 = -\alpha^{-1}x g'(t) \\
u_2 &= \alpha^{-2}(x^2/2!)g^{(2)}(t) \\
&\vdots \\
\phi_{m+1} &= \sum_{n=0}^m (-1)^n \alpha^{-n} (x^2/2!)g^{(n)}(t)
\end{aligned} \tag{3}$$

Either the t equation or the x equation represent the solution under general conditions.

ADVECTION-DIFFUSION EQUATION:

Let $\xi(x, y, z, t)$ represent concentration. Let the fluid velocity be $\bar{u}$ with components u, v, w in R^3 and assume an incompressible fluid

$$\partial \xi / \partial t + \bar{u} \cdot \nabla \xi = D \nabla^2 \xi$$

where D is the diffusion constant (which is a constant for a particular fluid or contaminant, temperature and pressure), $\xi(x, y, z, 0)$ is a given initial condition and various boundary conditions are possible. e.g., $\xi \rightarrow 0$ as $x, y, z \rightarrow \infty$, or $\xi(t)$ is specified on a boundary Γ , or, we have a preassigned flux at Γ . We have

$$\begin{aligned}
\partial \xi / \partial t &= D \{ \partial^2 \xi / \partial x^2 + \partial^2 \xi / \partial y^2 + \partial^2 \xi / \partial z^2 \} \\
&\quad - u \partial \xi / \partial x - v \partial \xi / \partial y - w \partial \xi / \partial z
\end{aligned} \tag{4}$$

By decomposition, using $L_t = \partial/\partial t$ and $L^{-1} = \int_0^t (\cdot) dt$

$$\begin{aligned}
\xi &= \xi(t=0) + DL_t^{-1}(\partial^2/\partial x^2) \sum_{n=0}^{\infty} \xi_n + DL_t^{-1}(\partial^2/\partial y^2) \sum_{n=0}^{\infty} \xi_n \\
&\quad + DL_t^{-1}(\partial^2/\partial z^2) \sum_{n=0}^{\infty} \xi_n - L_t^{-1}u(\partial/\partial x) \sum_{n=0}^{\infty} \xi_n \\
&\quad - L_t^{-1}v(\partial/\partial y) \sum_{n=0}^{\infty} \xi_n - L_t^{-1}w(\partial/\partial z) \sum_{n=0}^{\infty} \xi_n \\
\xi_0 &= \xi(t=0) = f(x, y, z) \\
\xi_{m+1} &= DL_t^{-1} \nabla^2 \xi_m - L_t^{-1} \bar{u} \cdot \nabla \xi_m
\end{aligned}$$

for $m \geq 0$. Now all components are determined and we can write $\phi_N(\xi) = \sum_{m=0}^{N-1} \xi_m$ as an approximation to ξ , improving as N increases.

If we have turbulent motion of the fluid, we can have random fluctuations of the concentration and hydrodynamical variables; hence statistical description becomes necessary. $\phi_N(\xi)$ becomes a series of stochastic terms and we form $\langle \phi_N(\xi) \rangle$ to get the expectation as a function of average velocity components. The customary treatments of turbulent motion lead to a lack of closure and concomitant assumptions which are avoided by using decomposition. Thus, in the above method, u, v, w , and ξ are replaced by their corresponding steady-state values plus quantities representing fluctuations from the steady-states. Thus $\xi = \bar{\xi} + \xi'$, $u = \bar{u} + u'$, $v = \bar{v} + v'$, $w = \bar{w} + w'$. Statistical averaging causes terms such as

$$\begin{array}{ll} D \nabla^2 \langle \xi' \rangle & (\partial/\partial t) \langle \xi' \rangle \\ \bar{u} (\partial/\partial x) \langle \xi' \rangle & \langle u' \rangle (\partial/\partial x) \bar{\xi} \end{array}$$

etc. to vanish. We then have

$$\begin{aligned} & (\partial/\partial t) \bar{\xi} + \bar{u} (\partial/\partial x) \bar{\xi} + \bar{v} (\partial/\partial y) \bar{\xi} + \bar{w} (\partial/\partial z) \bar{\xi} \\ & = D \nabla^2 \bar{\xi} - \overline{u' (\partial/\partial x) \xi'} - \overline{v' (\partial/\partial y) \xi'} - \overline{w' (\partial/\partial z) \xi'} \end{aligned}$$

The last three correlation terms involve correlations of velocities and concentration which are unknown. Then the procedure is to let u'_i for $i = 1, 2, 3$ denote u, v, w , and x_j for $j = 1, 2, 3$ represent x, y, z and to write terms as being proportional to a mean gradient of the concentration in terms of a "turbulent diffusion tensor" $-K_{ij}(x_j, t) \partial \xi / \partial x_j$. To clarify the difficulty, consider the operator format $Lu + Ru = g$ or $u = L^{-1}g - L^{-1}Ru$. If we average we have $\langle u \rangle = L^{-1}\langle g \rangle - L^{-1}\langle Ru \rangle$. We can think of g as an input to a system containing R . The output u can be statistically independent of g but not of R . To achieve closure, one must approximate. By decomposition one writes

$$u = L^{-1}g - L^{-1}R \sum_{n=0}^{\infty} u_n = L^{-1}g - L^{-1}R L^{-1}g + L^{-1}R L^{-1}R L^{-1}g - \dots$$

Averaging is no problem since g is statistically independent of R . We have

$$\langle u \rangle = L^{-1} \langle g \rangle - L^{-1} \langle R \rangle L^{-1} \langle g \rangle + \langle RL^{-1}R \rangle \langle g \rangle - \dots$$

NONLINEAR TRANSPORT:

Let's consider the equation $L\xi + R\xi + N\xi = g$ where

$$L = \partial / \partial t$$

$$N\xi = f(\xi)$$

$$R = \bar{u} \cdot \nabla - D \nabla^2$$

Let $\xi = \sum_{n=0}^{\infty} \xi_n$ and $N\xi = \sum_{n=0}^{\infty} A_n$. Then

$$\xi = L^{-1} g - L^{-1} R \sum_{n=0}^{\infty} \xi_n - L^{-1} \sum_{n=0}^{\infty} A_n$$

where

$$\xi_0 = L^{-1} g$$

$$\xi_{m+1} = -L^{-1} R \xi_m - L^{-1} A_m$$

for $m \geq 0$. Then $\phi_{m-1} = \sum_{n=0}^m \xi_n$ which converges to $\xi = \sum_{n=0}^{\infty} \xi_n$. Further generalizations are straightforward. We can, for example, consider $Fu = g$ where

$$Fu = L_t u + L_x u + L_y u + L_z u + Ru + Nu = g$$

and solve for $L_t u$, $L_x u$, $L_y u$, or $L_z u$ which would simply treat the other operator terms as the remainder operator R and would require the appropriate given boundary conditions.

The case of stochastic g or stochastic processes in the R term leads to a stochastic ϕ_N which can be averaged or from which expectations and covariances can be found. The solutions are verifiable by checking that the original equation and the given conditions are satisfied.

Since the concern here is solution of physical systems, inputs and conditions are assumed to be bounded. If the model equation and the conditions are physically correct and consistent, a solution is obtained which is unique and accurate. If numerical results are calculated, one sees the approach to a stable solution for the desired number of decimal places. If conditions on one variable are better known than the others, we consider the corresponding equation which can yield the solution most accurately.

