Attempting the oblate Q integral PhL 1.29.10

As usual, consideration of a relatively simple integral has led to massive misunderstandings of basic things at the kindergarten level.

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1. "Today's problem" is evaluation of this integral.

$$(u^{\mu}_{\nu}, u^{\mu}_{\nu'}) = \int_0^{\infty} d\zeta \; Q_{\nu}^{\mu}(i\zeta)^* \, Q_{\nu'}^{\mu}(i\zeta) = K_{\nu}^{\mu} \, \delta_{\nu,\nu'} \qquad // \; K_0^4 = 72\pi, \text{ example}$$

where $\nu = \mu - N$ $\nu' = \mu - N'$ N and N' are positive, even, and $\neq 0$.

This integral is purportedly an orthogonality integral for a certain ODE system described in "when are ODE solutions orthogonal.doc".

So here is the integral of interest:

$$I = \int_{0+}^{i\infty} dz \; Q_{\nu}^{\mu}(z)^* \, Q_{\nu}^{\mu}(z)$$

I tried doing this by "deformation of contour" methods and this made me realize that the integrand is not an analytic function f(z), nor in fact is it even a function of complex variable z.

2. Stupid Paradox. Here is an illustration of the paradox this leads to. Replace Q by the function $(z+1)^{-5}$. The integral up the z axis from the origin is I. It is a sum of positive elements and is therefore a positive number. This is equal to the integral up the negative imaginary axis. We close the contour to the right with a great circle that makes no contribution. The integrand has no singularities to the right of Re(z) = -1. So we find that the closed D-shaped integral equals 2 I and we close the contour to nothing and find I = 0.

3. Lars Kindergarten Revisited. So I went back to my ahlfors.doc, did some reading, and added lots of notes in an early section, and yes, it is true, the function $f(z) = Q_{\nu'}^{\mu}(z)^* \, Q_{\nu}^{\mu}(z)$ is not analytic in z, so the above "contour integration" approach is meaningless, and that explains my "paradox".

So, we can remove complex integration techniques from our tool kit for this integral. We just start over considering:

$$\int_0^{\infty} d\zeta \; Q_{\nu}^{\mu}(i\zeta)^* \, Q_{\nu'}^{\mu}(i\zeta)$$

How are ANY integrals of functions like this "done"?

Method 1: find a HG series form which converges for (0,i∞), multiply together, integrate term by term. This sounds pretty horrible. Is there a HG which includes this entire range: [but it truncates!]

$|z-1| < |z+1|$ distance from 1 less than from -1 = RH plane I think

But the imaginary axis is not included here. Forget this.

Method 2: use integral representation for Q's. Bateman p 155 (5) has one of the form

$$Q_\nu^\mu(i\zeta) = A\,(-\zeta^2-1)^{-\mu/2} \int_0^\pi dt\ f(t)\ (i\zeta + \cos t)^{\mu-\nu-1}$$

$$Q_\nu^\mu(i\zeta)^* = A\,(-\zeta^2-1)^{-\mu/2} \int_0^\pi dt\ f(t)\ (-i\zeta + \cos t)^{\mu-\nu-1}$$

$$Q_\nu^\mu(i\zeta)^*\, Q_\nu^\mu(i\zeta) = A^2\,(-\zeta^2-1)^{-\mu} \int_0^\pi dt\ f(t) \int_0^\pi dt'\ f(t')\ (i\zeta + \cos t)^{\mu-\nu-1}(-i\zeta + \cos t')^{\mu-\nu-1}$$

Comment: remember that our spectrum is $\nu = \mu - N$ $N = 2,4,6...$

which says $\mu-\nu-1 = \mu - [\mu-N]-1 = N-1 = 1,3,5,7...$ = odd integer.

Then we have to do this integral, where α = odd integer.

$$J \equiv \int_0^\infty d\zeta\ (i\zeta + \cos t)^{\mu-\nu-1}(-i\zeta + \cos t')^{\mu-\nu-1} = \int_0^\infty d\zeta\ (i\zeta + \cos t)^{\alpha}(-i\zeta + \cos t')^{\alpha}$$

where $\alpha = \mu-\nu-1$. Then factor out $(i)^\alpha$ from first, and $(-i)^\alpha$ from second

$$= (i)^\alpha\ (-i)^\alpha \int_0^\infty d\zeta\ (\zeta - i\cos t)^{\alpha}(\zeta + i\cos t')^{\alpha}$$

$$= \int_0^\infty dx\ (x - i\cos t)^{\alpha}(x + i\cos t')^{\alpha}$$

But we can stop right here because α = positive odd integer, this integral diverges. So our order interchange was not justified. Fugedit.

