Int 1 PhL 3.15.14

The integrals are these (all derived below)

1. $\int_0^{2\pi} d\theta \ \frac{1}{1 + b\cos\theta + c\sin\theta} = 2\pi/\sqrt{1-(b^2+c^2)}$ assuming $b^2+c^2 < 1$

2. $\int_0^{2\pi} d\theta \ \frac{1}{A + B\cos\theta + C\sin\theta} = 2\pi / \sqrt{A^2-B^2-C^2}$ assuming $B^2+C^2 < A^2$

3. $\int_0^{2\pi} d\theta \ \ln(1 + b\cos\theta + c\sin\theta) = 2\pi \ln [(1 + \sqrt{1-b^2-c^2})/2]$ assuming $b^2+c^2 < 1$

4. $\int_0^{2\pi} d\theta \ \ln(A + B\cos\theta + C\sin\theta) = 2\pi \ln [(A + \sqrt{A^2-B^2-C^2})/2]$ assuming $B^2+C^2 < A^2$

and you cannot find this thing in this form in GR7. Maple can do some of these, but messes up on others, so don't ever trust Maple on this kind of integral!

1. Condition on b and c for reasonableness. 1
2. Contour Approach 2
3. Maple does the integral 6
4. Another way 6
5. Obvious related integral 7
6. Another related integral 7

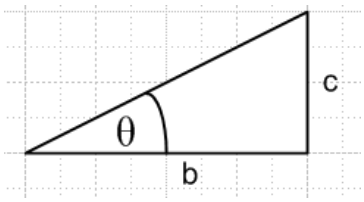
1. Condition on b and c for reasonableness.

In order to avoid an infinite integrand, we have to make sure the denominator does not vanish. If b and c are much smaller than 1 in magnitude, there is never a problem. But if b = 1 and c = 0 or vice versa, there is a problem. We suspect there is some condition which makes sure

$f(\theta) = 1 + b\cos\theta + c\sin\theta \geq 0$

We want to know where the min of $f(\theta)$ is located. To find that, we first set $f'(\theta) = 0$:

$-b\sin\theta + c\cos\theta = 0$ => $b\sin\theta = c\cos\theta$ => $\tan\theta = c/b$



=> one of two possibilities:

$$\sin\theta = \sigma c/\sqrt{b^2+c^2}$$
$$\cos\theta = \sigma b/\sqrt{b^2+c^2}$$

where σ is either +1 or -1. The curvature is

$$f''(\theta) = -b\cos\theta - c\sin\theta = -b[\sigma b/\sqrt{b^2+c^2}\,] - c[\sigma c/\sqrt{b^2+c^2}\,] = -2\sigma(b^2+c^2)/\sqrt{b^2+c^2} = -2\sigma\sqrt{b^2+c^2}$$

We want this to be positive to locate our min, so we need $\sigma = -1$. Thus our solution θ for the min is:

$$\sin\theta = -c/\sqrt{b^2+c^2}$$
$$\cos\theta = -b/\sqrt{b^2+c^2}$$

At this min location we want to have

$$1 + b\cos\theta + c\sin\theta \geq 0$$

or

$$1 + b[-b/\sqrt{b^2+c^2}\,] + c[-c/\sqrt{b^2+c^2}\,] \geq 0$$

or

$$1 - b^2/\sqrt{b^2+c^2}\,] - c^2/\sqrt{b^2+c^2}\,] \geq 0$$

or

$$1 - \sqrt{b^2+c^2} \geq 0$$

or

$$1 > \sqrt{b^2+c^2}$$

or

$$1 \geq b^2+c^2$$

Conclusion: $f(\theta)$ will be positive definite as long as $b^2+c^2 < 1$.