THE KDV EQUATION:

The equation is $u_t + \alpha uu_x + \beta u_{xxx} = 0$. In decomposition form, we write

$$L_t u = -\alpha uu_x - \beta u_{xxx}$$

where $u(x, t = 0) = f(x)$ is given. This equation was previously solved [2] also using the operator equation

$$L_x u = -\beta^{-1} u_t - \alpha \beta^{-1} uu_x$$

where $L_x = \partial^3 / \partial x^3$. This was done to ensure use of all the appropriate initial/boundary conditions. However, in Chapter 3 we saw that the solutions from each of the operator equations—called “partial solutions”—are actually the solution and identical in the general case when the t conditions depend on x , as above, and the x conditions depend on t . Therefore, we can simply use one or the other saving considerable computation. Since solving the $L_t u$ equation involves a single integration and the $L_x u$ equation involves three integrations, the optimal procedure is clear. We proceed by application of the L_t^{-1} which is a simple definite integration from 0 to t . Thus

$$u = u(t = 0) - \alpha L_t^{-1} uu_x - \beta L_t^{-1} (\partial^3 / \partial x^3) u$$

Identifying $u(t = 0)$ as u_0 in the decomposition $u = \sum_{n=0}^{\infty} u_n$ and writing the nonlinear term uu_x as $\sum_{n=0}^{\infty} A_n \{uu_x\}$ we have:

$$u = u_0 - \alpha L_t^{-1} \sum_{n=0}^{\infty} A_n - \beta L_t^{-1} (\partial^3 / \partial x^3) \sum_{n=0}^{\infty} u_n$$

if α and β are constants. We can now write the decomposition components

$$\begin{aligned} u_0 &= f(x) \\ u_1 &= -\alpha L_t^{-1} A_0 - \beta L_t^{-1} (\partial^3 / \partial x^3) u_0 \\ u_2 &= -\alpha L_t^{-1} A_1 - \beta L_t^{-1} (\partial^3 / \partial x^3) u_1 \\ &\vdots \\ u_m &= -\alpha L_t^{-1} A_{m-1} - \beta L_t^{-1} (\partial^3 / \partial x^3) u_{m-1} \end{aligned}$$

Indicating u_x as u' , we can write the A_n for uu' as:

$$\begin{aligned}
A_0 &= u_0 u'_0 \\
A_1 &= u_0 u'_0 + u_1 u'_0 \\
A_2 &= u_0 u'_2 + u_1 u'_1 + u_2 u'_0 \\
&\vdots \\
A_n &= u_0 u'_n + \cdots + u_n u'_0
\end{aligned}$$

We can now write the m -term approximant to the solution as $\phi_m = \sum_{n=0}^{m-1} u_n$ which converges to u .

THE NONLINEAR KLEIN-GORDON EQUATION:

$$u_{tt} - \nabla^2 u - f(u) = 0$$

Now $L_t = \partial^2 / \partial t^2$ and L_t^{-1} is a two-fold definite integration from 0 to t and we can write

$$\begin{aligned}
L_t u &= \nabla^2 u + f(u) \\
u &= A + Bt + L_t^{-1} \nabla^2 u + L_t^{-1} f(u)
\end{aligned}$$

Then,

$$\begin{aligned}
u_0 &= A + Bt \\
u_1 &= L_t^{-1} \nabla^2 u_0 + L_t^{-1} A_0 \{f(u)\} \\
u_2 &= L_t^{-1} \nabla^2 u_1 + L_t^{-1} A_1 \{f(u)\} \\
&\vdots \\
u_m &= L_t^{-1} \nabla^2 u_{m-1} + L_t^{-1} A_{m-1} \{f(u)\}
\end{aligned}$$

Thus, with specification of the explicit function $f(u)$ and the conditions on t assuming dependence on x, y, z , we can calculate ϕ_m . We can use one of the other possible operator equations if the appropriate conditions on x, y , or z are better known and do not give a vanishing u_0 term. If a forcing function g is also present, the u_0 term becomes $u_0 = A + Bt + L_t^{-1} g$.

NONLINEAR HEAT EQUATION:

$$c\rho u_t = [k(u)u_x]_x$$

With $L_t = \partial / \partial t$ and L_t^{-1} as integrations from 0 to t

$$L_t u = \frac{1}{c\rho} \frac{\partial}{\partial x} \left[k(u) \left(\frac{\partial}{\partial x} u \right) \right]$$

$$u = u(t=0) + \frac{1}{c\rho} L_t^{-1} \frac{\partial}{\partial x} \left[k(u) \frac{\partial}{\partial x} u \right]$$

We need to know $u(t=0)$ which we identify as u_0 in the decomposition, and also need the function $k(u)$ for the specific model being studied. For illustration, we use $k(u) = 1 + u$ which gives us a nonlinear term uu_x or uu' . Now

$$u = u_0 + \frac{1}{c\rho} L_t^{-1} \left[\frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} u_n + \frac{\partial}{\partial x} \sum_{n=0}^{\infty} A_n \{uu'\} \right]$$

$$A_0 = u_0 u'_0$$

$$A_1 = u_0 u'_1 + u_1 u'_0$$

$$A_2 = u_0 u'_2 + u_1 u'_1 + u_2 u'_0$$

$$\vdots$$

$$A_n = u_0 u'_n + \dots + u_n u'_0$$

Now

$$u_1 = \frac{1}{c\rho} L_t^{-1} \left[\frac{\partial^2}{\partial x^2} u_0 + \frac{\partial}{\partial x} A_0 \right]$$

$$u_2 = \frac{1}{c\rho} L_t^{-1} \left[\frac{\partial^2}{\partial x^2} u_1 + \frac{\partial}{\partial x} A_1 \right]$$

$$\vdots$$

$$\text{and } \phi_m = \sum_{n=0}^{m-1} u_n.$$

RANDOM NONLINEAR HEAT EQUATION:

We can write using $L_t = \partial/\partial t$,

$$L_t u = \frac{\partial}{\partial x} \left[k(u) \frac{\partial u}{\partial x} \right] + g$$

and assuming

$$k(u) = 1 + \alpha(t, \omega)u$$

$$L_t u = \frac{\partial}{\partial x} \left[(1 + \alpha u) \frac{\partial u}{\partial x} \right] + g$$

$$\begin{aligned}
 u_0 &= u(x, 0) + L_t^{-1} g \\
 u_1 &= L_t^{-1} \frac{\partial^2}{\partial x^2} u_0 + \frac{\partial}{\partial x} \left(\alpha u_0 \frac{\partial}{\partial x} u_0 \right) \\
 &\vdots
 \end{aligned}$$

For either random g or random α , for stationary or nonstationary cases, $\langle u \rangle$ is found by writing out the series and taking term by term expectations without closure approximation or perturbation. If g is random,

$$\begin{aligned}
 \langle u_0 \rangle &= u(x, 0) + L_t^{-1} \langle g \rangle \\
 \langle u_1 \rangle &= L_t^{-1} \frac{\partial^2}{\partial x^2} \langle u_0 \rangle + L_t^{-1} \frac{\partial}{\partial x} \langle \alpha \rangle \langle u_0 \rangle \frac{\partial}{\partial x} \langle u_0 \rangle
 \end{aligned}$$

where we note that the input of g would not be correlated with α , which is a system parameter. If α is random,

$$\langle u_1 \rangle = L_t^{-1} \frac{\partial^2}{\partial x^2} u_0 + L_t^{-1} \frac{\partial}{\partial x} \left(\langle \alpha \rangle \langle u_0 \rangle \frac{\partial}{\partial x} u_0 \right)$$

SINE-GORDON EQUATION:

$$\partial^2 u / \partial t^2 - \partial^2 u / \partial x^2 = \sin u$$

Letting $L_t = \partial / \partial t$ and $L_x u = u_x \equiv u'$,

$$L_t u' = \sin u$$

$$u' = u'(0) + L_t^{-1} \sin u = u'(0) + L_t^{-1} \sum_{n=0}^{\infty} A_n \{\sin u\}$$

$$u'(0) = u'_0 \quad \text{if} \quad u' = \sum_{n=0}^{\infty} u'_n$$

$$u_0 = \int u'(0) dx = u(0)$$

$$u'_1 = L_t^{-1} A_0$$

$$u_1 = \int (L_t^{-1} A_0) dx = L_x^{-1} L_t^{-1} A_0$$

$$\vdots$$

Thus the system is calculable using the A_n for $\sin u$:

$$A_0 = \sin u_0$$

$$A_1 = u_1 \cos u_0$$

$$A_2 = -(u_1^2/2) \sin u_0 + u_2 \cos u_0$$

$$A_3 = -(u_1^3/6) \cos u_0 - u_1 u_2 \sin u_0 + u_3 \cos u_0$$

$$\vdots$$

SCHRÖDINGER EQUATION WITH QUARTIC POTENTIAL:

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{1}{2} a x^4 \psi = E_1 \psi$$

describes the case of a particle in a quartic potential $V(x) = (1/2)ax^4$. It can be written

$$d^2 \psi / dx^2 + \alpha x^4 \psi + \beta \psi = 0$$

If $L = d^2/dx^2$, $L = \int \int (\cdot) dx dx$,

$$L\psi + \alpha x^4 \psi + \beta \psi = 0$$

$$\psi = \Phi - L^{-1}(\alpha x^4 + \beta)\psi$$

Thus

$$\psi_0 = \Phi$$

where Φ satisfies the conditions. Then for $n \geq 0$

$$\psi_{n+1} = -L^{-1}(\alpha x^4 + \beta)\psi_n$$

$\psi = \sum_{n=0}^{m-1} \psi_n$ is the solution.

