The Method of Euler Substitution PhL 11.1.16

Warning: The first 10 sections are valid. In the last 4 sections I was just flailing away trying to find the $\sin^{-1}$ form of my integral. Sometimes bad paths lead to lots of work and no payoff. This form has since been easily derived in my newer doc. I may publish this at some point, would be a very short doc. Just to save someone else from having to do the work.

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1. Introduction

The problem is how to do integrals of this form (see GR7 page 92)

$$\int dx\, R(x,\sqrt{a+bx+cx^2})$$

What does this notation mean? In GR7 this means a "rational function". Usually a rational function is a ratio of polynomials. But what does R(a,b) mean? It is never defined in GR7 as far as I know (nor on the web). I think what it means is that the integrand is a ratio of polynomials in the two variables a and b. Examples

$$R(x,\sqrt{a+bx+cx^2}) = \frac{Ax\sqrt{a+bx+cx^2} + Bx^7(\sqrt{a+bx+cx^2})^3 + x^3}{2\sqrt{a+bx+cx^2}\ x^3 + x^2} \qquad \text{typical}$$

The numerator and denominator are separately polynomials in these two variables.

Now since you can treat the terms separately, the numerator can be taken to be a monomial, so then

$$R(x,\sqrt{a+bx+cx^2}) = \frac{x^m(\sqrt{a+bx+cx^2})^n}{\text{Poly}(x,\sqrt{a+bx+cx^2})} \quad .$$

As I browse through the GR7 integrals starting at page 92 they seem to have this form.

The main issue is this: How to you get rid of the ugly radical $\sqrt{a+bx+cx^2}$.

Warning: Things are then confused when the same symbol R is redefined to mean

$$R \equiv a + bx + cx^2$$

Almost all integrals of interest, however, seem to be of the form

$$F(x,\sqrt{a+bx+cx^2}) = x^m (\sqrt{a+bx+cx^2})^n$$

where m and n are arbitrary integers of either sign. So this would be a special case of the above more general form.

2. Comments about $\sqrt{a+bx+cx^2}$

1. The a,b,c notation I (and GR) have chosen matches high-school usage. Some authors (Schuam) have a↔c so beware. I assume that a,b,c are all real, at least as an initial position.

2. If c > 0, we have no ambiguity in writing

$$\sqrt{a+bx+cx^2} = \sqrt{c}\,\sqrt{(a/c)+(b/c)x+x^2} = \sqrt{c}\,\sqrt{x^2 + (b/c)x + (a/c)}$$

If c < 0, we have to decide on a phase for $\sqrt{c} = \sqrt{-c}\ e^{i\pi/2}$ probably is what I would choose.

3. Now write

$R/c = x^2 + (b/c)x + (a/c) = (x-\alpha_+)(x-\alpha_-)$ where $\alpha_\pm$ are the roots. Then have

$$\sqrt{R} = \sqrt{a+bx+cx^2} = \sqrt{c}\,\sqrt{(x-\alpha_+)(x-\alpha_-)} \qquad R = a + bx + cx^2 = c(x-\alpha_+)(x-\alpha_-)$$

4. Values $\alpha_\pm$ are also the roots of $a + bx + cx^2$ regardless of the sign of c.

5. These roots of $x^2 + (b/c)x + (a/c) = 0$ are given by the quadratic formula,

$$\alpha_\pm = [\ -b/c \pm \sqrt{(b/c)^2 - 4\,(a/c)}\ ] / 2$$

$$= [\ -b/c \pm (1/c)\ \sqrt{b^2 - 4ac}\ ] / 2$$

$= [-b \pm \sqrt{b^2 - 4ac}] / 2c$

where b^2-4ac is called "the discriminant".

5a. If it happens that $b^2 = 4ac$, then we get $\alpha_\pm = -b/(2c)$ and then $R = c(x-\alpha_+)^2 = c(x + [b/2c])^2$ and $\sqrt{R} = \sqrt{c}\,(x + [b/2c])$. This is then for $\Delta = 0$ as defined in the next item.

6. GR7 make this definition

$-\Delta \equiv b^2 - 4ac$

7. If $b^2 - 4ac > 0$ then the roots are real. This means that the function $f(x) = a + bx + cx^2$ has two intersections with the x axis. If $c > 0$, the parabola cups up and intersects the x axis at the two roots. Therefore to the right of the upper root, we have $a + bx + cx^2$ and so $\sqrt{a+bx+cx^2}$ is real and well defined.

8. If $c < 0$, the parabola is cupping down. If $b^2 - 4ac > 0$ the roots are real as before, but now $\sqrt{a+bx+cx^2}$ is well defined and real only between the two roots.

9. When talking about integrals involving $\sqrt{a+bx+cx^2}$, I feel it is best to start with an integral over a range of x where $a + bx + cx^2$ is positive, so then $\sqrt{a+bx+cx^2}$ is real. Thus we have

Case 1: $c > 0$, $b^2 - 4ac > 0$ (real roots), cups up, $x > \alpha_+$ or $x < \alpha_-$ to have real $\sqrt{a+bx+cx^2}$

Case 2: $c > 0$, $b^2 - 4ac < 0$ (imag roots), cups up, all values of x give real $\sqrt{a+bx+cx^2}$ $\Delta > 0$

Case 3: $c < 0$, $b^2 - 4ac > 0$ (real roots), cups down, must have $\alpha_- < x < \alpha_+$ to have real $\sqrt{a+bx+cx^2}$

Case 4: $c < 0$, $b^2 - 4ac < 0$ (imag roots), cups down, no value of x gives real $\sqrt{a+bx+cx^2}$

It seems to me that Case 2 is the safest to deal with, since $\sqrt{a+bx+cx^2}$ is real for any value of x, so when you integrate you don't have to worry about what values x takes in the integration.

Fact: I will assume Case 2 from now on. This means $c > 0$ and $\Delta > 0$ in language of GR7.

3. Naive method, does not work

Ones first thought for "change of variables" would be to just set

$\sqrt{a+bx+cx^2} = t$

where t is a new variable. We would then be talking about

$R(x,\sqrt{a+bx+cx^2}) \rightarrow R(x(t),t)$

But we want x(t) to be simple! With the above method we get

$a + bx + x^2 = t^2$

$x^2+bx+(a-t^2) = 0$

and then x(t) is a quadratic formula

$x(t) = [-1 \pm \sqrt{b^2 - 4(a-t^2)}\,]/2$

and this then causes the x^m factors to be a mess still involving radicals. So this choice is no go. That is to say, when we change variables from x to t, we don't want to have radicals in the resulting functions of t.