2. Contour Approach

$$f = 1 + b\cos\theta + c\sin\theta = 1 + (b/2)(z+z^{-1}) + (c/2i)(z-z^{-1})$$

$$fz = z + (b/2)(z^2+1) + (c/2i)(z^2-1)$$

$$= z^2[\, b/2 + c/2i\,] + z + [\, 1 + b/2 - c/2i\,]$$

$$= [\, b/2 + c/2i\,](z-r_1)(z-r_2) = K(z-r_1)(z-r_2) = Kg$$

Write

$$I = \int_0^{2\pi} d\theta \ (1/f) \ = \oint dz/[iz] \ (1/f) = (1/i) \oint dz \ 1/(fz)$$

Then integral is

$$I = (1/iK) \oint dz \ \frac{1}{(z-r_1)(z-r_2)}$$

At this point see where poles are located and then result will be

$$I = 2\pi/K * \Sigma_{residues}$$

where there might be 0,1 or 2 residues in this sum!

The first step is to identify K and the roots r_1 and r_2. It's a little long-winded but :

```
f := (1 + b*cos(theta) + c*sin(theta));
```

$$f := 1 + b\cos(\theta) + c\sin(\theta)$$

```
f1 := convert(f,exp);
```

$$f1 := 1 + b\left(\frac{1}{2}e^{(I\theta)} + \frac{1}{2}\frac{1}{e^{(I\theta)}}\right) - \frac{1}{2}Ic\left(e^{(I\theta)} - \frac{1}{e^{(I\theta)}}\right)$$

```
f2 := subs(exp(I*theta)=z,f1);
```

$$f2 := 1 + b\left(\frac{1}{2}z + \frac{1}{2}\frac{1}{z}\right) - \frac{1}{2}Ic\left(z - \frac{1}{z}\right)$$

```
fz := collect(f2*z,z);
```

$$fz := \left(\frac{1}{2}b - \frac{1}{2}Ic\right)z^2 + z + \frac{1}{2}b + \frac{1}{2}Ic$$

```
K := coeff(fz,z^2);
```

$$K := \frac{1}{2}b - \frac{1}{2}Ic$$

```
g := fz/K: fz1 := collect(%,z);
```

$$fz1 := z^2 + \frac{z}{\frac{1}{2}b - \frac{1}{2}Ic} + \frac{\frac{1}{2}b + \frac{1}{2}Ic}{\frac{1}{2}b - \frac{1}{2}Ic}$$

```
s := solve(g,z);
```

$$s := \frac{1}{2}\frac{-2I + 2\sqrt{-1 + b^2 + c^2}}{Ib + c}, \frac{1}{2}\frac{-2I - 2\sqrt{-1 + b^2 + c^2}}{Ib + c}$$

```
r1 := s[1];
```

$$r1 := \frac{1}{2}\frac{-2I + 2\sqrt{-1 + b^2 + c^2}}{Ib + c}$$

```
r2 := s[2];
```

$$r2 := \frac{1}{2}\frac{-2I - 2\sqrt{-1 + b^2 + c^2}}{Ib + c}$$

Now we want to see where these poles are located, so we seek the mags of the roots:

```
assume(b^2+c^2 < 1, b,real,c,real);
abs(evalc(r1))^2: simplify(%);
```

$$\frac{(-1 + \sqrt{1 - b\sim^2 - c\sim^2})^2}{c\sim^2 + b\sim^2}$$

```
abs(evalc(r2))^2: simplify(%);
```

$$\frac{(1 + \sqrt{1 - b\sim^2 - c\sim^2})^2}{c\sim^2 + b\sim^2}$$

For r_1 we think this is true

$(-1+\sqrt{1-(b^2+c^2)}\,^2 < (b^2+c^2)$?

or

$-1 + 1 - (b^2+c^2) - 2\sqrt{1-(b^2+c^2)} < (b^2+c^2)$?

or

$- 2\sqrt{1-(b^2+c^2)} < 2\,(b^2+c^2)$?

Given our assumption noted above, this inequality is trivially true, so we find

$|r_1| < 1$ and this root lies inside the contour

For r_2 we think this is true

$(1+\sqrt{1-(b^2+c^2)}\,^2 > (b^2+c^2)$?

or

$+1 + 1 - (b^2+c^2) + 2\sqrt{1-(b^2+c^2)} > (b^2+c^2)$?

or

$2\sqrt{1-(b^2+c^2)} > 2\,(b^2+c^2) - 2$?

or

$2\sqrt{1-(b^2+c^2)} > -2\,[1 - (b^2+c^2)]$?

