Attempting the NEW oblate Q integral PhL 2.2.10

As usual, consideration of a relatively simple integral has led to massive misunderstandings of basic things at the kindergarten level.

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1. "Today's problem" is evaluation of this integral, where m is a negative integer:

$$(u^{m}_{\nu}, u^{m}_{\nu'}) = \int_0^{\infty} d\zeta \; Q_{\nu}^{m}(i\zeta)^* \, Q_{\nu',}^{m}(i\zeta) = K_{\nu}^{m} \, \delta_{\nu,\nu'}$$

where $\nu = |m| - N$ $\nu' = |m| - N'$ N and N' are positive, odd, and $\neq 0$.

This integral is purportedly an orthogonality integral for a certain ODE system described in "when are ODE solutions orthogonal.doc".

4. A Closer Look at Eigenfunctions

Let's go back to this again. Recall that our spectrum is $\nu = |m| - N$ with N = 1,3,5.... but $\nu > -1/2$. Therefore,

$1+\nu-|m| = 1 + |m| - N - |m| = 1-N = 0, -2, -4$ and so on.

Now look at our Q form given page 133 (36). The "a" argument of F(a,b,c,z) is then always a negative integer, which means the series truncates. We have:

$$Q_{\nu}^{m}(i\zeta) = 2^{\nu} [\Gamma(1+\nu) \Gamma(1+\nu+m)/ \Gamma(2+2\nu)] (i\zeta +1)^{m/2-\nu-1} (i\zeta -1)^{-m/2} F(1+\nu-m, 1+\nu, 2+2\nu; \; 2/(1+ i\zeta))$$

Assume for the moment that N = 1 so we have $1+\nu-|m| = 0$. Then ($\mu = m$, either symbol same)

$$F(1+\nu-\mu, 1+\nu, 2+2\nu; \; 2/(1+ i\zeta)) = F(0,1+\nu,2+2\nu; 2/(1+ i\zeta))$$

But we know that in this case, $F = 1 + abz/c + ... = 1$ so then ($m = \mu$)

$$Q_{\nu}^{m}(i\zeta) = 2^{\nu} [\Gamma(1+\nu) \Gamma(1+\nu+m)/ \Gamma(2+2\nu)] (i\zeta +1)^{m/2-\nu-1} (i\zeta -1)^{-m/2} \; [1 - (1+ i\zeta)^{-1}]$$

$$Q_{|m|-1}^{m}(z) = 2^{\nu} [\Gamma(1+\nu) \Gamma(1+\nu+\mu)/ \Gamma(2+2\nu)] (z +1)^{\mu/2-\nu-1} (z -1)^{-\mu/2} \; [1]$$

Let's try this for some simple case, say m = -2 :

$$Q_1^{-2}(z) = 2^1 [\ \Gamma(1+1)\ \Gamma(1+1-2)/\ \Gamma(2+2*1)\ ]\ (z+1)^{-2/2-1-1}\ (z-1)^{-(-2)/2}\ [1\ ]$$

$$= 2^1 [\ \Gamma(2)\ \Gamma(0)/\ \Gamma(4)\ ]\ (z+1)^{-3}\ (z-1)^1\ [1\ ]$$

STOP! Why are we getting a pole here?

and indeed there is no log. My formula above gives

$$Q_\nu^\mu(z) = 2^\nu [\ \Gamma(1+\nu)\ \Gamma(1+\nu+\mu)/\ \Gamma(2+2\nu)\ ]\ (z+1)^{\mu/2-\nu-1}\ (z-1)^{-\mu/2}\ [1 - (1+z)^{-1}\ ]$$

$$Q_0^2(z) = 2^0 [\ \Gamma(1+0)\ \Gamma(1+0+2)/\ \Gamma(2+2*0)\ ]\ (z+1)^{2/2-0-1}\ (z-1)^{-2/2}\ [1 - (1+z)^{-1}\ ]$$

$$= [\ 2!/1\ ]\ (z-1)^{-1}\ [1 - (1+z)^{-1}\ ]$$
$$= 2(z-1)^{-1}\ [1 - 1/(1+z)\ ] = 2(z-1)^{-1} [\ 1+z-1]/(z+1)$$
$$= 2(z-1)^{-1}\ z/(z+1) = 2z/\ [(z+1)(z-1)]$$ // agrees!