A LINEAR EXAMPLE:

$$d^2 u / dx^2 - kx^p u = g$$

$$u(1) = u(-1) = 0$$

We have, using decomposition,

$$Lu = g + kx^p u$$

Assume g is a constant. Then $u = c_1 + c_2x + gx^2/2 + L^{-1} kx^p u$ is decomposed into

$$\begin{aligned} u_0 &= c_1 + c_2x + gx^2/2 \\ u_{m+1} &= L^{-1} kx^p u_m \quad (m \geq 0) \end{aligned}$$

Since $u = \sum_{m=0}^{\infty} u_m$,

$$\begin{aligned} u &= \sum_{m=0}^{\infty} (L^{-1} kx^p)^m u_0 \\ u &= \sum_{m=0}^{\infty} (L^{-1} kx^p)^m c_1 + \sum_{m=0}^{\infty} (L^{-1} kx^p)^m c_2x \\ &\quad + \sum_{m=0}^{\infty} (L^{-1} kx^p)^m gx^2/2 \end{aligned}$$

We can write the result in the form

$$u = c_1 \alpha(x) + c_2 \beta(x) + \gamma(x)$$

where

$$\begin{aligned} \alpha(x) &= \sum_{m=0}^{\infty} k^m x^{mp+2m} / (mp+2m-1)(mp+2m) \\ \beta(x) &= \sum_{m=0}^{\infty} k^m x^{mp+2m+1} / (mp+2m)(mp+2m+1) \\ \gamma(x) &= \sum_{m=0}^{\infty} (1/2) g k^m x^{mp+2m+2} / (mp+2m+1)(mp+2m+2) \end{aligned}$$

We can use the conditions $u(1) = u(-1) = 0$ to evaluate c_1 and c_2 .

$$\begin{aligned} c_1 \alpha(1) + c_2 \beta(1) + \gamma(1) &= 0 \\ c_1 \alpha(-1) + c_2 \beta(-1) + \gamma(-1) &= 0 \end{aligned}$$

so that

$$\begin{aligned} c_1 &= \frac{\beta(1)\gamma(-1) - \beta(-1)\gamma(1)}{\alpha(1)\beta(-1) - \beta(1)\alpha(-1)} \\ c_2 &= \frac{\alpha(-1)\gamma(1) - \beta(1)\gamma(-1)}{\alpha(1)\beta(-1) - \beta(1)\alpha(-1)} \end{aligned}$$

(The case $g = 2$, $k = 40$, $p = 1$ was verified to seven-digit accuracy.)

NONLINEAR SCHRÖDINGER EQUATION:

$$iu_t + 2u|u|^2 + u_{xx} = 0$$

can be written

$$iL_t u + Nu + L_x u = 0$$

where $L_t = \partial/\partial t$, $L_x = \partial^2/\partial x^2$, and $Nu = 2u|u|^2$. We can solve either for the L_t operator or the L_x operator. If we solve for $L_t u$, we get immediately

$$u = u(x, 0) + iL_t^{-1}Nu + iL_t^{-1}L_x u$$

$$u_0 = u(x, 0)$$

$$u_1 = iL_t^{-1}A_0 + iL_t^{-1}L_x u_0$$

$$u_2 = iL_t^{-1}A_1 + iL_t^{-1}L_x u_1$$

$$\vdots$$

$$u_{n+1} = iL_t^{-1}A_n + iL_t^{-1}L_x u_n$$

and $\phi_n = \sum_{i=0}^{n-1} u_i$ as an n -term approximation converging to u . If we solve for $L_x u$, we have

$$u = u_0 - iL_x^{-1}L_t u - L_x^{-1}Nu$$

$$u_0 = \alpha + \beta x$$

$$u_{n+1} = -iL_x^{-1}L_t u_n - L_x^{-1}A_n$$

for $n \geq 0$. The given conditions determine α, β . If instead of a finite interval we have a limit such as $u \rightarrow 0$ as $x \rightarrow \infty$, then we set $\lim_{x \rightarrow \infty} u = 0$ and find the limit of the series.

TIME-DEPENDENT GENERALIZATION OF THE YUKAWA-COUPLED KLEIN-GORDON SCHRÖDINGER EQUATION:

$$\nabla^2 V - \partial^2 V / \partial t^2 - V + |\psi|^2 = 0$$

$$\nabla^2 V + i\partial \psi / \partial t + V\psi = 0$$

The equations involve operators in x, y, z and t . Our specific solution depends on the given initial/boundary conditions. Considering the equation for V , and assuming conditions on t are given,

$$L_t V = \nabla^2 V - V + |\psi|^2$$

$$V_0 = A + Bt$$

$$V_{n+1} = L_t^{-1} \nabla^2 V_n - L_t^{-1} V_n + L_t^{-1} A_n \{|\psi|^2\}$$

The A and B must satisfy given conditions, whether initial or boundary or limit conditions at infinity. The ψ equation, now using $L_t' = \partial/\partial t$, results in

$$\frac{\partial \psi}{\partial t} = iV\psi + i\nabla^2 \psi$$

$$\psi_0 = \psi(x, y, z, t = 0)$$

$$\psi_{n+1} = iL_x^{-1} \nabla^2 \psi_n + iL_t^{-1} A_n \{V\psi\}$$

To solve the system we find (ψ_0, V_0) , then (ψ_1, V_1) , then (ψ_2, V_2) , etc. Difficult integrations will arise because of difficult initial conditions for which solution is desired. This problem essentially vanishes by approximation of these functions by a few terms of an equivalent series, e.g., a Maclaurin series, or decomposition of the functions to arrive at elementary integrations.

THE N-BODY PROBLEM:

The problem of the interaction of N bodies is important in many connections and for many force laws which can involve attraction or repulsion, or even collisions. The problem is soluble by decomposition because the nonlinearity can be represented by the generalized polynomials (Chapter 3) which can be calculated. Here, we will consider N -body dynamics in a gravitational field. Assume a system of N point masses m_i where $i = 2, 3, \dots, N$, with positions specified by vectors $\vec{r}_i$ from a chosen origin. The distance between m_i and m_j will be denoted by $\vec{r}_{i,j} = |\vec{r}_i - \vec{r}_j| = [(\vec{r}_i - \vec{r}_j) \cdot (\vec{r}_i - \vec{r}_j)]^{1/2}$ where the multiplication indicates a scalar product.

The trivial one-body case is described simply by $m_1 \ddot{\vec{r}}_1 = 0$ with initial conditions

$$\vec{r}_1(0) = \rho_1$$

$$\dot{\vec{r}}_1(0) = v_1$$

TWO BODIES:

For two bodies in a gravitational field, we have two coupled equations

$$m_1 \ddot{\vec{r}}_1 = -G \frac{m_1 m_2}{|\vec{r}_{1,2}|^3} \vec{r}_{1,2}$$

$$m_2 \ddot{\vec{r}}_2 = -G \frac{m_2 m_1}{|\vec{r}_{2,1}|^3} \vec{r}_{2,1}$$

with initial conditions

$$\begin{aligned} \vec{r}_1(0) &= \vec{\rho}_1 & \vec{r}_2(0) &= \vec{\rho}_2 \\ \vec{r}_1'(0) &= \vec{v}_1 & \vec{r}_2'(0) &= \vec{v}_2 \end{aligned}$$

noting that $\vec{r}_{1,2} / |\vec{r}_{1,2}|^3 = \hat{r}_{1,2} / |\vec{r}_{1,2}|^2$ where $\hat{r}_{1,2}$ is the unit vector. We can for example, let one mass be a sufficiently large, say M , to be assumed as a fixed origin, then find the motion of the small mass m . Or, the origin can be the observer on earth considering the effect of the sun on the moon. Or, finally, the origin can be in a rocket traveling through the solar system.