4. Improved method A

Suppose instead you tried this

$\sqrt{R} = \sqrt{a+bx+cx^2} = t + \sqrt{c}\,x$ // $\sqrt{R} = \sqrt{a+bx+cx^2} = t - \sqrt{c}\,x$

With Case 2 above, $\sqrt{a+bx+cx^2}$ is real, $\sqrt{c}$, x is real and t is real, so there are no issues with ranges of x or t. Now we get

$R = a + bx + cx^2 = t^2 + cx^2 + 2\sqrt{c}\,t\,x$ // $R = a + bx + cx^2 = t^2 + cx^2 - 2\sqrt{c}\,t\,x$

Now you have arranged to have two terms cancel out with this result

$a + bx = t^2 + 2\sqrt{c}\,t\,x$

$(b - 2\sqrt{c}\,t\,)x = (t^2-a)$

$x(t) = (t^2-a)/(b - 2\sqrt{c}\,t\,)$ // $x(t) = (t^2-a)/(b + 2\sqrt{c}\,t\,)$

Maple verify:

```
R := a + b*x + c*x^2;
```

$$R := a + b\,x + c\,x^2$$

```
solve(sqrt(R) = t + sqrt(c)*x,x);
```

$$\frac{-t^2+a}{-b+2\,t\sqrt{c}}$$

Maple continues

```
sqrtR := t + sqrt(c)*X: simplify(%);
```

$$\frac{-t\,b+t^2\sqrt{c}+\sqrt{c}\,a}{-b+2\,t\sqrt{c}}$$

which says

$\sqrt{a+bx+cx^2}$ = (bt-$\sqrt{c}$ t^2-$\sqrt{c}$ a)/(b - 2$\sqrt{c}$ t) // $\sqrt{a+bx+cx^2}$ = (bt+$\sqrt{c}$ t^2+$\sqrt{c}$ a)/(b + 2$\sqrt{c}$ t)

You then end up with

R[x,$\sqrt{a+bx+cx^2}$] → R[(t^2-a)/(b - 2$\sqrt{c}$ t), (bt-$\sqrt{c}$ t^2-$\sqrt{c}$ a)/(b - 2$\sqrt{c}$ t)]

and your result then has no ugly radicals at least, though it is still a mess.

BUT, could an ugly radical reappear when you compute the connection between dx and dt?

x(t) = (t^2-a)(b - 2$\sqrt{c}$ t)$^{-1}$

dx/dt = (t^2-a)(-1) (b - 2$\sqrt{c}$ t)$^{-2}$ (- 2$\sqrt{c}$) + (2t) (b - 2$\sqrt{c}$ t)$^{-1}$

or

dx/dt = (2$\sqrt{c}$)(t^2-a) (b - 2$\sqrt{c}$ t)$^{-2}$ + 2t (b - 2$\sqrt{c}$ t)$^{-1}$

= 2 (b - 2$\sqrt{c}$ t)$^{-2}$ [($\sqrt{c}$)(t^2-a) + t(b - 2$\sqrt{c}$ t)]

= 2 (b - 2$\sqrt{c}$ t)$^{-2}$ [$\sqrt{c}$ t^2 - a $\sqrt{c}$ + tb - 2$t^2\sqrt{c}$]

= 2 (b - 2$\sqrt{c}$ t)$^{-2}$ (bt - a $\sqrt{c}$ -$\sqrt{c}$ t^2)

Maple verify:

```
X := S;
```

$$X := -\frac{-a+t^2}{-b+2\,t\sqrt{c}}$$

```
diff(X,t): simplify(%);
```

$$-2\,\frac{-t\,b+t^2\sqrt{c}+\sqrt{c}\,a}{(-b+2\,t\sqrt{c})^2}$$

OK

so then

x(t) = (t^2-a)(b - 2$\sqrt{c}$ t)$^{-1}$

$\sqrt{a+bx+cx^2}$ = (bt-$\sqrt{c}$ t^2-$\sqrt{c}$ a)/(b - 2$\sqrt{c}$ t)

$$dx = 2\,(b - 2\sqrt{c}\,t\,)^{-2}\,(bt - a\sqrt{c} - \sqrt{c}\,t^2\,)\,dt$$

No, no bad radicals have appeared here! That is because there were no radicals in x(t). So we then have

$$\int^{x} dx\; R(x,\sqrt{a+bx+cx^2}\,)$$

$$= \int^{t(x)} dt \;*\; 2\,(b - 2\sqrt{c}\,t\,)^{-2}\,(bt - a\sqrt{c} - \sqrt{c}\,t^2\,) \;*\; R[\,(t^2-a)/(b - 2\sqrt{c}\,t),\,(bt-\sqrt{c}\,a-\sqrt{c}\,t^2)/(b - 2\sqrt{c}\,t)\,]$$

It ain't purdy but at least there are no nasty radicals. But I also need to know t(x) so go back,

$$a + bx = t^2 + 2\sqrt{c}\,t\,x$$

$$t^2 + [2\sqrt{c}\;x]\,t + [-a-bx] = 0$$

$$t = [-\,2\sqrt{c}\;x \pm \sqrt{4cx^2 + 4a + 4bx}\;]/2$$

$$= -\sqrt{c}\;x \pm \sqrt{a+bx+cx^2}$$

so then the integration endpoint is

$$t_{\pm}(x) = -\sqrt{c}\;x \pm \sqrt{a+bx+cx^2} \qquad // \; t_{\pm}(x) = +\sqrt{c}\;x \pm \sqrt{a+bx+cx^2}$$

and we eventually have to worry about the sign. Both these functions $t_{\pm}(x)$ are real with our Case 2 conditions. Both these t functions satisfy $[\,\sqrt{a+bx+cx^2}\,]^2 = [\,t + \sqrt{c}\,x]^2$. But we want t to satisfy the equation $\sqrt{a+bx+cx^2} = t + \sqrt{c}\,x$! Which sign works?

$$\sqrt{a+bx+cx^2} = t_+ + \sqrt{c}\,x = [-\sqrt{c}\;x + \sqrt{a+bx+cx^2}\;] + \sqrt{c}\,x = \sqrt{a+bx+cx^2}$$

$$// \;\sqrt{a+bx+cx^2} = t_+ - \sqrt{c}\,x = [+\sqrt{c}\;x + \sqrt{a+bx+cx^2}\;] - \sqrt{c}\,x = \sqrt{a+bx+cx^2}$$

Therefore the only correct solution is t_+ which we will call t. Then

$$t(x) = -\sqrt{c}\;x + \sqrt{a+bx+cx^2}\,. \qquad // \; t(x) = +\sqrt{c}\;x + \sqrt{a+bx+cx^2}$$