I would like to see a table of such things. Maxima cannot do this kind of parameter combination (but it can differentiate one it can so) ! I can make Maple do it like this:

```
Q20 := (z^2 - 1) * diff(LegendreQ(0,0,z),z,z):
rationalize(%);
```

$$2\frac{z}{z^2-1}$$

so I could make my own table. Multiple derivatives: g := diff(f,x$n) n can be a variable.

So, back to our general formula for our "eigenfunctions" !

$$Q_\nu^\mu(z) = 2^\nu [\ \Gamma(1+\nu)\ \Gamma(1+\nu+\mu)/\ \Gamma(2+2\nu)\ ]\ (z+1)^{\mu/2-\nu-1}\ (z-1)^{-\mu/2}\ F(1+\nu-\mu,\ 1+\nu,\ 2+2\nu;\ 2/(1+z)\ )$$

Leave μ, but set in ν = μ-N, N = 2,4... > -1/2

$$Q_{\mu-N}^\mu(z) = 2^{\mu-N} [\ \Gamma(1+\mu-N)\ \Gamma(1+\mu-N+\mu)/\ \Gamma(2+2[\mu-N])\ ]$$
$$(z+1)^{\mu/2-[\mu-N]-1}\ (z-1)^{-\mu/2}\ F(1+(\mu-N)-\mu,\ 1+\mu-N,\ 2+2(\mu-N);\ 2/(1+z)\ )$$

$$= 2^{\mu-N} [\ \Gamma(1+\mu-N)\ \Gamma(1+2\mu-N)/\ \Gamma(2+2\mu-2N])\ ]\ (z+1)^{-\mu/2+N-1}\ (z-1)^{-\mu/2}$$
$$F(1-N,\ 1+\mu-N,\ 2+2\mu-2N;\ 2/(1+z)\ )$$

Let's try this differently using μ = ν + N

$$Q_\nu^\mu(z) = 2^\nu [\, \Gamma(1+\nu)\, \Gamma(1+\nu+\mu)/\, \Gamma(2+2\nu)\,]\, (z+1)^{\mu/2-\nu-1}\, (z-1)^{-\mu/2}\, F(1+\nu-\mu,\, 1+\nu,\, 2+2\nu;\; 2/(1+z))$$

$$Q_\nu^{\nu+N}(z) = 2^\nu [\, \Gamma(1+\nu)\, \Gamma(1+\nu+\nu+N)/\, \Gamma(2+2\nu)\,]\, (z+1)^{(\nu+N)/2-\nu-1}\, (z-1)^{-(\nu+N)/2}$$
$$F(1-N,\, 1+\nu,\, 2+2\nu;\; 2/(1+z))$$

$$Q_\nu^{\nu+N}(z) = 2^\nu [\, \Gamma(1+\nu)\, \Gamma(1+2\nu+N)/\, \Gamma(2+2\nu)\,]\, (z+1)^{(N-\nu)/2-1}\, (z-1)^{-(N+\nu)/2}$$
$$F(1-N,\, 1+\nu,\, 2+2\nu;\; 2/(1+z))$$

Well, μ is really pre-determined by the φ problem in a separation scenario, so better the other way:

$$Q_{\mu-N}^\mu(z) == 2^{\mu-N} [\, \Gamma(1+\mu-N)\, \Gamma(1+2\mu-N)/\, \Gamma(2+2\mu-2N)\,]\, (z+1)^{-\mu/2+N-1}\, (z-1)^{-\mu/2}$$
$$F(1-N,\, 1+\mu-N,\, 2+2\mu-2N);\; 2/(1+z))$$