or

$2 > -2\sqrt{1-(b^2+c^2)}$

and this is obviously true, so we find

$|r_2| > 1$ and this root lies outside the contour

The solution will then be

$I = 2\pi/K * \Sigma_{\mathtt{residue}}$ (at $z = r_1$)

Here then is this last step:

```
readlib(residue):
result := (2*Pi/K)*residue(1/fz1,z=r1):
evalc(result): simplify(%);
```

$$-2\,\frac{\sqrt{1-\tilde{b}^2-\tilde{c}^2}\,\pi}{-1+\tilde{b}^2+\tilde{c}^2}$$

So the final integral seems to be

$I = 2\pi/\sqrt{1-(b^2+c^2)}$

3. Maple does the integral

```
restart;
assume(b^2+c^2 < 1, b,real,c,real);
f := (1 + b*cos(theta) + c*sin(theta));
```

$$f := 1 + b\sim \cos(\theta) + c\sim \sin(\theta)$$

```
Int(1/f,theta=0..2*Pi);
```

$$\int_0^{2\pi} \frac{1}{1 + b\sim \cos(\theta) + c\sim \sin(\theta)}\, d\theta$$

```
value(%);
```

$$-2\,\frac{\pi\ \mathrm{signum}(-1 + b\sim)}{\sqrt{1 - b\sim^2 - c\sim^2}}$$

But we *know* that

$-1 < b < 1$ in order to guarantee that $1 > b^2+c^2$

so then

$-2 < -1+b < 0$

which $-1+b < 0$ so sign(-1+b) = -1, and then Maple agrees.

4. Another way

Once we know that $1 > b^2+c^2$ we can write

$b = \alpha\cos\varphi$ $\alpha < 1.$ $\alpha^2 = b^2+c^2 < 1$
$c = \alpha\sin\varphi$

Then

$f(\theta) = 1 + b\cos\theta + c\sin\theta = 1 + \alpha\cos\varphi\cos\theta + \alpha\sin\varphi\sin\theta$

$= 1 + \alpha\,[\cos\varphi\cos\theta + \sin\varphi\sin\theta]$

$= 1 + \alpha\cos(\theta-\varphi)$

Then our integral is

$$I = \int_0^{2\pi} d\theta\ \frac{1}{1 + \alpha\cos(\theta-\varphi)} = \int_0^{2\pi} d\theta\ \frac{1}{1 + \alpha\cos\theta}$$

We could to this using our contour method, but this one we can just look up in GR7,

$$1.^6 \qquad \int_0^{\pi} \frac{\cos nx\,dx}{1+a\cos x} = \frac{\pi}{\sqrt{1-a^2}}\left(\frac{\sqrt{1-a^2}-1}{a}\right)^n \qquad [a^2<1, \quad n\geq 0] \qquad \text{BI (64)(12)}$$

and we set n = 0 to get

$$I = 2\pi/\sqrt{1-\alpha^2} = 2\pi/\sqrt{1-(b^2+c^2)}$$

which agrees with the previous two solutions. Of course

```
assume(alpha,real,-1 < alpha, alpha < 1);
Int(1/(1+alpha*cos(theta)),theta = 0..2*Pi);
```

$$\int_0^{2\pi} \frac{1}{1+\alpha\sim\cos(\theta)}\,d\theta$$

```
value(%);
```

$$2\frac{\pi}{\sqrt{1-\alpha\sim^2}}$$

.

5. Obvious related integral

$$I = \int_0^{2\pi} d\theta\ \frac{1}{A+B\cos\theta+C\sin\theta} = 2\pi/\sqrt{A^2-B^2-C^2} \quad \text{assuming } B^2+C^2<A^2$$

Confirm with Maple:

```
restart;
assume(A,real,B,real,C,real,A^2 > B^2+C^2);
f := (A + B*cos(theta) + C*sin(theta));
Int(1/f,theta=0..2*Pi);
```

$$\int_0^{2\pi} \frac{1}{A\sim+B\sim\cos(\theta)+C\sim\sin(\theta)}\,d\theta$$

```
value(%);
```

$$2\frac{\pi\ \mathrm{signum}(A\sim-B\sim)}{\sqrt{A\sim^2-B\sim^2-C\sim^2}}$$

6. Another related integral

Consider

$$J = \int_0^{2\pi} d\theta \ \ln(A + B\cos\theta + C\sin\theta)$$

Notice that with our usual condition and also perhaps A > 0 the log argument is always positive, so the log is always some positive number, and the integral has to be real.