THREE BODIES:

For three bodies, we have:

$$m_1 \ddot{\vec{r}}_1 = -G \frac{m_1 m_2}{|\vec{r}_{1,2}|^3} \vec{r}_{1,2} - G \frac{m_1 m_3}{|\vec{r}_{1,3}|^3} \vec{r}_{1,3}$$

$$m_2 \ddot{\vec{r}}_2 = -G \frac{m_2 m_1}{|\vec{r}_{2,1}|^3} \vec{r}_{2,1} - G \frac{m_2 m_3}{|\vec{r}_{2,3}|^3} \vec{r}_{2,3}$$

$$m_3 \ddot{\vec{r}}_3 = -G \frac{m_3 m_1}{|\vec{r}_{3,1}|^3} \vec{r}_{3,1} - G \frac{m_3 m_2}{|\vec{r}_{3,2}|^3} \vec{r}_{3,2}$$

with initial conditions:

$$\begin{aligned} \vec{r}_1(0) &= \vec{\rho}_1 & \vec{r}_2(0) &= \vec{\rho}_2 & \vec{r}_3(0) &= \vec{\rho}_3 \\ \vec{r}_1'(0) &= \vec{v}_1 & \vec{r}_2'(0) &= \vec{v}_2 & \vec{r}_3'(0) &= \vec{v}_3 \end{aligned}$$

FOUR BODIES:

We now have four coupled equations and the corresponding initial conditions to consider.

$$\begin{aligned}
m_1 \ddot{\bar{r}}_1 &= -G \frac{m_1 m_2}{|\bar{r}_{1,2}|^3} \bar{r}_{1,2} - G \frac{m_1 m_3}{|\bar{r}_{1,3}|^3} \bar{r}_{1,3} - G \frac{m_1 m_4}{|\bar{r}_{1,4}|^3} \bar{r}_{1,4} \\
m_2 \ddot{\bar{r}}_2 &= -G \frac{m_2 m_1}{|\bar{r}_{2,1}|^3} \bar{r}_{2,1} - G \frac{m_2 m_3}{|\bar{r}_{2,3}|^3} \bar{r}_{2,3} - G \frac{m_2 m_4}{|\bar{r}_{2,4}|^3} \bar{r}_{2,4} \\
m_3 \ddot{\bar{r}}_3 &= -G \frac{m_3 m_1}{|\bar{r}_{3,1}|^3} \bar{r}_{3,1} - G \frac{m_3 m_2}{|\bar{r}_{3,2}|^3} \bar{r}_{3,2} - G \frac{m_3 m_4}{|\bar{r}_{3,4}|^3} \bar{r}_{3,4} \\
m_4 \ddot{\bar{r}}_4 &= -G \frac{m_4 m_1}{|\bar{r}_{4,1}|^3} \bar{r}_{4,1} - G \frac{m_4 m_2}{|\bar{r}_{4,2}|^3} \bar{r}_{4,2} - G \frac{m_4 m_3}{|\bar{r}_{4,3}|^3} \bar{r}_{4,3}
\end{aligned}$$

with initial conditions with $\bar{r}_i(0) = \bar{\rho}_i$ and $\dot{\bar{r}}_i(0) = \bar{v}_i$ with $i = 1, 2, 3, 4$.

GENERALIZING TO *N*-BODY DYNAMICS:

In a gravitational field, we first simplify notation by writing

$$\bar{f}(\bar{r}_i, \bar{r}_j) = (\bar{r}_i - \bar{r}_j)[(\bar{r}_i - \bar{r}_j) \cdot (\bar{r}_i - \bar{r}_j)]^{-3/2}$$

which is a convenient form for use of the generalized A_n for $f(u, v)$ discussed in Chapter 3. The indicated multiplication is a scalar product. Hence for N bodies,

$$\begin{aligned}
m_i \ddot{\bar{r}}_i &= -G m_i \sum_{\substack{j=1 \\ j \neq i}}^N \frac{m_j (\bar{r}_i - \bar{r}_j)}{|\bar{r}_i - \bar{r}_j|^3} \\
&= -G m_i \sum_{\substack{j=1 \\ j \neq i}}^N m_j f(\bar{r}_i, \bar{r}_j)
\end{aligned}$$

Let $L = d^2/dt^2$ to write the left side as $m_i L \bar{r}_i$ and apply L^{-1} , a two-fold definite integration from 0 to t , i.e., $\int_0^t \int_0^t (\cdot) dt dt$, to both sides. Also let

$$\bar{r}_i = \sum_{n=0}^{\infty} \bar{r}_n^{(i)} \quad \text{and} \quad \bar{r}_j = \sum_{n=0}^{\infty} \bar{r}_n^{(j)}$$

We can identify $\bar{r}_0^{(i)}$ as the solution of $L \bar{r}_0^{(i)} = 0$, or,

$$\bar{r}_0^{(i)} = \bar{r}^{(i)}(t=0) + t \bar{v}^{(i)}(t=0) = \bar{\rho}_i + t \bar{v}_i$$

Analogously,

$$\bar{r}_0^{(j)} = \bar{\rho}_j + t \bar{v}_j$$

The first decomposition component is determined and the following components are calculable from

$$\begin{aligned}\bar{r}_1^{(i)} &= -L^{-1} G \sum_{\substack{j=1 \\ j \neq i}}^{N-1} m_j A_0 \{f(\bar{r}_i, \bar{r}_j)\} \\ \bar{r}_2^{(i)} &= -L^{-1} G \sum_{\substack{j=1 \\ j \neq i}}^{N-1} m_j A_1 \{f(\bar{r}_i, \bar{r}_j)\} \\ &\vdots\end{aligned}$$

Thus $n \geq 0$, for

$$\bar{r}_{n+1}^{(i)} = -L^{-1} G \sum_{\substack{j=1 \\ j \neq i}}^{N-1} m_j A_n \{f(\bar{r}_i, \bar{r}_j)\}$$

so all components are calculable by determining the A_n using the methods of Chapter 3. For convenience, the necessary quantities are listed.

The A_n for u^{-m} are:

$$\begin{aligned}A_0 &= u_0^{-m} \\ A_1 &= -m u_0^{-(m+1)} u_1 \\ A_2 &= (1/2)m(m+1)u_0^{-(m+2)} u_1^2 - m u_0^{-(m+1)} u_2 \\ A_3 &= -(1/6)m(m+1)(m+2)u_0^{-(m+3)} u_1^3 \\ &\quad + m(m+1)u_0^{-(m+2)} u_1 u_2 - m u_0^{-(m+1)} u_3 \\ &\vdots\end{aligned}$$

The A_n for $f(u, v)$ are given by

$$A_n = \sum_{m+v=n}^n c(\mu, v; n) f_{\mu, v}$$

where

$$f_{\mu,\nu} = \frac{\partial^{\mu+\nu}}{\partial u_0^\mu \partial v_0^\nu} f(u_0, v_0)$$

$$A_0 = f_{0,0}$$

$$A_1 = u_1 f_{1,0} + v_1 f_{0,1}$$

$$A_2 = u_2 f_{2,0} + v_2 f_{0,2} + \frac{u_1^2}{2!} f_{2,0}$$

$$+ u_1 v_1 f_{1,1} + \frac{v_1^2}{2!} f_{0,2}$$

$$A_3 = u_3 f_{3,0} + v_3 f_{0,3} + u_1 u_2 f_{2,0}$$

$$+ [u_1 v_2 + u_2 v_1] f_{1,1} + v_1 v_2 f_{0,2}$$

$$+ \frac{u_1^3}{3!} f_{3,0} + \frac{u_1^2}{2!} v_1 f_{2,1}$$

$$+ \frac{v_1^2}{2!} u_1 f_{1,2} + \frac{v_1^3}{3!} f_{0,3}$$

$$\vdots$$

For scalar problems, $f(r) = r^{-2} = \sum_{n=0}^{\infty} A_n$

$$A_0 = r_0^{-2}$$

$$A_1 = -2r_0^{-3} r_1$$

$$A_2 = 3r_0^{-4} r_1^2 - 2r_0^{-3} r_2$$

$$A_3 = 4r_0^{-5} r_1^3 - 6r_1^{-4} r_1 r_2 - 2r_0^{-3} r_3$$

$$\vdots$$

For $f(r) = r^{-3} = \sum_{n=0}^{\infty} A_n$

$$A_0 = r_0^{-3}$$

$$A_1 = -3r_0^{-4} r_1$$

$$A_2 = 6r_0^{-5} r_1^2 - 3r_0^{-4} r_2$$

$$A_3 = -10r_0^{-6} r_1^3 + 12r_0^{-5} r_1 r_2 - 3r_0^{-4} r_3$$

$$\vdots$$

For the nonlinearity $\bar{f}(r_i, r_j)$ we define the necessary polynomials by $\bar{A}_n^{(i,j)}$, i.e.,

$$\bar{f}(\bar{r}_i - \bar{r}_j) = \sum_{n=0}^{\infty} \bar{A}_n^{(i,j)}$$

Since

$$\begin{aligned}\bar{f}(\bar{r}_i - \bar{r}_j) &= (\bar{r}_i - \bar{r}) \left[(\bar{r}_i - \bar{r}_j) \cdot (\bar{r}_i - \bar{r}_j) \right]^{-3/2} \\ &= \bar{g}(\bar{r}_i, \bar{r}_j) h(\bar{r}_i, \bar{r}_j)\end{aligned}$$