Comment: Think of this more traditional example

$$\int^{\theta} f(\sin\theta)d\theta \qquad u(\theta) = \sin\theta \qquad du = \cos\theta d\theta \qquad d\theta = (1-u^2)^{-1/2}du$$

$$\int^{\theta} f(\sin\theta)d\theta = \int^{u(\theta)} f(u)\,(1-u^2)^{-1/2}du$$

Example: f(x) = x so have

$$\int^{\theta} \sin\theta \, d\theta = \int^{u(\theta)} u \, (1-u^2)^{-1/2} \, du = -\sqrt{1-u^2}\,|^{u=\sin(\theta)} = -\cos\theta$$

Summary of Method A:

$$\sqrt{R} = \sqrt{a+bx+cx^2} = t + \sqrt{c}\, x$$

$$x(t) = (t^2-a)(b - 2\sqrt{c}\, t)^{-1}$$

$$\sqrt{a+bx+cx^2} = (bt-\sqrt{c}\, t^2-\sqrt{c}\, a)/(b - 2\sqrt{c}\, t)$$

$$dx = 2 (b - 2\sqrt{c}\, t)^{-2} (bt - a\sqrt{c} -\sqrt{c}\, t^2) \, dt$$

$$t(x) = -\sqrt{c}\, x + \sqrt{a+bx+cx^2}$$

$$\int^{x} dx \, R(x,\sqrt{a+bx+cx^2})$$

$$= \int^{t(x)} dt * 2 (b - 2\sqrt{c}\, t)^{-2} (bt - a\sqrt{c} -\sqrt{c}\, t^2) * R[(t^2-a)/(b - 2\sqrt{c}\, t), (bt-\sqrt{c}\, a-\sqrt{c}\, t^2)/(b - 2\sqrt{c}\, t)]$$

Question: How does Method A change if you take a different sign and say $\sqrt{R} = t - \sqrt{c}\, x$? I have reviewed lines above, and the answer is that you just take $\sqrt{c} \rightarrow -\sqrt{c}$ in ALL results. Of course this implies that $c \rightarrow c$ with no change. So the above would be +

$$\sqrt{R} = \sqrt{a+bx+cx^2} = t - \sqrt{c}\, x$$

$$x(t) = (t^2-a)(b + 2\sqrt{c}\, t)^{-1}$$

$$\sqrt{a+bx+cx^2} = (bt+\sqrt{c}\, t^2+\sqrt{c}\, a)/(b + 2\sqrt{c}\, t)$$

$$dx = 2 (b + 2\sqrt{c}\, t)^{-2} (bt + a\sqrt{c} +\sqrt{c}\, t^2) \, dt$$

$$t(x) = +\sqrt{c}\, x + \sqrt{a+bx+cx^2}$$

$$\int^{x} dx \, R(x,\sqrt{a+bx+cx^2})$$

$$= \int^{t(x)} dt * 2 (b + 2\sqrt{c}\, t)^{-2} (bt + a\sqrt{c} +\sqrt{c}\, t^2) * R[(t^2-a)/(b + 2\sqrt{c}\, t), (bt+\sqrt{c}\, a+\sqrt{c}\, t^2)/(b + 2\sqrt{c}\, t)]$$

Perhaps a little nicer in the blue form because the final result has all + signs.

5. Improved method B

Suppose instead you tried this

$$\sqrt{a+bx+cx^2} = xt + \sqrt{a}$$

Now we get

$$a + bx + cx^2 = x^2t^2 + a + 2\sqrt{a}\ xt$$

Now you have arranged to have two terms cancel out with this result

$$bx + cx^2 = x^2t^2 + 2\sqrt{a}\ xt$$

or

$$b + cx = xt^2 + 2\sqrt{a}\ t$$

$$x(c-t^2) = 2\sqrt{a}\ t - b$$

$$x(t) = (2\sqrt{a}\ t - b)/(c-t^2)$$

Maple verify:

```
restart;;
R := a + b*x + c*x^2;
```

$$R := a + b\,x + c\,x^2$$

```
S := solve(R = (x*t + sqrt(a))^2,x);
```

$$S := 0, \frac{-b + 2\,t\sqrt{a}}{c - t^2}$$

```
X :=S[2];
```

$$X := \frac{-b + 2\,t\sqrt{a}}{c - t^2}$$

```
diff(X,t): simplify(%);
```

$$2\,\frac{\sqrt{a}\,c + \sqrt{a}\,t^2 - t\,b}{(c - t^2)^2}$$

```
sqrtR := X*t+sqrt(a): simplify(%);
```

$$\frac{\sqrt{a}\,c + \sqrt{a}\,t^2 - t\,b}{c - t^2}$$

.

Maple has shown that

$$x(t) = (2\sqrt{a}\ t - b)/(c-t^2)$$

$$\sqrt{a+bx+cx^2} = (\sqrt{a}\, t^2-tb+\sqrt{a}\, c)/(c-t^2)$$

$$dx = 2\,(\sqrt{a}\, t^2-tb+\sqrt{a}\, c)(t^2-c)^{-2}\, dt$$

Then:

$$\int^{x} dx\; R(x,\sqrt{a+bx+cx^2}\,)$$

$$= \int^{t(x)} dt \; * 2\,(\sqrt{a}\, t^2-tb+\sqrt{a}\, c)(t^2-c)^{-2} * \; R\,(2\sqrt{a}\, t - b)/(c-t^2),\, (\sqrt{a}\, t^2-tb+\sqrt{a}\, c)/(c-t^2)]$$

We then also need t(x):

$$b + cx = xt^2 + 2\sqrt{a}\; t \qquad // \; b + cx = xt^2 - 2\sqrt{a}\; t$$

$$xt^2 + 2\sqrt{a}\; t + [-b-cx] = 0$$

$$t_{\pm} = [-2\sqrt{a} \pm \sqrt{4a + 4x(b+cx)}\,] /2x$$

$$= [\, -\sqrt{a} \pm \sqrt{a + bx + cx^2}\,] /x \;=\; [\, -\sqrt{a} \pm \sqrt{R}\,] /x \qquad // \; t_{\pm} = [\, +\sqrt{a} \pm \sqrt{R}\,] /x$$

We now confirm which is the correct solution going back to

$$\sqrt{R} = xt_{+} + \sqrt{a} = x\,\{\,[\, -\sqrt{a} + \sqrt{a + bx + cx^2}\,] /x\} + \sqrt{a}\,\}$$

$$= \{\,[\, -\sqrt{a} + \sqrt{a + bx + cx^2}\,]\} + \sqrt{a} = \sqrt{a + bx + cx^2}$$

Repeat this in blue:

$$\sqrt{R} = xt_{+} - \sqrt{a} = x\,\{\,[\, +\sqrt{a} + \sqrt{a + bx + cx^2}\,] /x\} - \sqrt{a}\,\}$$

$$= \{\,[\, +\sqrt{a} + \sqrt{a + bx + cx^2}\,]\} - \sqrt{a} = \sqrt{a + bx + cx^2}$$