4. A Closer Look at Eigenfunctions

Let's go back to this again. Recall that our spectrum is $\nu = \mu - N$ with $N = 2,4,6....$ but $\nu > -1/2$. Therefore,

$1+\nu-\mu = 1 + \mu - N - \mu = 1-N = -1, -3, -5$ and so on.

Now look at our Q form given page 133 (36). The "a" argument of F(a,b,c,z) is then always a negative integer, which means the series truncates. We have:

$$Q_\nu^\mu(i\zeta) = 2^\nu\,[\,\Gamma(1+\nu)\,\Gamma(1+\nu+\mu)/\,\Gamma(2+2\nu)\,]\,(i\zeta+1)^{\mu/2-\nu-1}\,(i\zeta-1)^{-\mu/2}\,F(1+\nu-\mu,\ 1+\nu,\ 2+2\nu;\ 2/(1+i\zeta))$$

Assume for the moment that N = 2 so we have $1+\nu-\mu = -1$. Then:

$$F(1+\nu-\mu,\ 1+\nu,\ 2+2\nu;\ 2/(1+i\zeta)) = F(-1,1+\nu,2+2\nu;\ 2/(1+i\zeta))$$

But we know that

$$F(-1,b,c,x) = 1 + (-1)bx/c$$

so

$$F(-1,1+\nu,2+2\nu;\ 2/(1+i\zeta)) = 1 + (-1)(1+\nu)x/(2+2\nu) = 1 - x/2 = 1 - (1+i\zeta)^{-1}$$

and then

$$Q_\nu^\mu(i\zeta) = 2^\nu [\ \Gamma(1+\nu)\ \Gamma(1+\nu+\mu)/\ \Gamma(2+2\nu)\ ]\ (i\zeta+1)^{\mu/2-\nu-1}\ (i\zeta-1)^{-\mu/2}\ [1 - (1+i\zeta)^{-1}]$$

$$Q_{\mu-2}^\mu(z) = 2^\nu [\ \Gamma(1+\nu)\ \Gamma(1+\nu+\mu)/\ \Gamma(2+2\nu)\ ]\ (z+1)^{\mu/2-\nu-1}\ (z-1)^{-\mu/2}\ [1 - (1+z)^{-1}]$$

Hey, what happened to the log part of the Q function? Go look for example at $Q_0^2(z)$. But this choice of parameters is "out of range" in the usual P_n^m sense, so not in Smythe's small table. We are out in a brave new world here where authors generally don't venture. We do have a formula like this:

$$Q_\nu^m(z) = (z^2-1)^{m/2}\partial_m Q_\nu(z) \qquad m = 1,2,3....$$

When applied here you get

$$Q_0^2(z) = (z^2-1)^{2/2}\partial_2 Q_0(z) = (z^2-1)\,\partial_2 Q_0(z)$$

$$= (1/2)(z^2-1)\,\partial_2[\ln(z+1) - \ln(z-1)] \qquad \text{AS p 333}$$
$$= (1/2)(z^2-1)\partial\,[(z+1)^{-1} - (z-1)^{-1}]$$
$$= (1/2)(z^2-1)[(-1)(z+1)^{-2} - (-1)(z-1)^{-2}]$$
$$= -(1/2)(z^2-1)[(z+1)^{-2} - (z-1)^{-2}]$$
$$= -(1/2)(z^2-1)[1/(z+1)^2 - 1/(z-1)^2]$$
$$= -(1/2)(z^2-1)[(z-1)^2 - (z+1)^2]\ /\ [(z+1)^2(z-1)^2]$$
$$= -(1/2)(z^2-1)[-4z]\ /\ [(z+1)^2(z-1)^2]$$
$$= -(1/2)[-4z]\ /\ [(z+1)(z-1)]$$
$$= 2z\ /\ [(z+1)(z-1)] = 2z/(z^2-1)$$