We can now write

$$\bar{g}(\bar{r}_i, \bar{r}_j) = (\bar{r}_i - \bar{r}_j) = \sum_{n=0}^{\infty} \bar{a}_n^{(i,j)}$$

$$h(\bar{r}_i, \bar{r}_j) = \sum_{n=0}^{\infty} b_n^{(i,j)}$$

so that

$$\sum_{n=0}^{\infty} \bar{A}_n^{(i,j)} = \left\{ \sum_{n=0}^{\infty} \bar{a}_n^{(i,j)} \right\} \left\{ \sum_{n=0}^{\infty} b_n^{(i,j)} \right\}$$

which can be rewritten as

$$\bar{A}_n^{(i,j)} = \sum_{m=0}^n \bar{a}_{n-m}^{(i,j)} b_m^{(i,j)}$$

We can decompose the $\bar{r}_i$ and $\bar{r}_j$ thus:

$$\begin{aligned}\bar{r}_i &= \sum_{n=0}^{\infty} \bar{r}_n^{(i)} \\ \bar{r}_j &= \sum_{n=0}^{\infty} \bar{r}_n^{(j)}\end{aligned}$$

Then

$$\begin{aligned}\bar{g}(\bar{r}_i, \bar{r}_j) &= (\bar{r}_i - \bar{r}_j) = \sum_{n=0}^{\infty} (\bar{r}_n^{(i)} - \bar{r}_n^{(j)}) = \sum_{n=0}^{\infty} \bar{a}_n^{(i,j)} \\ h(\bar{r}_i, \bar{r}_j) &= \sum_{n=0}^{\infty} b_n^{(i,j)}\end{aligned}$$

Since the h function is a nonlinear function of two variables, we can write it in terms of the generalized polynomials $A_n\{f(u, v)\}$ in Chapter 3 (and also in reference [4]). Thus

$$b_n^{(i,j)} = \sum_{\mu+\nu=n} c^{(i,j)}(\mu, \nu, n) \frac{\partial^{\mu+\nu} h(\bar{r}_0^{(i)}, \bar{r}_0^{(j)})}{\partial^{\mu} \bar{r}_0^{(i)} \partial^{\nu} \bar{r}_0^{(j)}}$$

Thus we have a system of N coupled nonlinear second-order equations, or Volterra integral equations when the L^{-1} operator is applied, which are solvable by the decomposition method since the nonlinearity, though difficult, can be expressed by methods of Chapter 3 and reference [2]. Details for specific special cases will appear in a forthcoming paper as they are programmed and calculated.

NONLINEAR RELATIVISTIC PARTIAL DIFFERENTIAL EQUATIONS:

A class of nonlinear equations occurring in mathematical approaches to elementary particle theory [5] is given by

$$\nabla^2 \phi - \partial^2 \phi / \partial t^2 = m^2 \phi + g \phi^3 \quad (m, g > 0)$$

The objective is to obtain a non-perturbative time-development of the field of this and similar equations involving other nonlinear interactions, without the use of cutoff functions or truncations. In decomposition form [2-5]

$$(L_x + L_y + L_z)\phi - L_t \phi = m^2 \phi + g \phi^3$$

where $L_x = \partial^2 / \partial x^2$, $L_y = \partial^2 / \partial y^2$, $L_z = \partial^2 / \partial z^2$, $L_t = -\partial^2 / \partial t^2$. The possible operator equations for partial solutions [4] are:

$$\begin{aligned} L_x \phi &= m^2 \phi + g \phi^3 - (L_y + L_z - L_t) \phi \\ L_y \phi &= m^2 \phi + g \phi^3 - (L_x + L_z - L_t) \phi \\ L_z \phi &= m^2 \phi + g \phi^3 - (L_x + L_y - L_t) \phi \\ L_t \phi &= -m^2 \phi - g \phi^3 + (L_x + L_y + L_z) \phi \end{aligned}$$

The solution $\phi(x, y, z, t)$ will be approximated by the n -term decomposition series Φ_n

$$\Phi_n = \sum_{m=0}^{n-1} \phi_m(x, y, z, t)$$

We visualize a manifold in a four-dimensional coordinate system with x, y, z, t axes. On the coordinate planes we have curves $\phi(x)$, $\phi(y)$, $\phi(z)$, and $\phi(t)$ representing intersection s of the ϕ "surface" with the planes. We can start from any of these functions (i.e., the initial-boundary conditions) to generate $\phi(x, y, z, t)$. We can begin with any one of these equations to yield the solution

and in general all of the results are identical. (In special cases, they are asymptotically equal.)

Suppose we consider the fourth equation. Then operating with L_t^{-1} or the two-fold definite integration from 0 to t and identifying the initial term

$$\varphi = \varphi(t=0, x, y, z) + t\varphi'(t=0, x, y, z)$$

we have

$$\varphi = \varphi_0 + L_t^{-1} m^2 \varphi - L_t^{-1} g \varphi^3 + L_t^{-1} [L_x + L_y + L_z] \varphi$$

We now apply the decomposition

$$\varphi = \sum_{n=0}^{\infty} \varphi_n$$

We can now determine all components of φ ; thus,

$$\varphi_1 = -L_t^{-1} m^2 \varphi_0 - L_t^{-1} g A_0 + L_t^{-1} (L_x + L_y + L_z) \varphi_0$$

$$\varphi_2 = -L_t^{-1} m^2 \varphi_1 - L_t^{-1} g A_1 + L_t^{-1} (L_x + L_y + L_z) \varphi_1$$

$$\vdots$$

$$\varphi_{m+1} = -L_t^{-1} m^2 \varphi_m - L_t^{-1} g A_m + L_t^{-1} (L_x + L_y + L_z) \varphi_m$$

Now

$$\Phi_{m+1} = \sum_{n=0}^{\infty} \varphi_n$$

is a convergent approximation to $\varphi = \sum_{n=0}^{\infty} \varphi_n$, i.e., to the solution. Which of the four equations we use depends on which initial/boundary conditions are best known or measured. When the nonlinearity is of higher degree, e.g. in the scalar relativistic equation:

$$\nabla^2 \varphi - (\partial^2 / \partial t^2) \varphi = m^2 \varphi + g \varphi^p$$

the same procedure applies with the appropriate A_n for φ^p for which rules are given in [2].

TIME-DEPENDENT SCHRÖDINGER EQUATION IN CONFIGURATION SPACE:

The equation is [6]

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r}, t) + V(\vec{r}) \Psi(\vec{r}, t) = -\frac{\hbar}{i} \frac{\partial \Psi}{\partial t}$$

The vector $\vec{r}$ is the position vector of the particle referred to a convenient origin. We can introduce unit vectors along axes of rectangular (x, y, z) or spherical (r, θ, ϕ) coordinates. In terms of rectangular Cartesian coordinates, the operator $\nabla^2 = L_x + L_y + L_z$ where $L_x = \partial^2 / \partial x^2$, $L_y = \partial^2 / \partial y^2$, $L_z = \partial^2 / \partial z^2$.

Then

$$[L_x + L_y + L_z] \Psi = \frac{2m}{i\hbar} \frac{\partial \Psi}{\partial t} + \frac{2m}{\hbar^2} V \Psi$$

Using decomposition, we can solve for any of the operators L_x, L_y, L_z , or $L_t = \partial / \partial t$ as long as we know the appropriate conditions on x, y, z , or t . The inverse operators $L_x^{-1}, L_y^{-1}, L_z^{-1}$ are two-fold indefinite integrations respectively in x, y, z . The inverse L_t^{-1} is a single definite integration from zero to t representing an initial condition problem for which $\Psi(t=0)$ is required. In the other cases, we have boundary-value problems for which we need values of Ψ at two values of x or y or z . Suppose we solve in terms of L_x , then

$$L_x \Psi = \alpha \frac{\partial \Psi}{\partial t} + \beta V \Psi - L_y \Psi - L_z \Psi$$

where $\alpha = 2m / i\hbar$ and $\beta = 2m / \hbar^2$. Then we must operate on all terms with L_x^{-1} . The left side becomes $L_x^{-1} L_x \Psi = \Psi - A - Bx$. Rearranging, we have $\Psi = A + Bx - \alpha L_x^{-1} \partial / \partial t \Psi - \beta L_x^{-1} V \Psi - L_x^{-1} [L_y + L_z] \Psi$. Let $\Psi = \sum_{m=0}^{\infty} \Psi_m$ and identify Ψ_0 as $A + Bx$. Then an n -term approximant to Ψ denoted by ϕ_n will be

$$\phi_n[\Psi] = \sum_{m=0}^{n-1} \Psi_m$$

which becomes Ψ in the limit as $n \rightarrow \infty$. Because of rapid convergence, a few terms are generally sufficient. (When this is not the case, one can use Padé approximants or other acceleration techniques or the method of asymptotic decomposition.) The decomposition components are

$$\Psi_0 = A + Bx$$

$$\Psi_1 = \alpha L_x^{-1} \frac{\partial}{\partial x} \Psi_0 - \beta L_x^{-1} V \Psi_0 - L_x^{-1} [L_y + L_z] \Psi_0$$

$$\Psi_2 = \alpha L_x^{-1} \frac{\partial}{\partial x} \Psi_1 - \beta L_x^{-1} V \Psi_1 - L_x^{-1} [L_y + L_z] \Psi_1$$

$$\vdots$$

$$\Psi_n = \alpha L_x^{-1} \frac{\partial}{\partial x} \Psi_{n-1} - \beta L_x^{-1} V \Psi_{n-1} - L_x^{-1} [L_y + L_z] \Psi_{n-1}$$

and we can now write $\phi_n[\Psi]$. For a particular potential, e.g. an isotropic harmonic oscillator, $V = 1/2 m \omega^2 (x^2 + y^2 + z^2)$, we can now calculate the solution. Evidently, we can also deal with nonlinear potential functions or nonlinear Schrödinger equations. The decomposition method is not restricted to potentials varying slowly in a de Broglie wavelength.