Summary:

$$\sqrt{a+bx+cx^2} = xt + \sqrt{a}$$

$$x(t) = (2\sqrt{a}\; t - b)/(c-t^2)$$

$$t(x) = [\, -\sqrt{a} + \sqrt{a + bx + cx^2}\,] /x$$

$$\sqrt{a+bx+cx^2} = (\sqrt{a}\, t^2-tb+\sqrt{a}\, c)/(c-t^2)$$

$$dx = 2\,(\sqrt{a}\,t^2-tb+\sqrt{a}\,c)(t^2-c)^{-2}\,dt$$

$$\int^{x} dx\, R(x,\sqrt{a+bx+cx^2}\,)$$

$$= \int^{t(x)} dt \;* 2\,(\sqrt{a}\,t^2-tb+\sqrt{a}\,c)(t^2-c)^{-2} *\; R\,(2\sqrt{a}\,t - b)/(c-t^2),\,(\sqrt{a}\,t^2-tb+\sqrt{a}\,c)/(c-t^2)]$$

I think you could replace $\sqrt{a} \rightarrow -\sqrt{a}$ (and a→a) to get a "blue form" for the above equations.

6. Improved method C

Write

$$a + bx + cx^2 = c\,[\,x^2 + (b/c)x + (a/c)\,] = c\,(x-\alpha)(x-\beta)$$

where α and β are the roots of $a + bx + cx^2$. Then try this:

$$\sqrt{a+bx+cx^2} = (x-\alpha)t$$

Then

$$a + bx + cx^2 = (x-\alpha)^2t^2$$

or

$$c\,(x-\alpha)(x-\beta) = (x-\alpha)^2t^2$$

or

$$c(x-\beta) = (x-\alpha)t^2$$

where once again we have cancellation on the two sides. Then

$$cx - c\beta = xt^2-\alpha t^2$$

or

$$x(c - t^2) = c\beta-\alpha t^2$$

$$x(t) = (\,c\beta-\alpha t^2)/(c-t^2)$$

```
restart;
R := c*(x-alpha)*(x-beta);
```

$$R := c\,(x-\alpha)\,(x-\beta)$$

```
S := solve(R = (x-alpha)^2*t^2,x);
```

$$S := \alpha, \frac{-t^2\,\alpha + c\,\beta}{c - t^2}$$

```
X :=S[2];
```

$$X := \frac{-t^2\,\alpha + c\,\beta}{c - t^2}$$

```
sqrtR := (X-alpha)*t: simplify(%);
```

$$\frac{c\,(\beta-\alpha)\,t}{c - t^2}$$

```
diff(X,t): simplify(%);
```

$$2\,\frac{c\,(\beta-\alpha)\,t}{(c - t^2)^2}$$

So we have the following

$x(t) = (c\beta - \alpha t^2)/(c - t^2)$

$\sqrt{a+bx+cx^2} = c(\beta-\alpha)t/(c-t^2)$

$dx = [\,2c(\beta-\alpha)t / (c-t^2)^2\,]\,dt$

Then,

$$\int^{x} dx\, R\left(x, \sqrt{a+bx+cx^2}\right) =$$

$$\int^{t(x)} dt \;*\; [\,2c(\beta-\alpha)t / (c-t^2)^2\,] \;*\; R[\,(c\beta-\alpha t^2)/(c-t^2)\,,\; c(\beta-\alpha)t/(c-t^2)\,]$$

This one is perhaps more complicated because you have to compute the two roots α and β, but you can see that it gives a result with no radicals, as in the previous two cases. In the case of my Example below this will give

$$\int^{x} dx/\sqrt{a+bx+cx^2} = \int^{t(x)} dt \;*\; [\,2c(\beta-\alpha)t / (c-t^2)^2\,] \;*\; [\,c(\beta-\alpha)t/(c-t^2)\,]^{-1}$$

$$= \int^{t(x)} dt \;*\; [\,2 / (c-t^2)\,] \;=\; \text{same result as Method B shown below.}$$

7. Improved method D (my own, no good)

Suppose instead you tried this

$$R = \sqrt{a+bx+cx^2} \quad = t + \sqrt{bx}$$

Now we get

$$R = a + bx + cx^2 = t^2 + bx + 2t\sqrt{bx}$$

Again we get two terms cancelling, leaving

$$a + cx^2 = t^2 + 2t\sqrt{bx}$$

But this method does not give a simple expression for x(t), so it is a no go.

8. Summary of the three methods that are useful (these are Euler's three)

Method A:

$$\sqrt{R} = \sqrt{a+bx+cx^2} \quad = t + \sqrt{c}\, x$$

$$x(t) = (t^2-a)(b - 2\sqrt{c}\, t)^{-1} \qquad\qquad t(x) = -\sqrt{c}\ \ x + \sqrt{a+bx+cx^2}$$

$$\sqrt{a+bx+cx^2} \quad = (bt-\sqrt{c}\, t^2-\sqrt{c}\, a)/(b - 2\sqrt{c}\, t)$$

$$dx = \ 2\,(b - 2\sqrt{c}\, t)^{-2}\,(bt - a\sqrt{c} -\sqrt{c}\, t^2)\, dt$$

$$\int^{x} dx\ \mathcal{R}(x,\sqrt{a+bx+cx^2})$$

$$= \int^{t(x)} dt\ *\ 2\,(b - 2\sqrt{c}\, t)^{-2}\,(bt - a\sqrt{c} -\sqrt{c}\, t^2)\ *\ \mathcal{R}[\,(t^2-a)/(b - 2\sqrt{c}\, t), (bt-\sqrt{c}\, a-\sqrt{c}\, t^2)/(b - 2\sqrt{c}\, t)\,]$$

Method B:

$$\sqrt{a+bx+cx^2} \quad = xt + \sqrt{a}$$

$$x(t) = (2\sqrt{a}\ \ t - b)/(c-t^2) \qquad\qquad t(x) = [\ -\sqrt{a}\ \ + \sqrt{a + bx + cx^2}\ ]\ /x$$

$$\sqrt{a+bx+cx^2} \ = \ (\sqrt{a}\, t^2-tb+\sqrt{a}\, c)/(c-t^2)$$

$$dx = 2\,(\sqrt{a}\, t^2-tb+\sqrt{a}\, c)(t^2-c)^{-2}\, dt$$

=

$$\int^{x} dx\ \mathcal{R}(x,\sqrt{a+bx+cx^2})$$

$$= \int^{t(x)} dt \ * 2 (\sqrt{a}\, t^2\text{-}tb\text{+}\sqrt{a}\, c)(t^2\text{-}c)^{-2} * \mathcal{R}\,(2\sqrt{a}\, t - b)/(c\text{-}t^2), (\sqrt{a}\, t^2\text{-}tb\text{+}\sqrt{a}\, c)/(c\text{-}t^2)]$$

