Correcting GR7 Integral 3_197_2 PhL 1.30.11

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Overview In this document I derive from scratch an integral in GR7, and I find that there are in fact two errors which I then report in an email to Dan Z which appears as the last section (mailed 1.31.11). I needed this integral as one step in the process of computing a certain Mehler sech^2 integral, as detailed elsewhere.

1. Verify Bateman page 115 (6) starting from Bateman page 59 (10).

Bateman page 115 (6) makes this claim:

$$(6)\quad F(a,\ b;\ c;\ z^{-1}) = \frac{\Gamma(c)}{\Gamma(b)\ \Gamma(c-b)} \int_1^\infty (s-1)^{c-b-1}\ s^{a-c}\ (s-z^{-1})^{-a}\ ds$$

$$1 + \operatorname{Re} a > \operatorname{Re} c > \operatorname{Re} b, \quad |\arg(z-1)| < \pi$$

Let's try to derive this from the "fundamental" integral representation page 59 (10) which is this

2.1.3. The fundamental integral representations

If $\operatorname{Re} c > \operatorname{Re} b > 0$, we have Euler's formula

$$(10)\quad F(a,b;\ c;\ z) = \Gamma(c)\ [\Gamma(b)\ \Gamma(c-b)]^{-1} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\ dt.$$

which I will write out as

$$B(b,c-b)\ F(a,b;c;z) = \int_0^1 dt\ \ t^{b-1}\ (1-t)^{c-b-1}\ (1-tz)^{-a}$$

Now do the pain:

$$t = x^{-1} \qquad dt = -x^{-2}dx\ = -t^2 dx \qquad \int_0^1 dt = \int_\infty^1 (-x^{-2}dx)\ = \int_1^\infty dx\ x^{-2}$$

$$t^{b-1} = x^{1-b}$$

$(1-t)^{c-b-1} = t^{c-b-1}(t^{-1}-1)^{c-b-1} = x^{1+b-c}\,(x-1)^{c-b-1}$

$(1-tz)^{-a} = t^{-a}\,(t^{-1}-z)^{-a} = x^{a}\,(x-z)^{-a}$

So we then have

$$B(b,c-b)\,F(a,b;c;z) = \int_1^\infty dx\; x^{-2}\; x^{1-b}\, x^{1+b-c}\,(x-1)^{c-b-1}\, x^{a}\,(x-z)^{-a}$$

$$= \int_1^\infty dx\; [x^{-2}\; x^{1-b}\, x^{1+b-c}\, x^{a}\,]\,(x-1)^{c-b-1}\,(x-z)^{-a}$$

$$= \int_1^\infty dx\; [x^{-c}\, x^{a}\,]\,(x-1)^{c-b-1}\,(x-z)^{-a}$$

$$= \int_1^\infty dx\; x^{a-c}\,(x-1)^{c-b-1}\,(x-z)^{-a}$$

which then says

$$B(b,c-b)\,F(a,b;c;z^{-1}) = \int_1^\infty (x-1)^{c-b-1}\, x^{a-c}\,(x-z^{-1})^{-a}\,dx$$

and this agrees with the page 115 result quoted above. Thus the thing is verified.

2. Convert lower endpoint from 1 to u

We start with our verified result

$$B(b,c-b)\,F(a,b;c;\,z) = \int_1^\infty (x-1)^{c-b-1}\, x^{a-c}\,(x-z)^{-a}\,dx \qquad (2.1)$$

Now do the pain:

$$y = ux \qquad x = y/u \quad dx = dy/u \qquad \int_1^\infty dx \;=\; \int_u^\infty dy\; u^{-1}$$