and indeed there is no log. My formula above gives

$$Q_\nu^\mu(z) = 2^\nu [\ \Gamma(1+\nu)\ \Gamma(1+\nu+\mu)/\ \Gamma(2+2\nu)\ ]\ (z+1)^{\mu/2-\nu-1}\ (z-1)^{-\mu/2}\ [1 - (1+z)^{-1}]$$

$$Q_0^2(z) = 2^0 [\ \Gamma(1+0)\ \Gamma(1+0+2)/\ \Gamma(2+2*0)\ ]\ (z+1)^{2/2-0-1}\ (z-1)^{-2/2}\ [1 - (1+z)^{-1}]$$

$$= [\ 2!/1\ ]\ (z-1)^{-1}\ [1 - (1+z)^{-1}]$$
$$= 2(z-1)^{-1}\ [1 - 1/(1+z)] = 2(z-1)^{-1}\ [1+z-1]/(z+1)$$

$= 2(z-1)^{-1}\, z/(z+1) = 2z/[(z+1)(z-1)]$ // agrees!

I would like to see a table of such things. Maxima cannot do this kind of parameter combination (but it can differentiate one it can so) ! I can make Maple do it like this:

```
Q20 := (z^2 - 1) * diff(LegendreQ(0,0,z),z,z):
rationalize(%);
```

$$2\frac{z}{z^2-1}$$

so I could make my own table. Multiple derivatives: g := diff(f,x$n) n can be a variable.

So, back to our general formula for our "eigenfunctions" !

$$Q_\nu^\mu(z) = 2^\nu [\, \Gamma(1+\nu)\, \Gamma(1+\nu+\mu)/\, \Gamma(2+2\nu)\,]\, (z+1)^{\mu/2-\nu-1}\, (z-1)^{-\mu/2}\, F(1+\nu-\mu,\, 1+\nu,\, 2+2\nu;\; 2/(1+z))$$

Leave μ, but set in ν = μ-N, N = 2,4... > -1/2

$$Q_{\mu-N}^\mu(z) = 2^{\mu-N} [\, \Gamma(1+\mu-N)\, \Gamma(1+\mu-N+\mu)/\, \Gamma(2+2[\mu-N])\,]$$
$$(z+1)^{\mu/2-[\mu-N]-1}\, (z-1)^{-\mu/2}\, F(1+(\mu-N)-\mu,\, 1+\mu-N,\, 2+2(\mu-N);\; 2/(1+z))$$

$$= 2^{\mu-N} [\, \Gamma(1+\mu-N)\, \Gamma(1+2\mu-N)/\, \Gamma(2+2\mu-2N])\,]\, (z+1)^{-\mu/2+N-1}\, (z-1)^{-\mu/2}$$
$$F(1-N,\, 1+\mu-N,\, 2+2\mu-2N;\; 2/(1+z))$$

Let's try this differently using μ = ν + N

$$Q_\nu^\mu(z) = 2^\nu [\, \Gamma(1+\nu)\, \Gamma(1+\nu+\mu)/\, \Gamma(2+2\nu)\,]\, (z+1)^{\mu/2-\nu-1}\, (z-1)^{-\mu/2}\, F(1+\nu-\mu,\, 1+\nu,\, 2+2\nu;\; 2/(1+z))$$

$$Q_\nu^{\nu+N}(z) = 2^\nu [\, \Gamma(1+\nu)\, \Gamma(1+\nu+\nu+N)/\, \Gamma(2+2\nu)\,]\, (z+1)^{(\nu+N)/2-\nu-1}\, (z-1)^{-(\nu+N)/2}$$
$$F(1-N,\, 1+\nu,\, 2+2\nu;\; 2/(1+z))$$

$$Q_\nu^{\nu+N}(z) = 2^\nu [\, \Gamma(1+\nu)\, \Gamma(1+2\nu+N)/\, \Gamma(2+2\nu)\,]\, (z+1)^{(N-\nu)/2-1}\, (z-1)^{-(N+\nu)/2}$$
$$F(1-N,\, 1+\nu,\, 2+2\nu;\; 2/(1+z))$$

Well, μ is really pre-determined by the φ problem in a separation scenario, so better the other way:

$$Q_{\mu-N}^\mu(z) == 2^{\mu-N} [\, \Gamma(1+\mu-N)\, \Gamma(1+2\mu-N)/\, \Gamma(2+2\mu-2N)\,]\, (z+1)^{-\mu/2+N-1}\, (z-1)^{-\mu/2}$$
$$F(1-N,\, 1+\mu-N,\, 2+2\mu-2N;\; 2/(1+z))$$