Method C: where α,β are the roots of $a + bx + cx^2 = 0$ "

$$\sqrt{a+bx+cx^2} = (x-\alpha)t$$

$$x(t) = (c\beta-\alpha t^2)/(c-t^2) \qquad t(x) = \sqrt{(\beta-x)/(\alpha-x)}$$

$$\sqrt{a+bx+cx^2} = c(\beta-\alpha)t/\,(c-t^2)$$

$$dx = [\ 2c(\beta-\alpha)t\ /\ (c-t^2)^2]\ dt$$

$$\int^{x} dx\ \mathcal{R}(x,\sqrt{a+bx+cx^2}\ ) =$$

$$\int^{t(x)} dt \ * \ [\ 2c(\beta-\alpha)t\ /(c-t^2)^2] \ * \ \mathcal{R}[\ (c\beta-\alpha t^2)/(c-t^2)\ ,\ c(\beta-\alpha)t/\,(c-t^2)]$$

9. Application of Method A

First I quote the Method A results

$$x(t) = (t^2\text{-}a)(b - 2\sqrt{c}\, t\ )^{-1} \qquad t(x) = -\sqrt{c}\ \ x + \sqrt{a+bx+cx^2}$$

$$\sqrt{a+bx+cx^2} = (bt\text{-}\sqrt{c}\, t^2\text{-}\sqrt{c}\, a)/(b - 2\sqrt{c}\, t)$$

$$dx = \ 2\ (b - 2\sqrt{c}\, t\ )^{-2}\ (bt - a\,\sqrt{c}\ \text{-}\sqrt{c}\, t^2\ )\ dt$$

$$\int^{x} dx\ R(x,\sqrt{a+bx+cx^2}\ )$$

$$= \int^{t(x)} dt \ * \ 2\ (b - 2\sqrt{c}\, t\ )^{-2}\ (bt - a\,\sqrt{c}\ \text{-}\sqrt{c}\, t^2\ ) * \ R[\ (t^2\text{-}a)/(b - 2\sqrt{c}\, t), (bt\text{-}\sqrt{c}\, a\text{-}\sqrt{c}\, t^2)/(b - 2\sqrt{c}\, t)\ ]$$

Now the application is this:

$$R(x,\sqrt{a+bx+cx^2}\ ) \ = \ 1/\sqrt{a+bx+cx^2}$$

so

$$\int^{x} dx/\sqrt{a+bx+cx^2}$$

$$= \int^{t(x)} dt * \ 2\ (b - 2\sqrt{c}\, t\ )^{-2}\ (bt - a\,\sqrt{c}\ \text{-}\sqrt{c}\, t^2\ ) \ * 1/\ [\ \ (bt\text{-}\sqrt{c}\, t^2\text{-}\sqrt{c}\, a)/(b - 2\sqrt{c}\, t)\ ]$$

$$= \int^{t(x)} dt * \ 2\ (b - 2\sqrt{c}\, t\ )^{-2}\ (bt - a\,\sqrt{c}\ \text{-}\sqrt{c}\, t^2\ ) \ * (b - 2\sqrt{c}\, t) /\ \ (bt\text{-}\sqrt{c}\, t^2\text{-}\sqrt{c}\, a)$$

$= \int^{t(x)} dt * \ 2\,(b - 2\sqrt{c}\,t)^{-1}\,(bt - a\sqrt{c} - \sqrt{c}\,t^2) \ * 1 / \ (bt - \sqrt{c}\,t^2 - \sqrt{c}\,a)$

$= \int^{t(x)} dt * \ 2\,(b - 2\sqrt{c}\,t)^{-1} \ * 1 / \ 1$

$= 2 \int^{t(x)} dt \ / (b - 2\sqrt{c}\,t)$

$= 2/(2\sqrt{c}) \int^{t(x)} dt \ / (b/(2\sqrt{c}) - t)$

$= - \ (1/\sqrt{c}) \int^{t(x)} dt \ / (t - r) \qquad\qquad r \equiv b/(2\sqrt{c})$

$= - \ (1/\sqrt{c}) \ln(t-r)|^{t(x)}$

$= - \ (1/\sqrt{c}) \ln(t(x) - r) \quad .$ // algebra seems OK

But recall that

$t(x) = -\sqrt{c}\ x + \sqrt{a+bx+cx^2} = -\sqrt{c}\ x + \sqrt{R}$

so we then have

$= - \ (1/\sqrt{c}) \ln[-\sqrt{c}\ x + \sqrt{R} - b/(2\sqrt{c})]$

$= - \ (1/\sqrt{c}) \ln[-\sqrt{c}\ x - b/(2\sqrt{c}) + \sqrt{R}\,]$

Now think of this as $\ln[\frac{*}{1}]$. You are allowed to multiply top and bottom by whatever you like that does not depend on x. Then you can throw away the bottom since it is just a constant indep of x. So mult top and bottom by $+ 2\sqrt{c}$ to get

$= - \ (1/\sqrt{c}) \ln [- 2cx - b + 2\sqrt{cR}\,]$

Again, I have done this in Case 2 where $c > 0, \quad -\Delta = b^2 - 4ac < 0$ so $\Delta > 0$

There are two official GR7 answers for this case

2.261[11] For $n = -1$

$$\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{c}} \ln\left(\frac{2\sqrt{cR} + 2cx + b}{\sqrt{\Delta}}\right) \qquad [c > 0] \qquad \text{TI (127)}$$

$$= \frac{1}{\sqrt{c}} \operatorname{arcsinh}\left(\frac{2cx + b}{\sqrt{\Delta}}\right) \qquad [c > 0, \quad \Delta > 0] \qquad \text{DW}$$

The result I got is similar to the first result here, but it is NOT THE SAME even adding constants. I seem to have multiple sign errors! The denominator factor is irrelevant since it just represents a constant, so here is the GR7 result compared with my result

$$GR7(x) = + (1/\sqrt{c}) \ln [+2cx + b + 2\sqrt{cR}]$$

$$PL(x) = - (1/\sqrt{c}) \ln [- 2cx - b + 2\sqrt{cR}]$$

So I claim to have shown that

$$\int^{x} dx/\sqrt{a+bx+cx^2} = PL(x) + \text{constant}$$

A check would be to show that

$$dPL(x)/dx = 1/\sqrt{a+bx+cx^2} \quad ?$$

OK,

$$dPL(x)/dx = - (1/\sqrt{c}) \, d_x \ln [- 2cx - b + 2\sqrt{cR}]$$

$$= - (1/\sqrt{c}) [- 2cx - b + 2\sqrt{cR}]^{-1} (-2c + 2\sqrt{c} \, d_x\sqrt{R})$$