$(x-1)^{c-b-1} = u^{1+b-c}\;(ux-u)^{c-b-1} = u^{1+b-c}\;(y-u)^{c-b-1}$

$x^{a-c} = u^{c-a}\,(ux)^{a-c} = u^{c-a}\,y^{a-c}$

$(x-z)^{-a} = u^{a}\,(ux-uz)^{-a} = u^{a}(y-uz)^{-a}$

Then we have

$$B(b,c\text{-}b)\ F(a,b;c;z) = \int_1^\infty dx\ (x-1)^{c-b-1}\ x^{a-c}\ (x-z)^{-a}$$

$$= \int_u^\infty dy\ u^{-1}\ u^{1+b-c}\ (y-u)^{c-b-1}\ u^{c-a}\ y^{a-c}\ u^{a}(y-uz)^{-a}$$

$$= \int_u^\infty dy\ [u^{-1}\ u^{1+b-c}\ u^{c-a}\ u^{a}\ ]\ (y-u)^{c-b-1}\ y^{a-c}\ (y-uz)^{-a}$$

$$= \int_u^\infty dy\ [u^{b}\ ]\ (y-u)^{c-b-1}\ y^{a-c}\ (y-uz)^{-a}$$

So we have shown that

$$B(b,c\text{-}b)\ F(a,b;c;z) = u^{b}\ \int_u^\infty dy\ y^{a-c}\ (y-u)^{c-b-1}\ (y-uz)^{-a}$$

We now set z' = -uz so z = -z'/u and we have

$$B(b,c\text{-}b)\ F(a,b;c;\ -z'/u) = u^{b}\ \int_u^\infty dy\ y^{a-c}\ (y-u)^{c-b-1}\ (y+z')^{-a}$$

and we remove the primes to get

$$B(b,c\text{-}b)\ F(a,b;c;\ -z/u) = u^{b}\ \int_u^\infty dy\ y^{a-c}\ (y-u)^{c-b-1}\ (y+z)^{-a}$$

which gives us this integral evaluation

$$\int_u^\infty dy\ y^{a-c}\ (y-u)^{c-b-1}\ (y+z)^{-a} = u^{-b}\ B(b,c\text{-}b)\ F(a,b;c;\ -z/u) \qquad (2.2)$$

3. Try to make the F function look like that in GR7 page 317

Ignore this section, skip to the next section! Results in this section might be wrong! This just gives irrelevant forms of the integral.

That integral is this:

$$2.^{11} \quad \int_u^\infty x^{-\lambda}(x+\beta)^{\nu}(x-u)^{\mu-1}\,dx = u^{-\lambda}(\beta+u)^{\mu+\nu}\ \mathrm{B}(\lambda-\mu-\nu,\mu)\ {}_2F_1\left(\lambda,\mu;\lambda-\mu;-\frac{\beta}{u}\right)$$
$$\left[\left|\arg\frac{u}{\beta}\right| < \pi \text{ or } \left|\frac{\beta}{u}\right| < 1, \quad 0 < \operatorname{Re}\mu < \operatorname{Re}(\lambda-\nu)\right] \quad \text{ET II 201(8)}$$

I will start with (2.2) and do various edits. First, replace y with x and z with β

$$\int_{u}^{\infty} dy\, y^{a-c} (y-u)^{c-b-1} (y+z)^{-a} = u^{-b} B(b,c-b)\, F(a,b;c; -z/u)$$

$$\int_{u}^{\infty} dx\, x^{a-c} (x-u)^{c-b-1} (x+\beta)^{-a} = u^{-b} B(b,c-b)\, F(a,b;c; -\beta/u)$$

(3a) Now I will try these replacements in my formula (trying to match the GR F function)

$a = \lambda \qquad b = \mu \qquad c = \lambda-\mu$

Then we find

$a-c = \lambda - \lambda+\mu = \mu$

$c-b-1 = \lambda-\mu - \mu -1 = \lambda - 1 - 2\mu$

$-a = -\lambda$

This gives

$$\int_{u}^{\infty} dx\, x^{a-c} (x-u)^{c-b-1} (x+\beta)^{-a} = u^{-b} B(b,c-b)\, F(a,b;c; -\beta/u)$$

$$\int_{u}^{\infty} dx\, x^{\mu} (x-u)^{\lambda-1-2\mu} (x+\beta)^{-\lambda} = u^{-\mu} B(\mu,\lambda-2\mu)\, F(\lambda,\mu;\lambda-\mu; -\beta/u) \qquad (3.2)$$

(3b) Alternatively, I could have done this:

$a = \mu \qquad b = \lambda \qquad c = \lambda-\mu$

Then we find

$a-c = \mu - \lambda+\mu = -\lambda$

$c-b-1 = \lambda-\mu - \lambda -1 = -\mu-1$

$-a = -\mu$

This gives

$$\int_{u}^{\infty} dx\, x^{a-c} (x-u)^{c-b-1} (x+\beta)^{-a} = u^{-b} B(b,c-b)\, F(a,b;c; -\beta/u)$$

$$\int_{u}^{\infty} dx\, x^{-\lambda} (x-u)^{-\mu-1} (x+\beta)^{-\mu} = u^{-\lambda} B(\lambda,-\mu)\, F(\mu,\lambda; \lambda-\mu; -\beta/u) \qquad (3.3)$$

(3c) Summary of results and comparison with GR7:

$$\int_u^\infty dx\, x^{\mu} (x-u)^{\lambda-1-2\mu} (x+\beta)^{-\lambda} = u^{-\mu} B(\mu,\lambda-2\mu) F(\lambda,\mu;\lambda-\mu; -\beta/u) \qquad (3.2)$$

$$\int_u^\infty dx\, x^{-\lambda} (x-u)^{-\mu-1} (x+\beta)^{-\mu} = u^{-\lambda} B(\lambda,-\mu) F(\lambda,\mu; \lambda-\mu; -\beta/u) \qquad (3.3)$$

$$\int_u^\infty dx\, x^{-\lambda} (x-u)^{\mu-1} (x+\beta)^{\nu} =?= u^{-\lambda} (\beta+u)^{u+\nu} B(\lambda-\mu-\nu,\mu) F(\lambda,\mu; \lambda-\mu; -\beta/u) \qquad (GR7)$$

I think the GR7 result is wrong! Let's continue:

(3d) What does the Kummer 1 formula do (Bateman p 105)

$$(1) \quad u_1 = F(a, b; c; z)$$

$$(2) \quad = (1-z)^{c-a-b} F(c-a, c-b; c; z)$$

Notice that it leaves c and z unchanged. If we apply this to $F(\lambda,\mu; \lambda-\mu; -\beta/u)$ we have

$c-a = \lambda-\mu-\lambda = -\mu$
$c-b = \lambda-\mu-\mu = \lambda-2\mu$
$c-a-b = \lambda-\mu-\lambda-\mu = -2\mu$

$$F(\lambda,\mu; \lambda-\mu; -\beta/u) = (1+\beta/u)^{-2\mu} F(-\mu,\lambda-2\mu; \lambda-\mu; -\beta/u)$$

But this is some very different F function from what appears in the Bateman formula. My conclusion is that the GR7 result is just plain wrong!

4. Derive the PhL version of the GR7 result:

Well, we already have this result from (2.2) above

$$\int_u^\infty dy\, y^{a-c} (y-u)^{c-b-1} (y+z)^{-a} = u^{-b} B(b,c-b) F(a,b;c; -z/u) \qquad (2.2)$$

which I rewrite as

$$\int_u^\infty dx\, x^{a-c} (x-u)^{c-b-1} (x+\beta)^{-a} = u^{-b} B(b,c-b) F(a,b;c; -\beta/u) \qquad (4.1)$$

The LHS of the GR7 integral is this

$$\int_u^\infty dx\, x^{-\lambda} (x-u)^{\mu-1} (x+\beta)^{\nu} =$$

So I need to solve these equations (left column) for a,b,c: [notice that c = λ-ν, not λ-μ !]

-a = ν	=> a = -ν		a = -ν
a-c = -λ	=> -ν-c = -λ =>	c = λ-ν	c = λ-ν
c-b = μ	=> b = c-μ = λ-μ-ν		b = λ-μ-ν
			c-a-b = μ+ν

Then my result 4.1 becomes

$$\int_u^\infty dx\, x^{-\lambda} (x-u)^{\mu-1} (x+\beta)^{\nu} = u^{-(\lambda-\mu-\nu)}\, B(\lambda-\mu-\nu,\mu)\, F(-\nu, \lambda-\mu-\nu; \lambda-\nu; -\beta/u) \qquad (4.2)$$

This *agrees* with page 286 of GR of my hardcopy GR4!