Notice that

$$\Gamma(2+2\mu-2N)\,/\,\Gamma(1+\mu-N) = \Gamma(2x)/\Gamma(x) \qquad x = 1+\mu-N$$

$$= (1/\sqrt{\pi})\, 2^{2x-1}\Gamma(x+1/2) = (1/\sqrt{\pi})\, 2^{2(1+\mu-N)-1}\Gamma(3/2+\mu-N)$$

$$= (1/\sqrt{\pi})\, 2^{2(\mu-N)+1}\Gamma(3/2+\mu-N)$$

so we could replace:

$$\Gamma(1+\mu-N)/\,\Gamma(2+2\mu-2N) = \sqrt{\pi}\;\; 2^{-2(\mu-N)+1}\; 1/\,\Gamma(3/2+\mu-N)$$

to get

$$Q_{\mu-N}^{\mu}(z) = 2^{\mu-N}\,[\sqrt{\pi}\;\; 2^{-2(\mu-N)+1}\,\Gamma(1+2\mu-N)/\,\Gamma(3/2+\mu-N)\,]\,(z+1)^{-\mu/2+N-1}\,(z-1)^{-\mu/2}$$
$$F(1-N,\, 1+\mu-N,\, 2+2\mu-2N);\;\; 2/(1+z)\,)$$

$$= \sqrt{\pi}\; 2^{-\mu+N+1}\,[\Gamma(1+2\mu-N)/\,\Gamma(3/2+\mu-N)\,]\,(z+1)^{-\mu/2+N-1}\,(z-1)^{-\mu/2}$$
$$F(1-N,\, 1+\mu-N,\, 2+2\mu-2N);\;\; 2/(1+z)\,)$$

which is perhaps a little more compact. This IS our eigenfunction set for $N = 2,4...N_{max}$.

Let's go back to our earlier (safer) formula:

$$Q_{\mu-N}^{\mu}(z) = 2^{\mu-N}\,[\,\Gamma(1+\mu-N)\,\Gamma(1+2\mu-N)/\,\Gamma(2+2\mu-2N)\,]\,(z+1)^{-\mu/2+N-1}\,(z-1)^{-\mu/2}$$
$$F(1-N,\, 1+\mu-N,\, 2+2\mu-2N);\;\; 2/(1+z)\,)$$

and define an ancillary parameter $\kappa \equiv 1+\mu-N = 1+\nu$. Then our formula reads

$$Q_{\mu-N}^{\mu}(z) = 2^{\mu-N}\,[\,\Gamma(\kappa)\,\Gamma(\kappa+\mu)/\,\Gamma(2\kappa)\,]\,(z+1)^{-\mu/2+N-1}\,(z-1)^{-\mu/2}\;\; F(1-N,\, \kappa,\, 2\kappa;\;\; 2/(1+z)\,)$$

We can then evaluate:

N=2

$$F(-1,\, \kappa,\, 2\kappa;\, \psi) = 1 + (-1)(\kappa)/(2\kappa)\,\psi = 1-(\kappa/2)\psi$$

N=4

$$F(-3,\, \kappa,\, 2\kappa;\, \psi) = 1 + (-3)(\kappa)/(2\kappa)\,\psi + (-3)\,(-2)\,(\kappa)\,(\kappa+1)/\,[2!(2\kappa)\,(2\kappa+1)]\;\; \psi^2$$

$$= 1 - (3/2)\psi + (3/2)\,(\kappa+1)/(2\kappa+1)\;\psi^2 \;+ \text{a } \psi^3 \text{ term}$$

<u>Statement:</u> The eigenfunctions are all of this form which shows only powers, no logs:

$$Q_{\mu-N}^{\mu}(z) = A_{\mu N}\;\,(z+1)^{-\mu/2+N-1}\,(z-1)^{-\mu/2}\,\Sigma_{n=1}^{N-1}\,a_{n\mu}\;\,\psi^{n} \qquad\qquad \psi = 2/(1+z)$$

If $\mu = m$ is a positive integer, this formula is still correct, but we have a simpler way to compute things:

$$Q_{m-N}^{m}(z) = (z^2-1)^{m/2}\partial_m Q_{m-N}(z)$$

For example, if m = 2 we have

$$Q_{2-N}^{2}(z) = (z^2-1)\,\partial_2 Q_{2-N}(z)$$

and this is the only allowed value of N for m=2: // if N = 2, get ν = -2, no good!