Now

$$d_x\sqrt{R} = (1/2)(1/\sqrt{R})(b + 2cx)$$

So then,

$$dPL(x)/dx = - (1/\sqrt{c}) [- 2cx - b + 2\sqrt{cR}]^{-1} [-2c + 2\sqrt{c} (1/2)(1/\sqrt{R})(b + 2cx)]$$

$$= - (1/\sqrt{c}) [- 2cx - b + 2\sqrt{cR}]^{-1} [-2c\sqrt{R} + \sqrt{c} (b + 2cx)] /\sqrt{R}$$

$$= (1/\sqrt{c}) [- 2cx - b + 2\sqrt{cR}]^{-1} [2c\sqrt{R} - \sqrt{c} (b + 2cx)] /\sqrt{R}$$

$$= (1/\sqrt{c}) [- 2cx - b + 2\sqrt{cR}]^{-1} [2c\sqrt{R} - \sqrt{c} b - 2\sqrt{c} cx)] /\sqrt{R}$$

$$= (1/\sqrt{c}) [- 2cx - b + 2\sqrt{cR}]^{-1} [2\sqrt{c}\sqrt{c}\sqrt{R} - \sqrt{c} b - 2\sqrt{c} cx)] /\sqrt{R}$$

$$= [- 2cx - b + 2\sqrt{cR}]^{-1} [2\sqrt{c}\sqrt{R} - b - 2cx)] /\sqrt{R}$$

$$= [- 2cx - b + 2\sqrt{cR}]^{-1} [- 2cx - b + 2\sqrt{c}\sqrt{R})] /\sqrt{R}$$

$= 1/\sqrt{R}$

This would seem to say that my result is correct. Now let's try their result:

$dGR7(x)/dx = (1/\sqrt{c})\ d_x \ln[+2cx + b + 2\sqrt{cR}]$

$= (1/\sqrt{c})[+2cx + b + 2\sqrt{cR}]^{-1}(2c + 2\sqrt{c}\ d_x\sqrt{R})$

$= (1/\sqrt{c})[+2cx + b + 2\sqrt{cR}]^{-1}(2c + 2\sqrt{c}\ [(1/2)(1/\sqrt{R})(b + 2cx)])$

$= (1/\sqrt{c})[+2cx + b + 2\sqrt{cR}]^{-1}(2c + \sqrt{c}\ (1/\sqrt{R})(b + 2cx))$

$= (1/\sqrt{c})[+2cx + b + 2\sqrt{cR}]^{-1}(2c\sqrt{R} + \sqrt{c}(b + 2cx))/\sqrt{R}$

$= (1/\sqrt{c})[+2cx + b + 2\sqrt{cR}]^{-1}(2c\sqrt{R} + \sqrt{c}\,b + 2\sqrt{c}\,cx))/\sqrt{R}$

$= (1/\sqrt{c})[+2cx + b + 2\sqrt{cR}]^{-1}(2\sqrt{c}\sqrt{c}\sqrt{R} + \sqrt{c}\,b + 2\sqrt{c}\,cx))/\sqrt{R}$

$= [+2cx + b + 2\sqrt{cR}]^{-1}(2\sqrt{c}\sqrt{R} + b + 2cx))/\sqrt{R}$

$= 1/\sqrt{R}$

So both the PL and GR7 solutions are valid! How can that be?

$GR7(x) = +(1/\sqrt{c})\ln[+2cx + b + 2\sqrt{cR}]$

$PL(x) = -(1/\sqrt{c})\ln[-2cx - b + 2\sqrt{cR}]$

$= (1/\sqrt{c})\ln[(-2cx - b + 2\sqrt{cR})^{-1}]$

$= (1/\sqrt{c})\ln\left[\frac{1}{(-2cx - b + 2\sqrt{cR})}\right]$

Now mult top and bottom by $(-2cx - b - 2\sqrt{cR})$, certainly allowed to do that

$= (1/\sqrt{c})\ln\left[\frac{-2cx - b - 2\sqrt{cR}}{(-2cx - b + 2\sqrt{cR})(-2cx - b - 2\sqrt{cR})}\right]$

Now evaluate the bottom:

$(-2cx - b + 2\sqrt{cR})(-2cx - b - 2\sqrt{cR})$

$= (2cx + b - 2\sqrt{cR})(2cx + b + 2\sqrt{cR})$

$= (2cx + b)^2 - 4cR$

$= (2cx + b)^2 - 4c(a+bx+cx^2)$

$= 4c^2x^2 + b^2 + 4cxb - 4ca - 4cbx - 4c^2x^2$

$= b^2 - 4ca$

Thus I have shown that

$$PL(x) = (1/\sqrt{c}) \ln\left[\frac{-2cx - b - 2\sqrt{cR}}{b^2-4ac}\right]$$

$$= (1/\sqrt{c}) \ln\left[\frac{2cx + b + 2\sqrt{cR}}{4ac-b^2}\right]$$

$$= (1/\sqrt{c}) \ln\left[\frac{2cx + b + 2\sqrt{cR}}{1}\right] \quad \text{// dropping constant term}$$

$$= (1/\sqrt{c}) \ln\left[\frac{2cx + b + 2\sqrt{cR}}{1}\right] \quad \text{// dropping constant term}$$

$$= (1/\sqrt{c}) \ln[+2cx + b + 2\sqrt{cR}]$$

$= GR7(x)$

So finally I have shown that my result agrees with the GR7 result.

Question: What happens to this example if we take the blue form? My black result was

$$PL(x) = - (1/\sqrt{c}) \ln[-2cx - b + 2\sqrt{cR}]$$

so following the simple rule $\sqrt{c} \rightarrow -\sqrt{c}$ this becomes

$$PL(x) = + (1/\sqrt{c}) \ln[-2cx - b - 2\sqrt{cR}]$$

Now use the fact that $\ln[-s] = \ln(e^{i\pi}s) = i\pi + \ln(x)$ and throw out the constant to get

$$PL(x) = + (1/\sqrt{c}) \ln[2cx + b + 2\sqrt{cR}]$$

Recall from above

$$GR7(x) = + (1/\sqrt{c}) \ln[+2cx + b + 2\sqrt{cR}]$$

so using the blue method gets you the GR7(x) result, good!

Here is Maple's version of the integral:

```
Int( 1/sqrt(a+b*x+c*x^2),x);
```

$$\int \frac{1}{\sqrt{a+b\,x+c\,x^2}}\,dx$$

```
value(%);
```

$$\frac{\ln\left(\frac{\frac{1}{2}b+c\,x}{\sqrt{c}}+\sqrt{a+b\,x+c\,x^2}\right)}{\sqrt{c}}$$

which I can write as (inside log, to top and bottom times $2\sqrt{c}$) and then drop bottom constant :

$(1/\sqrt{c})\ \ln(\ b + 2cx + 2\sqrt{cR}\ )$

and this agrees with the GF7 function stated above.

