

Euler Substitutions

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Euler substitutions are used to evaluate integrals of the form

$$\int R(x, \sqrt{ax^2 + bx + c}) dx,$$

by “removing” the radical. There are three specific substitutions suggested by Euler. In each one of them the idea is to eliminate the term with x^2 . We assume that $a \neq 0$ and that the polynomial $ax^2 + bx + c$ is not strictly negative. Note also that if $b = 0$ the radical can be removed by a trigonometric or hyperbolic substitution.

1. First Euler substitution.

When $a > 0$ we introduce a new variable t by setting

$$\sqrt{ax^2 + bx + c} = x\sqrt{a} + t. \quad (1)$$

Then

$$ax^2 + bx + c = ax^2 + 2xt\sqrt{a} + t^2,$$

so that the term ax^2 cancels out and solving for x we find

$$x = \frac{t^2 - c}{b - 2t\sqrt{a}}.$$

Thus the integral takes the form

$$\int Q(t) dt, \quad (2)$$

where $Q(t)$ is a rational function.

2. Second Euler substitution.

Suppose the polynomial $ax^2 + bx + c$ has two different real roots, i.e.

$$ax^2 + bx + c = a(x - \alpha)(x - \beta), \alpha \neq \beta.$$

Then we set

$$\sqrt{ax^2 + bx + c} = (x - \alpha)t, \quad (3)$$

which yields

$$a(x - \alpha)(x - \beta) = (x - \alpha)^2 t^2,$$

$$a(x - \beta) = (x - \alpha)t^2,$$

$$x = \frac{\alpha t^2 - a\beta}{t^2 - a},$$

and again the integral is reduced to the form (2).

3. Third Euler substitution

The third Euler substitution can be used when $c \geq 0$. In such case we set

$$\sqrt{ax^2 + bx + c} = xt + \sqrt{c}, \quad (4)$$

and then

$$ax^2 + bx + c = x^2 t^2 + 2xt\sqrt{c} + c,$$

$$ax + b = xt^2 + 2t\sqrt{c},$$

$$x = \frac{b - 2t\sqrt{c}}{t^2 - a}, \text{ etc,}$$

leading to the form (2).

4. Remarks.

The first two Euler substitutions are sufficient to cover all possible cases, because if $a < 0$, then the roots of the polynomial $ax^2 + bx + c$ are real and different (the graph of this

polynomial is a parabola opening down and with vertex above the x-axis). Nevertheless, the third Euler substitution is useful because it often requires less computations.

It is good to keep in mind that the radical $\sqrt{ax^2 + bx + c}$ can be simplified by completing the polynomial $ax^2 + bx + c$ to a perfect square and then using a trigonometric or hyperbolic substitution.

5.Examples.

1. We shall evaluate

$$\int \frac{dx}{\sqrt{x^2 + x + 1}}, \quad (5)$$

by the first Euler substitution. Set

$$\sqrt{x^2 + x + 1} = x + t.$$

Then

$$x = \frac{t^2 - 1}{1 - 2t},$$

$$dx = \frac{2(t - t^2 - 1)}{(1 - 2t)^2} dt,$$

$$\sqrt{x^2 + x + 1} = \frac{t - t^2 - 1}{1 - 2t},$$

and

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 + x + 1}} &= 2 \int \frac{dt}{1 - 2t} = -\ln|1 - 2t| + C \\ &= -\ln|1 + 2x - 2\sqrt{x^2 + x + 1}| + C. \end{aligned} \quad (6)$$

2. The same integral can be evaluated by the third Euler substitution. For this we set

$$\sqrt{x^2 + x + 1} = xt + 1,$$

and therefore, solving for x ,

$$\begin{aligned}
 x &= \frac{2t-1}{1-t^2}, \\
 dx &= \frac{2(t^2-t+1)}{(1-t^2)^2} dt, \\
 \int \frac{dx}{\sqrt{x^2+x+1}} &= 2 \int \frac{dt}{1-t^2} = \ln \left| \frac{1+t}{1-t} \right| + C \\
 &= \ln \left| \frac{x-1+\sqrt{x^2+x+1}}{x+1-\sqrt{x^2+x+1}} \right| + C,
 \end{aligned} \tag{7}$$

which is (6) in a different form.

3. Now consider

$$F(x) = \int \frac{dx}{(x-2)\sqrt{3x-x^2-2}}, \tag{8}$$

for $1 < x < 2$. We shall use the second Euler substitution. Since $3x-x^2-2 = (2-x)(x-1)$ we set

$$\sqrt{3x-x^2-2} = t(x-1),$$

which yields

$$\begin{aligned}
 x &= \frac{t^2+2}{t^2+1}, \quad dx = \frac{-2t}{(t^2+1)^2} dt, \quad t = \sqrt{\frac{2-x}{x-1}}, \\
 \int \frac{dx}{(x-2)\sqrt{3x-x^2-2}} &= 2 \int \frac{dt}{t^2(t^2+1)} = 2 \int \left(\frac{1}{t^2} - \frac{1}{t^2+1} \right) dt \\
 &= \frac{-2}{t} - 2 \arctan t + C.
 \end{aligned}$$

Finally,

$$F(x) = -2\sqrt{\frac{x-1}{2-x}} - 2\arctan\sqrt{\frac{2-x}{x-1}} + C.$$

Of course, in this integral we can use the second Euler substitution in the form

$$\sqrt{3x-x^2-2} = t(2-x),$$

with the same success.