Notice that

$$\Gamma(2+2\mu-2N)\,/\,\Gamma(1+\mu-N) = \Gamma(2x)/\Gamma(x) \qquad x = 1+\mu-N$$

$$= (1/\sqrt{\pi})\, 2^{2x-1}\Gamma(x+1/2) = (1/\sqrt{\pi})\, 2^{2(1+\mu-N)-1}\Gamma(3/2+\mu-N)$$
$$= (1/\sqrt{\pi})\, 2^{2(\mu-N)+1}\Gamma(3/2+\mu-N)$$

so we could replace:

$$\Gamma(1+\mu-N)/\, \Gamma(2+2\mu-2N) = \sqrt{\pi}\; 2^{-2(\mu-N)+1}\; 1/\, \Gamma(3/2+\mu-N)$$

to get

$$Q_{\mu-N}^\mu(z) == 2^{\mu-N} [\sqrt{\pi}\; 2^{-2(\mu-N)+1}\, \Gamma(1+2\mu-N)/\, \Gamma(3/2+\mu-N)\,]\, (z+1)^{-\mu/2+N-1}\, (z-1)^{-\mu/2}$$
$$F(1-N,\, 1+\mu-N,\, 2+2\mu-2N);\; 2/(1+z))$$

$$= \sqrt{\pi}\, 2^{-\mu+N+1}\, [\Gamma(1+2\mu-N)/\, \Gamma(3/2+\mu-N)\,]\, (z+1)^{-\mu/2+N-1}\, (z-1)^{-\mu/2}$$
$$F(1-N,\, 1+\mu-N,\, 2+2\mu-2N);\; 2/(1+z))$$

which is perhaps a little more compact. This IS our eigenfunction set for $N = 2,4...N_{max}$.

Let's go back to our earlier (safer) formula:

$$Q_{\mu-N}^\mu(z) == 2^{\mu-N} [\, \Gamma(1+\mu-N)\, \Gamma(1+2\mu-N)/\, \Gamma(2+2\mu-2N)\,]\, (z+1)^{-\mu/2+N-1}\, (z-1)^{-\mu/2}$$
$$F(1-N,\, 1+\mu-N,\, 2+2\mu-2N);\; 2/(1+z))$$

and define an ancillary parameter $\kappa \equiv 1+\mu-N = 1+\nu$. Then our formula reads

$$Q_{\mu-N}^\mu(z) = 2^{\mu-N} [\, \Gamma(\kappa)\, \Gamma(\kappa+\mu)/\, \Gamma(2\kappa)\,]\, (z+1)^{-\mu/2+N-1}\, (z-1)^{-\mu/2}\; F(1-N,\, \kappa,\, 2\kappa;\; 2/(1+z))$$

We can then evaluate:

N=2

$F(-1, \kappa, 2\kappa; \psi) = 1 + (-1)(\kappa)/(2\kappa)\,\psi = 1-(\kappa/2)\psi$

N=4

$F(-3, \kappa, 2\kappa; \psi) = 1 + (-3)(\kappa)/(2\kappa)\,\psi + (-3)(-2)(\kappa)(\kappa+1)/[2!(2\kappa)(2\kappa+1)]\,\psi^2$

$= 1 - (3/2)\psi + (3/2)(\kappa+1)/(2\kappa+1)\,\psi^2 + \text{a } \psi^3 \text{ term}$

Statement: The eigenfunctions are all of this form which shows only powers, no logs:

$$Q_{\mu-N}^{\mu}(z) = A_{\mu N}\,(z+1)^{-\mu/2+N-1}\,(z-1)^{-\mu/2}\,\Sigma_{n=1}^{N-1}\,a_{n\mu}\,\psi^n \qquad \psi = 2/(1+z)$$

If μ= m is a positive integer, this formula is still correct, but we have a simpler way to compute things:

$$Q_{m-N}^{m}(z) = (z^2-1)^{m/2}\partial_m Q_{m-N}(z)$$

For example, if m = 2 we have

$$Q_{2-N}^{2}(z) = (z^2-1)\,\partial_2 Q_{2-N}(z)$$

and this is the only allowed value of N for m=2: // if N = 2, get ν = -2, no good!