10. Redo the same Application using Method B

First pull in the Method B summary,

$\sqrt{R} = \sqrt{a+bx+cx^2} = xt + \sqrt{a}$ // $\sqrt{R} = xt - \sqrt{a}$

$x(t) = (2\sqrt{a}\ t - b)/(c-t^2)$ $t(x) = [\ -\sqrt{a} + \sqrt{a + bx + cx^2}\ ]\ /x = (-\sqrt{a} + \sqrt{R}\ )/x$

$\sqrt{a+bx+cx^2} = (\sqrt{a}\ t^2-tb+\sqrt{a}\ c)/(c-t^2)$

$dx = 2\ (\sqrt{a}\ t^2-tb+\sqrt{a}\ c)(t^2-c)^{-2}\ dt$

$\int^{x} dx\ R(x,\sqrt{a+bx+cx^2}\ )$

$= \int^{t(x)} dt\ * 2\ (\sqrt{a}\ t^2-tb+\sqrt{a}\ c)(t^2-c)^{-2} *\ R\ (2\sqrt{a}\ t - b)/(c-t^2),\ (\sqrt{a}\ t^2-tb+\sqrt{a}\ c)/(c-t^2)]$

Now let $R(x,\sqrt{R}\ ) = 1/\sqrt{R}$ so the above becomes

$\int^{x} dx/\sqrt{R} = \int^{t(x)} dt\ * 2\ (\sqrt{a}\ t^2-tb+\sqrt{a}\ c)(t^2-c)^{-2} *\ \ 1/\ (\sqrt{a}\ t^2-tb+\sqrt{a}\ c)/(c-t^2)$

$$= \int^{t(x)} dt \ * 2\,(\sqrt{a}\,t^2 - tb + \sqrt{a}\,c)(t^2-c)^{-2} * \ (c-t^2)/\,(\sqrt{a}\,t^2 - tb + \sqrt{a}\,c)$$

$$= - \int^{t(x)} dt \ * 2\,(t^2-c)^{-1} \ = - 2 \int^{t(x)} dt\,/(t^2 - c)$$

I can have Maple do this integral to get

```
Int(1/(t^2-c),t);
```

$$\int \frac{1}{t^2 - c}\,dt$$

```
value(%);
```

$$-\frac{\operatorname{arctanh}\left(\frac{t}{\sqrt{c}}\right)}{\sqrt{c}}$$

Continuing then from above

$$\int^{x} dx/\sqrt{R} \ = - 2 \int^{t(x)} dt\,/(t^2 - c) \ = + (2/\sqrt{c}\,)\tanh^{-1}(t/\sqrt{c}\,)\Big|^{t=t(x)}$$

$$= (2/\sqrt{c}\,)\tanh^{-1}\,[\,(-\sqrt{a} + \sqrt{R}\,)/(x\sqrt{c}\,)]$$

$$= (2/\sqrt{c}\,)\tanh^{-1}\,[\,\frac{(-\sqrt{a} + \sqrt{R}\,)}{x\sqrt{c}}\,]$$

I guess I have assumed that a > 0 and c > 0 just by writing $\sqrt{a}$ and $\sqrt{c}$. GR7 does not give a result in this form. Let's first just check doing the derivative

$$d_x\,\{\,(2/\sqrt{c}\,)\tanh^{-1}\,[\,\frac{(-\sqrt{a} + \sqrt{R}\,)}{x\sqrt{c}}\,]\,\} \ = (2/\sqrt{c}\,)\,d_x\,\{\ \tanh^{-1}\,[\,\frac{(-\sqrt{a} + \sqrt{R}\,)}{x\sqrt{c}}\,]\,\}$$

$$= (2/\sqrt{c}\,)\,d_x\,\{\ \tanh^{-1}\,u(x)\,\} \qquad \text{// recall } d_x\sqrt{R} \ = (1/2)(1/\sqrt{R}\,)(b + 2cx)$$

$$= (2/\sqrt{c}\,)\,(1-u^2)^{-1} d_x\,[\,\frac{(-\sqrt{a} + \sqrt{R}\,)}{x\sqrt{c}}\,] \ = \ (2/\sqrt{c}\,)\,(1-u^2)^{-1}\{\,x\sqrt{c}\ \ d_x\sqrt{R} \ - (-\sqrt{a} + \sqrt{R}\,)\sqrt{c}\,\}\,/\,x^2c$$

$$= (2/\sqrt{c}\,)\,(1-u^2)^{-1}\{\,x\sqrt{c} \ \ (1/2)(1/\sqrt{R}\,)(b + 2cx) - (-\sqrt{a} + \sqrt{R}\,)\sqrt{c}\,\}\,/\,x^2c$$

$$= (2/\sqrt{c}\,)\,(1-u^2)^{-1}\sqrt{c}\,\{\,x \ \ (1/2)(1/\sqrt{R}\,)(b + 2cx) - (-\sqrt{a} + \sqrt{R}\,)\}\,/\,x^2c$$

$$= 2\,(1-u^2)^{-1}\,\{\,x \ \ (1/2)(1/\sqrt{R}\,)(b + 2cx) - (-\sqrt{a} + \sqrt{R}\,)\}\,/\,x^2c$$

Meanwhile:

$$u = \frac{(-\sqrt{a} + \sqrt{R})}{x\sqrt{c}}$$

$$1 - u^2 = 1 - [\frac{(-\sqrt{a} + \sqrt{R})}{x\sqrt{c}}]^2 = [\, x^2c - (-\sqrt{a} + \sqrt{R})^2 \,] / x^2c$$

$$= [\, x^2c - (a - 2\sqrt{a}\sqrt{R} + R) \,] / x^2c$$

$$= [\, x^2c - a + 2\sqrt{a}\sqrt{R} - R) \,] / x^2c$$

Now we end up with

$$d_x(sol) = 2 \{ x^2c / [\, x^2c - a + 2\sqrt{a}\sqrt{R} - R) \,]\} * \{ x\ (1/2)(1/\sqrt{R})(b + 2cx) - (-\sqrt{a} + \sqrt{R})\} / x^2c$$

$$= 2 \{1/ [\, x^2c - a + 2\sqrt{a}\sqrt{R} - R) \,]\} * \{ x\ (1/2)(1/\sqrt{R})(b + 2cx) + \sqrt{a} - \sqrt{R})\}$$

$$= 2 \{1/ [\, x^2c - a + 2\sqrt{a}\sqrt{R} - R) \,]\} * \{ x\ (1/2)(b + 2cx) + \sqrt{a}\sqrt{R} - R)\} * (1/\sqrt{R})$$