The problem solution is complete when A and B are evaluated by matching to the given conditions. This requires matching the ϕ_n to the boundary conditions for each value of n as previously discussed.

Other interesting examples (Ginzburg-Landau equation, Euler equations for inviscid flow, isentropic flow) as well as further results on Navier-Stokes equations and on numerical computation will appear in future publications.

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PADÉ AND SHANKS TRANSFORMS

PADÉ APPROXIMANTS:

The objective here is to find a solution in the large, i.e., in the range $(0, \infty)$ from the decomposition series which normally has a finite circle of convergence for initial-value problems. The procedure is to seek a rational function for the series. Given a function $f(z)$ expanded in a Maclaurin series $f(z) = \sum_{n=0}^{\infty} c_n z^n$, we can use the coefficients of the series to represent the function by a ratio of two polynomials

$$\frac{a_0 + a_1 z + \dots + a_L z^L}{b_0 + b_1 z + \dots + b_M z^M}$$

symbolized by $[L/M]$ and called the Padé approximant. The basic idea is to match the series coefficients as far as possible. Even though the series has a finite region of convergence, we can obtain the limit of the function as $x \rightarrow \infty$ if $L = M$.

Notice that if we are satisfied with $[1/1]$, we will have

$$(a_0 + a_1 z) = (b_0 + b_1 z)(c_0 + c_1 z + c_2 z^2 + \dots)$$

so that coefficients of z^2 are zero, i.e.,

$$b_1 c_1 + b_0 c_2 = 0$$

Taking $b_0 = 1$, we have

$$b_1 c_1 + c_2 = 0$$

Now consider $[2/2]$ or

$$(a_0 + a_1 z + a_2 z^2) = (b_0 + b_1 z + b_2 z^2)(c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \dots)$$

Clearly the coefficients of z^3 are zero, so that we can write

$$b_2 c_1 + b_1 c_2 + b_0 c_3 = 0$$

In general, we note that there are $L + 1$ independent coefficients in the numerator and $M + 1$ coefficients in the denominator. To make the system determinable, it is customary to let $b_0 = 1$. We then have M independent

coefficients in the denominator and $L + M + 1$ independent coefficients in all. Now the $[L/M]$ approximant can fit the power series through orders $1, z, z^2, \dots, z^{L+M}$ with an error of $O(z^{L+M+1})$. For example, for

$$f(z) = 1 - \frac{1}{2}z + \frac{1}{3}z^2 + \dots$$

we have

$$[1/1] = \frac{1 + (1/6)z}{1 + (2/3)z} = f(z) + O(z^3)$$

Consequently,

$$(a_0 + a_1z + \dots + a_Lz^L) = (b_0 + b_1z + \dots + b_Mz^M)(c_0 + c_1z + \dots)$$

Equating coefficients of $z^{L+1}, z^{L+2}, \dots, z^{L+M}$ in turn, we can write

$$b_M c_{L-M+1} + b_{M-1} c_{L-M+2} + \dots + b_0 c_{L+1} = 0$$

$$b_M c_{L-M+2} + b_{M-1} c_{L-M+3} + \dots + b_0 c_{L+2} = 0$$

$$\vdots$$

$$b_M c_L + b_{M-1} c_{L+1} + \dots + b_0 c_{L+M} = 0$$

Setting $b_0 = 1$, we have M linear equations for the M coefficients in the denominator.

$$\begin{bmatrix} c_{L-M+1} & c_{L-M+2} & \dots & c_L \\ c_{L-M+2} & c_{L-M+3} & \dots & c_{L+1} \\ \vdots & & & \\ c & c_L & \dots & c_{L+M-1} \end{bmatrix} \cdot \begin{bmatrix} b_{M-1} \\ b_{M-2} \\ \vdots \\ b_1 \end{bmatrix} = - \begin{bmatrix} c_{L+1} \\ c_{L+2} \\ \vdots \\ c_{L+M} \end{bmatrix}$$

We invert the matrix on the left and solve for the b_i for $i = 1, \dots, M$. Since we know the $c_0, c_1, c_2, \dots$, we can equate coefficients of $1, z, z^2, \dots, z^L$ to get $a_0, a_1, \dots, a_L$. Thus

$$a_0 = c_0$$

$$a_1 = c_1 + b_1 c_0$$

$$a_2 = c_2 + b_1 c_1 + b_2 c_0$$

$$\vdots$$

$$a_L = c_L + \sum_{i=1}^{\min(L, M)} b_i c_{L-i}$$

Thus the numerator and denominator of the Padé approximant are determined and we have agreement with the original series through order z^{L+M} . From the matrix equation, we can write the lower-order approximants. (For higher orders, one can use symbolic programs.)

$$1) \quad [L/M] = [1/1]$$

$$2) \quad [L/M] = [2/2] \quad b_1 = -c_2$$

$$\begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix} \cdot \begin{pmatrix} b_2 \\ b_1 \end{pmatrix} = - \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}$$

or

$$\begin{pmatrix} b_2 \\ b_1 \end{pmatrix} = \begin{pmatrix} c_3/D & -c_2/D \\ -c_2/D & c_1/D \end{pmatrix} \cdot \begin{pmatrix} -c_3 \\ -c_4 \end{pmatrix}$$

where $D = c_1c_3 - c_2^2$ so that we have

$$b_2 = \frac{c_2c_4 - c_2^2}{c_1c_3 - c_2^2}$$

$$b_1 = \frac{c_2c_3 - c_1c_4}{c_1c_3 - c_2^2}$$

For $f(z) = c_0 + c_1z + c_2z^2 + \dots$ we have

$$\begin{aligned} [1/1] &= \frac{a_0 + a_1z}{b_0 + b_1z} & \lim_{z \rightarrow \infty} [1/1] &= a_1/b_1 \\ [2/2] &= \frac{a_0 + a_1z + a_2z^2}{b_0 + b_1z + b_2z^2} & \lim_{z \rightarrow \infty} [2/2] &= a_2/b_2 \\ [3/3] &= \frac{a_0 + a_1z + a_2z^2 + a_3z^3}{b_0 + b_1z + b_2z^2 + b_3z^3} & \lim_{z \rightarrow \infty} [3/3] &= a_3/b_3 \\ &\vdots & & \\ \lim_{z \rightarrow \infty} [m/m] &= a_m/b_m \end{aligned}$$

EXAMPLE: Find the limit for $e^{-x/(1+x)}$.

$$[1/1] = .333\dots$$

$$[2/2] = .368\dots$$

$$[3/3] = .368\dots$$

EXAMPLE: For e^x , we have

$$[1/1] = \frac{2+x}{2-x}$$

$$[2/2] = \frac{12+6x+x^2}{12-6x+x^2}$$

$$[3/3] = \frac{120+60x+12x^2+x^3}{120-60x+12x^2-x^3}$$

Note that if we let $x = 1$ to consider the series for e , we get the correct limit.

$$[1/1] = 3$$

$$[2/2] = 2.714$$

$$[3/3] = 2.718$$

$$[4/4] = 2.718$$

PROBLEM: Noting that the limit is correct for $x = 1$ but the limit at ∞ for $[1/1]$, $[2/2]$, $[3/3]$ fluctuates between ± 1 for both e^x and e^{-x} , i.e., $a_m/b_m = \pm 1$ as m increases, explain the lack of convergence to a limit since we know $e^{-x} \rightarrow 0$ as $x \rightarrow \infty$ and $e^x \rightarrow \infty$ as $x \rightarrow \infty$. Try the Shanks (or Wynn*) transformation. These transformations are effective in accelerating convergence of many slowly convergent series.