N=2: $Q_0^2(z) = (z^2-1)^2\partial_2\, Q_0(z) = 2z/(z^2-1)$ as Maple verified

If m = 4, we then have

$$Q_{4-N}^{4}(z) = (z^2-1)^2\,\partial_4 Q_{4-N}(z)$$

Then we have several cases:

N=0: $Q_4^4(z) = (z^2-1)^2\,\partial_4 Q_4(z) + ??$ // not a legal N!

N=2: $Q_2^4(z) = (z^2-1)^2\,\partial_4 Q_2(z) = 48z/(z^2-1)^2$ Maple AND Maxima!

N=4: $Q_0^4(z) = (z^2-1)^2\,\partial_4 Q_0(z) = 24z(z^2+1)/(z^2-1)^2$ Maple

Notice that the N=2 and N=4 legal EF's vanish at z = 0, as our homo BC requires.

I skipped m = 3, but it will have ν = 3-2 = 1 as the sole eigenfunction.

From Maxima I find that

$$Q_4^4(z) = \{ \ln[(z+1)/(z-1)] * 105(z^8 - 4z^6 + 6z^4 - 4z^2 + 1)$$

$$+ 2(-105z^7 + 385z^5 - 511z^3 + 279z) \} / [2(z^2-1)^2]$$

This is illegal because it is not in my spectrum, and also you can see it does not vanish at z = 0 so it violates the homo BC (which is why it is not in the spectrum).

4. Back to our integral.

Now we can at least look at the examples I did in "when are ODE solutions orthogonal.doc".

First, we should find that:

$$I = \int_0^\infty d\zeta\ Q_2^4(i\zeta)^*\, Q_0^4(i\zeta) = 0$$

Now we know exactly how to do this integral:

$$I = \int_0^\infty d\zeta\ [48z/(z^2-1)^2]^*\ [24z(z^2+1)/(z^2-1)^2]$$

$$= \int_0^\infty d\zeta \;\; [48(i\zeta)/(-\zeta^2-1)^2]^* \;\; [24(i\zeta)(-\zeta^2+1)/(-\zeta^2-1)^2]$$

$$= \int_0^\infty d\zeta \;\; [48(-i\zeta)/(-\zeta^2-1)^2] \;\; [24(i\zeta)(-\zeta^2+1)/(-\zeta^2-1)^2]$$

$$= 48*24 \;\; \int_0^\infty d\zeta \; \zeta^2(1-\zeta^2) / (1+\zeta^2)^4$$

$$= 48*24 \;\; \int_0^\infty dx \; x^2(1-x^2) / (1+x^2)^4$$

$$= 48*24 \; [\;\; \int_0^\infty dx \; x^2/ \; (1+x^2)^4 \;\; - \;\; \int_0^\infty dx \; x^4/ (1+x^2)^4 \;\;]$$

and Maple tells us that:

```
int(x^2/(1+x^2)^4,x = 0..infinity);
```

$$\frac{1}{32}\pi$$

```
int(x^4/(1+x^2)^4,x = 0..infinity);
```

$$\frac{1}{32}\pi$$

so amazingly enough, we really do get 0, to full accuracy. Now let's try this integral

$$\int_0^\infty d\zeta \;\; Q_2^4(i\zeta)^* \; Q_2^4(i\zeta) \;\; =$$

$$= \int_0^\infty d\zeta \;\; 48z^*/(z^2-1)^2 \;\; 48z/(z^2-1)^2 \;\; = \;\; 48^2 \;\; \int_0^\infty d\zeta \;\; \zeta^2 / (-\zeta^2-1)^4$$

$$= 48^2 \;\; \int_0^\infty dx \;\; x^2 / (x^2+1)^4 \;\; = 48^2 \;\; \pi/32 = (3/2)*48*\pi \;\; = 72\pi$$

```
evalf(72*Pi);
```

226.1946711

This agrees exactly with the numeric integration in "when are ODE solutions orthog"

```
evalf(Int(g, z=0..infinity));
```

226.1946711