Now we can get an alternate version using our Kummer 1

$$(1) \quad u_1 = F(a, b; c; z)$$

$$(2) \quad = (1-z)^{c-a-b} F(c-a, c-b; c; z)$$

which then says

$$F(-\nu, \lambda-\mu-\nu; \lambda-\nu; -\beta/u) = F(a,b;c; -\beta/u) = (1+\beta/u)^{\mu+\nu} F(\lambda,\mu; \lambda-\nu; -\beta/u)$$

$$= u^{-\nu-\mu} (u+\beta)^{\mu+\nu} F(\lambda,\mu; \lambda-\nu; -\beta/u)$$

Then my alternate form is

$$\int_u^\infty dx\, x^{-\lambda} (x-u)^{\mu-1} (x+\beta)^{\nu} = u^{-(\lambda-\mu-\nu)}\, B(\lambda-\mu-\nu,\mu)\, F(-\nu, \lambda-\mu-\nu; \lambda-\nu; -\beta/u)$$

$$= u^{-(\lambda-\mu-\nu)}\, B(\lambda-\mu-\nu,\mu)\, u^{-\nu-\mu} (u+\beta)^{\mu+\nu} F(\lambda,\mu; \lambda-\nu; -\beta/u)$$

$$= u^{-\lambda+\mu+\nu}\, u^{-\nu-\mu}\, (\beta+u)^{\mu+\nu}\, B(\lambda-\mu-\nu,\mu)\, F(\lambda,\mu; \lambda-\nu; -\beta/u)$$

$$= u^{-\lambda} (\beta+u)^{\mu+\nu}\, B(\lambda-\mu-\nu,\mu)\, F(\lambda,\mu; \lambda-\nu; -\beta/u)$$

This then agrees with Bateman GR7 except for one symbol I show in red! GR7 has it wrong!

Does this appear in the errata? His latest errata is April 2008 and no, it does not appear there. I should report this if I can get more confirmation. Well I just got it from GR4. I will soon send in a comment.

5. Comparing the GR7 and the Bateman ET II results

Here is the GR7 result

$$2.^{11}\quad \int_u^\infty x^{-\lambda}(x+\beta)^\nu (x-u)^{\mu-1}\,dx = u^{-\lambda}(\beta+u)^{\mu+\nu}\,\mathrm{B}(\lambda-\mu-\nu,\mu)\;{}_2F_1\left(\lambda,\mu;\lambda-\mu;-\frac{\beta}{u}\right)$$
$$\left[\left|\arg\frac{u}{\beta}\right|<\pi \text{ or } \left|\frac{\beta}{u}\right|<1,\quad 0<\operatorname{Re}\mu<\operatorname{Re}(\lambda-\nu)\right] \quad \text{ET II 201(8)}$$

and here I have finally found the Bateman result on page 146 of my Russian ET II. The Russian page has the wrong heading on it, that is why I was having trouble finding the result. Here is what ET II really says

	$f(x)$	$[\Gamma(\mu)]^{-1}\int_y^\infty f(x)\,(x-y)^{\mu-1}\,dx = h(y;\ \mu)$
(8)	$x^{-\lambda}(x+\alpha)^\nu$, $0<\operatorname{Re}\mu<\operatorname{Re}(\lambda-\nu)$	$y^{\mu+\nu-\lambda}\frac{\Gamma(\lambda-\mu-\nu)}{\Gamma(\lambda-\nu)}\times$ $\times{}_2F_1(-\nu,\ \lambda-\mu-\nu;\ \lambda-\nu;\ -\alpha/y)$, $\lvert\arg(\alpha/y)\rvert<\pi$ или $\lvert\alpha/y\rvert<1$

Let's copy the ET II result as follows:

$\Gamma(\mu)^{-1}\ \int_y^\infty dx\ x^{-\lambda}\ (x+\alpha)^\nu\ (x-y)^{\mu-1} = y^{\mu+\nu-\lambda}\,\Gamma(\lambda-\mu-\nu)/\Gamma(\lambda-\nu) * F(-\nu,\lambda-\mu-\nu;\ \lambda-\nu;\ -\alpha/y)$