For cases where the Padé approximant appears inapplicable, we can sometimes use transformations of the series. Consider

$$f(z) = c_0 + c_3 z^3 + c_6 z^6 + \dots$$

i.e., $c_{3m} \neq 0$ but $c_{3m+1} = c_{3m+2} = 0$. We let $y = z^3$ so that

$$f(y) = c_0 + c_1 y + c_2 y^2 + \dots$$

is now in a form for Padé transformation. In general for a series

$$f(z) = \sum_{m=0}^{\infty} c_{Nm} z^{Nm}$$

where N is a positive integer (or "skip factor") with $c_{Nm} \neq 0$ but $c_{Nm+v} = 0$ for $1 \leq v \leq N-1$. We substitute $y = z^N$ to get

* Other useful transforms used to accelerate convergence are the Euler, Wynn, and Van Wijngaarden transforms.

$$f(y) = c_0 + c_N y + c_{2N} y^2 + \dots$$

or

$$f(y) = \sum_{n=0}^{\infty} c_n y^n$$

where $n = Nm$.

EXAMPLE:
$$f(x) = \left(\frac{1 + (1/2)x}{1 + 2x} \right)^{1/2} = 1 - \frac{3}{4}x + \frac{39}{32}x^2 - \dots$$

To approximate by $[1/1]$ we have

$$c_1 b_1 = -c_2$$

$$-\frac{3}{4}b_1 = -\frac{39}{32}$$

$$b_1 = 13/8$$

$$a_0 = c_0 = 1$$

$$a_1 = c_1 + b_1 c_0 = -\frac{3}{4} + \frac{13}{8} \cdot 1 = 7/8$$

Consequently

$$[1/1] = \frac{1 + (7/8)x}{1 + (13/8)x} = 0.54$$

(which is within 8% of the correct limit of 0.5 for the function).

VERIFICATION:

$$\left(1 + \frac{13}{8}x \right) \left(1 - \frac{3}{4}x + \frac{39}{32}x^2 \right) = 1 + \frac{7}{8}x + O(x^3)$$

EXERCISE: Calculate $[2/2]$ and $[3/3]$ to show that the limit approaches 0.5 more and more closely as we go to higher-order $[L/M]$.

EXAMPLE: $f(x) = 1 - x + x^2/2! - x^3/3! + \dots$

$$[1/1] = \frac{1 - (1/2)x}{1 + (1/2)x} \rightarrow -1 \text{ as } x \rightarrow \infty$$

EXAMPLE: $f(x) = e^x$

$$[1/1] = \frac{2+x}{2-x}$$

$$[2/2] = \frac{12+6x+x^2}{12-6x+x^2}$$

EXAMPLE:

$$f(x) = \left[\frac{1+2x}{1+x} \right]^{1/2} = 1 + \frac{1}{2}x - \frac{5}{8}x^2 + \frac{13}{16}x^3 - \frac{141}{128}x^4 + \dots$$

We note that even though the series has a limited region of convergence ($x \leq 1/2$), the function is smooth for $0 \leq x < \infty$. If we write a ratio

$$(a+bx)/(c+dx)$$

it is clear that we get a finite limit as x approaches ∞ . Calculating $[L/M] = [1/1]$, we have

$$[1/1] = \frac{1+(7/4)x}{1+(5/4)x} \rightarrow 1.4 \text{ as } x \rightarrow \infty$$

and carrying out the division, we get

$$[1/1] = 1 + \frac{1}{2}x - \frac{5}{8}x^2 + \frac{25}{32}x^3 - \frac{125}{128}x^4 + \dots$$

which exactly matches the first three terms of the Taylor series. If we go as high as $[5/5]$, we match the first 11 terms of the Taylor series

$$[2/2] = \frac{1+(13/4)x+(41/16)x^2}{1+(11/14)x+(29/16)x^2} \rightarrow 1.4137 \text{ as } x \rightarrow \infty$$

which is the correct limit of $2^{1/2}$ to three-place accuracy.

EXERCISE: Since $\cos x = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n)!$, show that

$$[2/2] = (12-5x^2)/(12+x^2)$$

uses the first five terms of the Taylor series. Show that $[2/2]$ is closer to the exact value of $\cos x$ than the sum of the first 5 terms.

SERIES NOT SUITABLE FOR PADÉ TRANSFORM:

Sometimes the series is not in a convenient form for the Padé transform which is designed for a series $\sum_{n=0}^{\infty} c_n x^n$ with non-zero c_0, c_1, c_2 .

If we have missing terms, e.g., $f(x) = \sum_{n=0}^{\infty} c_{Nn} x^{Nn}$ where N is a positive integer and $c_0 \neq 0$, e.g., $\sum_{n=0}^{\infty} c_0 x^0 + c_3 x^3 + c_6 x^6 + \dots$ then we use the transform $z = x^N$ (or $z = x^3$ in this specific case) and let $b_n = c_{Nn}$ to write $f(z) = \sum_{n=0}^{\infty} b_n z^n$ and $b_0 \neq 0$ which is now suitable for the Padé transform.

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ and $c_0 = 0$, we can apply a translation $z = x - \xi$ with $\xi < \rho$ with ρ symbolizing the radius of convergence. If we use $|\xi| < 1$, the result will be simpler for manual computation because the resulting series for each new coefficient in $f(z) = \sum_{n=0}^{\infty} b_n z^n$ will converge rapidly. The result for b_n will be $b_0 \neq 0$ and

$$b_n = \sum_{v=0}^{\infty} \binom{v+n}{n} c_{v+n} \xi^v$$

which is equivalent to an analytic continuation.

THE SHANKS TRANSFORM:

This is a nonlinear transform which can be very effective, particularly in accelerating convergence of slowly converging series. It has even been applied to diverging series which seems contradictory. However, if a power series has been obtained by dividing out a rational function, this nonlinear transform is a means of inverting the procedure to obtain the rational function.

The Shanks transform is related to the Padé approximant. It is more accurate; however, the Padé approximant is more explicitly expressed in terms of the coefficients of the original series. Let's write the sequence of partial sums $\{S_n\}$ for a series and define the Shanks transform by

$$T\{S_n\} = \frac{S_{n+1}S_{n-1} - S_n^2}{S_{n+1} + S_{n-1} - 2S_n}$$

We often want repeated transforms, called the iterated Shanks transform, so it is convenient to write $\{A_n\}$ for the $\{S_n\}$. The iterated transforms will often lead to an extremely accurate solution. The first-order transform is written as:

$$B_a = T\{A_a\} = \frac{A_{a+1}A_{a-1} - A_a^2}{A_{a+1} + A_{a-1} - 2A_a}$$

$$C_a = T\{B_a\}$$

$$D_a = T\{C_a\}$$

We now write the sequence simply as

$$\begin{array}{ccccccc} A_0 & & & & & & \\ A_1 & B_1 & & & & & \\ A_2 & B_2 & C_2 & & & & \\ A_3 & B_3 & C_3 & D_3 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \end{array}$$

where $A_a \equiv S_a$.

EXAMPLE : In the continued fraction representing $\sqrt{2}$, the sequence of partial sums $\{S_n\}$ is $S_1 = 1$, $S_2 = 3/2$, $S_3 = 7/5$. Find the limit. Calculation yields:

n	A_n	B_n	C_n
1	1		
2	3/2	17/12	
3	7/5	99/70	19601/13860
4	17/12	577/408	
5	41/29		

We note that $A_5 = 1.4137$ which is correct to 3 figures, while $C_3 = 1.414213564$ is correct to 9 figures.

EXERCISE: Verify all results.

EXAMPLE: The Leibnitz series for π is $\pi = 4 - 4/3 + 4/5 - 4/7 + \dots$. The results are:

n	A_n	B_n	C_n	D_n	E_n
0	4.0000000				
1	2.6666667	3.1666667			
2	3.4666667	3.1333333	3.1421053		
3	2.8952381	3.1452381	3.1414502	3.1415993	
4	3.3396825	3.1396825	3.1416433	3.1415909	3.1415928
5	2.9760462	3.1427129	3.1415713	3.1415933	3.1415927
6	3.2837385	3.1408814	3.1416029	3.1415925	
7	3.0170718	3.1420718	3.1415873		
8	3.2523659	3.1412548			
9	3.0418396				

We see that the tenth partial sum A_9 , or S_9 , is correct only to one figure. Shanks [1] points out that to give an answer correct to eight figures would require $n = 40$ million in S_n while we note that $e_5(S_n)$, or E_n , is already correct to eight figures.