N=2: $Q_0^2(z) = (z^2-1)^2\partial_2\, Q_0(z) = 2z/(z^2-1)$ as Maple verified

If m = 4, we then have

$$Q_{4-N}^{4}(z) = (z^2-1)^2\,\partial_4 Q_{4-N}(z)$$

Then we have several cases:

N=0: $Q_4^4(z) = (z^2-1)^2\,\partial_4 Q_4(z) + ??$ // not a legal N!

N=2: $Q_2^4(z) = (z^2-1)^2\,\partial_4 Q_2(z) = 48z/(z^2-1)^2$ Maple AND Maxima!

N=4: $Q_0^4(z) = (z^2-1)^2\,\partial_4 Q_0(z) = 24z(z^2+1)/(z^2-1)^2$ Maple

Notice that the N=2 and N=4 legal EF's vanish at z = 0, as our homo BC requires.

I skipped m = 3, but it will have ν = 3-2 = 1 as the sole eigenfunction.

From Maxima I find that

$$Q_4^4(z) = \{ \ln[(z+1)/(z-1)] * 105(z^8 - 4z^6 + 6z^4 - 4z^2 + 1)$$

$$+ 2(-105z^7 + 385z^5 - 511z^3 + 279z) \} / [2(z^2-1)^2]$$

This is illegal because it is not in my spectrum, and also you can see it does not vanish at z = 0 so it violates the homo BC (which is why it is not in the spectrum).

4. Back to our integral.

Now we can at least look at the examples I did in "when are ODE solutions orthogonal.doc".

First, we should find that:

$$I = \int_0^\infty d\zeta \; Q_2^4(i\zeta)^* \, Q_0^4(i\zeta) = 0$$

Now we know exactly how to do this integral:

$$I = \int_0^\infty d\zeta \; [48z/(z^2-1)^2]^* \; [24z(z^2+1)/(z^2-1)^2]$$

$$= \int_0^\infty d\zeta \; [48(i\zeta)/(-\zeta^2-1)^2]^* \; [24(i\zeta)(-\zeta^2+1)/(-\zeta^2-1)^2]$$

$$= \int_0^\infty d\zeta \; [48(-i\zeta)/(-\zeta^2-1)^2] \; [24(i\zeta)(-\zeta^2+1)/(-\zeta^2-1)^2]$$

$$= 48*24 \int_0^\infty d\zeta \, \zeta^2(1-\zeta^2)/(1+\zeta^2)^4$$

$$= 48*24 \int_0^\infty dx \, x^2(1-x^2)/(1+x^2)^4$$

$$= 48*24 \left[\int_0^\infty dx \, x^2/(1+x^2)^4 - \int_0^\infty dx \, x^4/(1+x^2)^4 \right]$$

and Maple tells us that:

```
int(x^2/(1+x^2)^4,x = 0..infinity);
```

$$\frac{1}{32}\pi$$

```
int(x^4/(1+x^2)^4,x = 0..infinity);
```

$$\frac{1}{32}\pi$$

so amazingly enough, we really do get 0, to full accuracy. Now let's try this integral

$$\int_0^\infty d\zeta \; Q_2^4(i\zeta)^* \, Q_2^4(i\zeta) \;=$$

$$= \int_0^\infty d\zeta \; 48z^*/(z^2-1)^2 \; 48z/(z^2-1)^2 \;= 48^2 \int_0^\infty d\zeta \; \zeta^2 / (-\zeta^2-1)^4$$

$$= 48^2 \int_0^\infty dx \; x^2 / (x^2+1)^4 \;= 48^2 \;\; \pi/32 = (3/2)*48*\pi \;= 72\pi$$

```
evalf(72*Pi);
```

226.1946711

This agrees exactly with the numeric integration in "when are ODE solutions orthog"

```
evalf(Int(g, z=0..infinity));
```

226.1946711