Then

$$dJ/dA = \int_0^{2\pi} d\theta \ \frac{1}{A + B\cos\theta + C\sin\theta} = 2\pi / \sqrt{A^2-B^2-C^2}$$

$$J = 2\pi \int dA/\sqrt{A^2-B^2-C^2} + k$$

But this is an easy integral

```
restart;
Int(1/sqrt(x^2-d^2),x);
```

$$\int \frac{1}{\sqrt{x^2 - d^2}}\,dx$$

```
value(%);
```

$$\ln\left(x + \sqrt{x^2 - d^2}\right)$$

$d^2 = B^2+C^2$

Then we have

$$J = 2\pi \ln(A + \sqrt{A^2-B^2-C^2}) + k$$

To determine the constant k, we set B = C = 0 and find

$$J = \int_0^{2\pi} d\theta \ \ln(A + 0\cos\theta + 0\sin\theta) = 2\pi \ln(A)$$

But our formula says

$$J = 2\pi \ln(2A) + k = 2\pi\ln(A) + 2\pi\ln(2) + k \quad \Rightarrow \quad k = -2\pi\ln(2)$$

So our integral must then be

$$J = 2\pi \ln(A + \sqrt{A^2-B^2-C^2}) - 2\pi\ln(2)$$

$$= 2\pi \ln [(1/2) (A + \sqrt{A^2-B^2-C^2})]$$

Somehow Maple gets a result which does not depend on C! Maple says

J = 2π [ln(A+B)-Iπ] [wrong!]

but this is garbage because we know the result is real. I will show that my result is correct by doing a few numerical integrations:

```
restart;
f := ln(A + B*cos(theta) + C*sin(theta));
```

$$f := \ln(A + B\cos(\theta) + C\sin(\theta))$$

```
A := 2.1;  B := 0.3; C := 0.5;
```

$$A := 2.1$$
$$B := .3$$
$$C := .5$$

```
evalf(Int(f,theta=0..2*Pi));
```

4.536966021

```
2*Pi*ln((1/2)*(A + sqrt(A^2-B^2-C^2))):evalf(%);
```

4.536966022

```
C := 0.6;
```

$$C := .6$$

```
evalf(Int(f,theta=0..2*Pi));
```

4.494938281

```
2*Pi*ln((1/2)*(A + sqrt(A^2-B^2-C^2))):evalf(%);
```

4.494938280

Clearly the result does depend on C. So here is what I have shown:

$$\int_0^{2\pi} d\theta \ \ln(A + B\cos\theta + C\sin\theta) = 2\pi \ln[(A + \sqrt{A^2-B^2-C^2})/2] \quad \text{assuming} \quad B^2+C^2 < A^2$$

This could be confirmed by looking at this simpler case

$$\int_0^{2\pi} d\theta \ \ln(1 + b\cos\theta + c\sin\theta) = 2\pi \ln[(1 + \sqrt{1-b^2-c^2})/2] \quad \text{assuming} \quad b^2+c^2 < 1$$

and then we do our same reparameterization

$$f(\theta) = 1 + b\cos\theta + c\sin\theta = 1 + \alpha\cos\varphi\cos\theta + \alpha\sin\varphi\sin\theta$$

$$= 1 + \alpha[\cos\varphi\cos\theta + \sin\varphi\sin\theta]$$

$$= 1 + \alpha\cos(\theta-\varphi) \qquad \alpha^2 < 1$$

Then we get

$$J = \int_0^{2\pi} d\theta \ \ln(1 + \alpha \cos\theta \ )$$

Maple can't even do this one, but GR7 can:

$$9.^8 \qquad \int_0^{\pi} \ln(a + b\cos x)\, dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \qquad [a \geq |b| > 0] \qquad \text{GW (322)(15)}$$

This says

$$J = 2\pi \ln [\ (1 + \sqrt{1-\alpha^2}\ )/2]$$

which then agrees with my result.