$\int_y^\infty dx\ x^{-\lambda}\ (x+\alpha)^\nu\ (x-y)^{\mu-1} = y^{\mu+\nu-\lambda}\,\Gamma(\lambda-\mu-\nu)\,\Gamma(\mu)/\Gamma(\lambda-\nu) * F(-\nu,\lambda-\mu-\nu;\ \lambda-\nu;\ -\alpha/y)$

$\int_y^\infty dx\ x^{-\lambda}\ (x+\alpha)^\nu\ (x-y)^{\mu-1} = y^{\mu+\nu-\lambda}\,\mathrm{B}(\lambda-\mu-\nu,\mu) * F(-\nu,\lambda-\mu-\nu;\ \lambda-\nu;\ -\alpha/y)$

We then have to do the Kummer thing

$$(1)\quad u_1 = F(a,\ b;\ c;\ z)$$
$$(2)\qquad = (1-z)^{c-a-b}\,F(c-a,\ c-b;\ c;\ z)$$

$F(-\nu,\ \lambda-\mu-\nu;\ \lambda-\nu;\ -\alpha/y) = F(a,b;c;\ -\alpha/y) = (1+\alpha/y)^{\mu+\nu}\,F(\lambda,\mu;\ \lambda-\nu;\ -\alpha/y)$

$= y^{-\nu-\mu}\,(y+\alpha)^{\mu+\nu}\,F(\lambda,\mu;\ \lambda-\nu;\ -\alpha/y)$

so the Bateman result becomes

$$\int_y^\infty dx\, x^{-\lambda}\ (x+\alpha)^\nu\ (x-y)^{\mu-1} = y^{\mu+\nu-\lambda}\, B(\lambda-\mu-\nu,\mu) * F(-\nu,\lambda-\mu-\nu;\ \lambda-\nu;\ -\alpha/y)$$

$$= y^{\mu+\nu-\lambda}\, B(\lambda-\mu-\nu,\mu)\ y^{-\nu-\mu}\ (y+\alpha)^{\mu+\nu}\ F(\lambda,\mu;\ \lambda-\nu;\ -\alpha/y)$$

$$= y^{-\lambda}\ (y+\alpha)^{\mu+\nu}\ B(\lambda-\mu-\nu,\mu)\ F(\lambda,\mu;\ \lambda-\nu;\ -\alpha/y)$$

So this is what Bateman has to say. We change now y→u and α → β to get

$$= u^{-\lambda}\ (u+\beta)^{\mu+\nu}\ B(\lambda-\mu-\nu,\mu)\ F(\lambda,\mu;\ \lambda-\nu;\ -\beta/u)$$

So we have now transcribed Bateman to read

$$\int_u^\infty dx\, x^{-\lambda}\ (x+\beta)^\nu\ (x-u)^{\mu-1} = u^{-\lambda}\ (\beta+u)^{\mu+\nu}\ B(\lambda-\mu-\nu,\mu)\ F(\lambda,\mu;\ \lambda-\nu;\ -\beta/u)$$

which confirms the typo

$$\int_u^\infty x^{-\lambda}(x+\beta)^\nu(x-u)^{\mu-1}\,dx = u^{-\lambda}(\beta+u)^{\mu+\nu}\ \mathrm{B}(\lambda-\mu-\nu,\mu)\ {}_2F_1\left(\lambda,\mu;\lambda-\mu;-\frac{\beta}{u}\right)$$

Now while we are here, let's look at the Bateman conditions:

$$|\arg(\alpha/y)| < \pi \qquad |(\alpha/y)| < 1$$

=>

$$|\arg(\beta/u)| < \pi \qquad |(\beta/u)| < 1$$

Amazingly, the GR7 result has the left one upside down.

$$\left[\left|\arg\frac{u}{\beta}\right| < \pi \text{ or } \left|\frac{\beta}{u}\right| < 1, \quad 0 < \mathrm{Re}\,\mu < \mathrm{Re}(\lambda-\nu)\right] \quad \text{ET II 201(8)}$$

6. Email to Dan Zwillinger

See separated doc "dan z email 2.2.11.doc" .