EXERCISE: In the example

$$f(x) = 1 - \frac{3}{4}x + \frac{39}{32}x^2 - \dots$$

in the previous section on Padé transforms, we obtained $[1/1] = .54$ which was close to the correct limit of .5. Show that the Shanks transform $T(S_2) = .54$ also. Investigate the relationship between the two procedures.

Another related transform (which can also be iterated like the Shanks transform) is the Aitken transform defined by

$$T\{S_n\} = S_n - \{(S_{n+1} - S_n)^2 / (S_{n+2} - 2S_{n+1} + S_n)\}$$

The first-order Shanks transform is equivalent to the Aitken (δ^2) process and the m th order Shanks transform of the n th partial sum is equivalent to the $[m/n]$ Padé approximant. We can view the Shanks transform as a unifying concept subsuming the Aitken process and the Padé approximant.

SUGGESTED READING

1. Daniel Shanks, Nonlinear Transformations of Divergent and Slowly Converging Series *J. Math. and Physics*, **34**, (1-42) (1953).
2. A.C. Aitken, On Bernoulli's Numerical Solution of Algebraic Equations, *Proc. Roy Soc.*, Edinburgh, (289-305) (1926).

ON STAGGERED SUMMATION OF DOUBLE DECOMPOSITION SERIES

$$\begin{aligned}
 u_0 &= u_0^{(0)} \\
 u_1 &= u_0^{(1)} + u_1^{(0)} \\
 u_2 &= u_0^{(2)} + u_1^{(1)} + u_2^{(0)} \\
 u_3 &= u_0^{(3)} + u_1^{(2)} + u_2^{(1)} + u_3^{(0)} \\
 u_4 &= u_0^{(4)} + u_1^{(3)} + u_2^{(2)} + u_3^{(1)} + u_4^{(0)} \\
 &\vdots
 \end{aligned}
 \tag{1}$$

The first column on the right of the equation is equal to u_0 . The second column is equal to u_1 . The third column is equal to u_2 , etc. These sums of each column will be denoted by $u_0^i, u_1^i, u_2^i, \dots$ respectively where i indicates the initial-value format. The sum of the column to the left of the equal sign is denoted by $\sum_{n=0}^{\infty} u_n^b$ for boundary-value format. We have

$$u = \sum_{n=0}^{\infty} u_n^b = u_0^i + u_1^i + \dots = \sum_{n=0}^{\infty} u_n^i$$

Thus in writing the approximants ϕ_m we can write

$$\begin{aligned}
 \phi_1[u^b] &= \phi_1[u_0^i] \\
 \phi_2[u^b] &= \phi_2[u_0^i] + \phi_1[u_1^i] \\
 \phi_3[u^b] &= \phi_3[u_0^i] + \phi_2[u_1^i] + \phi_1[u_2^i] \\
 &\vdots \\
 \phi_m[u^b] &= \phi_m[u_0^i] + \phi_{m-1}[u_1^i] + \dots + \phi_1[u_{m-1}^i]
 \end{aligned}$$

which can be written

$$\phi_m[u^b] = \sum_{n=0}^{m-1} \phi_{m-n}[u_n^i] \tag{2}$$

Referring again to (1), we have

$$u_m = u_0^{(m)} + u_1^{(m-1)} + u_2^{(m-1)} + \dots + u_{m-2}^{(2)} + u_{m-1}^{(1)} + u_m^{(0)}$$

or $u_m = \sum_{n=0}^m u_n^{m-n}$ in our boundary-value format. Returning to (1) we can write in initial-value format,

$$u_0 = u_0^{(0)} + u_0^{(1)} + u_0^{(2)} + \dots$$

$$u_1 = u_1^{(0)} + u_1^{(1)} + u_1^{(2)} + \dots$$

$$\vdots$$

$$u_n = u_n^{(0)} + u_n^{(1)} + u_n^{(2)} + \dots$$

or $u_n = \sum_{m=0}^{\infty} u_n^{(m)}$ (in initial-value format). By decomposition $u = \sum_{n=0}^{\infty} u_n$.

By double decomposition $u = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)}$.

FORMULAS OF INTEREST:

$$\sum_{m=0}^{\infty} \sum_{n=0}^m u_n^{(m)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)}$$

$$\phi_m[u^b] = \sum_{n=0}^{m-1} \phi_{m-n}[u_n^i]$$

$$u_m^b \equiv \sum_{n=0}^m u_n^{(m-n)}$$

$$u_n^i \equiv \sum_{m=0}^{\infty} u_n^{(m)}$$

$$u^b = \sum_{m=0}^{\infty} \sum_{n=0}^m u_n^{(m-n)}$$

$$u^i = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)}$$

$$u^b = u^i$$

$$\lim_{m \rightarrow \infty} \phi_m[u^b] = u^b$$

$$\lim_{m \rightarrow \infty} \sum_{n=0}^{m-1} \phi_{m-n}[u_n^i] = \sum_{n=0}^{\infty} u_n^i = u^i$$

CAUCHY PRODUCTS OF INFINITE SERIES

In one dimension for $u = \sum_{\mu=0}^{\infty} a_{\mu} x^{\mu}$ and $\beta = \sum_{m=0}^{\infty} \beta_m x^m$, we can write

$$\beta u = \sum_{m=0}^{\infty} x^m \sum_{\mu=0}^m \beta_{m-\mu} a_{\mu}$$

In two dimensions,

$$\begin{aligned} \beta &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \beta_{m_1, m_2} x_1^{m_1} x_2^{m_2} \\ u &= \sum_{\mu_1=0}^{\infty} \sum_{\mu_2=0}^{\infty} a_{\mu_1, \mu_2} x_1^{\mu_1} x_2^{\mu_2} \\ \beta u &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} x_1^{m_1} x_2^{m_2} \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \beta_{m_1-\mu_1, m_2-\mu_2} a_{\mu_1, \mu_2} \end{aligned}$$

In three dimensions,

$$\begin{aligned} \beta &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \beta_{m_1, m_2, m_3} x_1^{m_1} x_2^{m_2} x_3^{m_3} \\ u &= \sum_{\mu_1=0}^{\infty} \sum_{\mu_2=0}^{\infty} \sum_{\mu_3=0}^{\infty} a_{\mu_1, \mu_2, \mu_3} x_1^{\mu_1} x_2^{\mu_2} x_3^{\mu_3} \\ \beta u &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} x_1^{m_1} x_2^{m_2} x_3^{m_3} \\ &\quad \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \sum_{\mu_3=0}^{m_3} \beta_{m_1-\mu_1, m_2-\mu_2, m_3-\mu_3} a_{\mu_1, \mu_2, \mu_3} \end{aligned}$$

In four dimensions,

$$\begin{aligned} \beta &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \sum_{m_4=0}^{\infty} \beta_{m_1, m_2, m_3, m_4} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4} \\ u &= \sum_{\mu_1=0}^{\infty} \sum_{\mu_2=0}^{\infty} \sum_{\mu_3=0}^{\infty} \sum_{\mu_4=0}^{\infty} a_{\mu_1, \mu_2, \mu_3, \mu_4} x_1^{\mu_1} x_2^{\mu_2} x_3^{\mu_3} x_4^{\mu_4} \end{aligned}$$

$$\beta u = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \sum_{m_4=0}^{\infty} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4} \\ \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \sum_{\mu_3=0}^{m_3} \sum_{\mu_4=0}^{m_4} \beta_{m_1-\mu_1, m_2-\mu_2, m_3-\mu_3, m_4-\mu_4} a_{\mu_1, \mu_2, \mu_3, \mu_4}$$

In N dimensions,

$$\beta = \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} \beta_{m_1, \dots, m_N} x_1^{m_1} \cdots x_N^{m_N} \\ u = \sum_{\mu_1=0}^{\infty} \cdots \sum_{\mu_N=0}^{\infty} a_{\mu_1, \dots, \mu_N} x_1^{\mu_1} \cdots x_N^{\mu_N} \\ \beta u = \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} x_1^{m_1} \cdots x_N^{m_N} \sum_{\mu_1=0}^{m_1} \cdots \sum_{\mu_N=0}^{m_N} \beta_{m_1-\mu_1, \dots, m_N-\mu_N} a_{\mu_1, \dots, \mu_N}$$

These can all be programmed using "do-loops" and stopping rules. These product rules should be useful in programming solutions where the system input and system coefficients are known only as power series.

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