$$= 1 \{ [\, x(b + 2cx) + 2\sqrt{a}\sqrt{R} - 2R)] / [\, x^2c - a + 2\sqrt{a}\sqrt{R} - R) \,]\} * (1/\sqrt{R})$$

$$= N/D * (1/\sqrt{R})$$

To make things work, I would need to have N = D. But,

$$N = x(b + 2cx) + 2\sqrt{a}\sqrt{R} - 2R) = xb + 2xc^2 + 2\sqrt{a}\sqrt{R} - R - a - bx - cx^2$$

$$= xc^2 + 2\sqrt{a}\sqrt{R} - R - a$$

Meanwhile from above we have

$$D = x^2c - a + 2\sqrt{a}\sqrt{R} - R$$

Wow, so I have N = D, and the result is verified! So I have a "new form" of the integral,

$$I \equiv \int^x dx/\sqrt{R} = (2/\sqrt{c})\, \tanh^{-1}[\frac{(-\sqrt{a} + \sqrt{R})}{x\sqrt{c}}]$$

which I have directly verified by differentiation.

Question: Why would this still be valid if $\sqrt{a} \rightarrow -\sqrt{a}$ doing the "blue" pathway. Maybe it would help to write the above in its log form using

$\tanh^{-1}u = (1/2)\ln[(1+u)/(1-u)]$ $\qquad u = (-\sqrt{a} + \sqrt{R})/(x\sqrt{c})$

$$= (1/2)\ln\left[\frac{1 + (-\sqrt{a} + \sqrt{R})/(x\sqrt{c})}{1 - (-\sqrt{a} + \sqrt{R})/(x\sqrt{c})}\right]$$

$$= (1/2)\ln\left[\frac{x\sqrt{c} + (-\sqrt{a} + \sqrt{R})}{x\sqrt{c} - (-\sqrt{a} + \sqrt{R})}\right] = (1/2)\ln\left[\frac{x\sqrt{c} - \sqrt{a} + \sqrt{R}}{x\sqrt{c} + \sqrt{a} - \sqrt{R}}\right]$$

Now make the change $\sqrt{a} \rightarrow -\sqrt{a}$ to get

$$[\text{blue}] = (1/2)\ln\left[\frac{x\sqrt{c} + \sqrt{a} + \sqrt{R}}{x\sqrt{c} - \sqrt{a} - \sqrt{R}}\right] =$$

To show these are equivalent, I have to show that

$$\ln\left[\frac{x\sqrt{c} - \sqrt{a} + \sqrt{R}}{x\sqrt{c} + \sqrt{a} - \sqrt{R}}\right] - \ln\left[\frac{x\sqrt{c} + \sqrt{a} + \sqrt{R}}{x\sqrt{c} - \sqrt{a} - \sqrt{R}}\right] = \text{constant independent of x}$$

which says

$$\ln\left[\frac{x\sqrt{c} - \sqrt{a} + \sqrt{R}}{x\sqrt{c} + \sqrt{a} - \sqrt{R}} * \frac{x\sqrt{c} - \sqrt{a} - \sqrt{R}}{x\sqrt{c} + \sqrt{a} + \sqrt{R}}\right] = \text{const} \;?? \quad = \ln[\text{num/den}]$$

$$\text{num} = (x\sqrt{c} - \sqrt{a})^2 - R = x^2c + a - 2x\sqrt{c}\sqrt{a} - a - bx - cx^2 = -2x\sqrt{c}\sqrt{a} - bx$$

$$\text{den} = (x\sqrt{c} + \sqrt{a})^2 - R = x^2c + a + 2x\sqrt{c}\sqrt{a} - a - bx - cx^2 = +2x\sqrt{c}\sqrt{a} - bx$$

Note that

$$\text{num/den} = (-2x\sqrt{c}\sqrt{a} - bx)/(2x\sqrt{c}\sqrt{a} - bx) = (-2\sqrt{c}\sqrt{a} - b)/(2\sqrt{c}\sqrt{a} - b)$$

$$= -4ac + b^2$$

Thus we have

$\ln[\text{num/den}] = \ln[-4ac + b^2]$ and this IS in fact a constant independent of x.

Fact: If you do the blue path and take $\sqrt{a} \rightarrow -\sqrt{a}$, you get another valid solution!

$$I \equiv \int^{x} dx/\sqrt{R} = (2/\sqrt{c})\tanh^{-1}\left[\frac{(-\sqrt{a} + \sqrt{R})}{x\sqrt{c}}\right] + K$$

$$I \equiv \int^{x} dx/\sqrt{R} = (2/\sqrt{c})\,\tanh^{-1}\left[\frac{(+\sqrt{a}+\sqrt{R})}{x\sqrt{c}}\right] + K'$$

Here is Maple confirming that these are both valid forms:

```
J1 := (2/sqrt(c))* arctanh((sqrt(a) + sqrt(R))/(x*sqrt(c)));
```

$$J1 := 2\,\frac{\operatorname{arctanh}\left(\frac{\sqrt{a}+\sqrt{a+b\,x+c\,x^2}}{x\,\sqrt{c}}\right)}{\sqrt{c}}$$

```
diff(J1,x): simplify(%);
```

$$\frac{1}{\sqrt{a+b\,x+c\,x^2}}$$

```
J2 := (2/sqrt(c))* arctanh((-sqrt(a) + sqrt(R))/(x*sqrt(c)))
```

$$J2 := 2\,\frac{\operatorname{arctanh}\left(\frac{-\sqrt{a}+\sqrt{a+b\,x+c\,x^2}}{x\,\sqrt{c}}\right)}{\sqrt{c}}$$

```
diff(J2,x): simplify(%);
```

$$\frac{1}{\sqrt{a+b\,x+c\,x^2}}$$

How to I express this in terms of other hyperbolics? Start this way with the second solution above,

$$(\sqrt{c}/2)\,I = \tanh^{-1}\left[\frac{(\sqrt{a}+\sqrt{R})}{x\sqrt{c}}\right] \qquad \text{OK}$$

$$\tanh\{(\sqrt{c}/2)\,I\} = \frac{(\sqrt{a}+\sqrt{R})}{x\sqrt{c}} \qquad \text{OK}$$

$$\tanh^2\{(\sqrt{c}/2)\,I\} = \left[\frac{(\sqrt{a}+\sqrt{R})}{x\sqrt{c}}\right]^2$$

Then

$$\operatorname{sech}^2\{(\sqrt{c}/2)\,I\} = 1 - \tanh^2\{(\sqrt{c}/2)\,I\}$$

$$= 1 - \left[\frac{(\sqrt{a}+\sqrt{R})}{x\sqrt{c}}\right]^2$$

$$= [\,cx^2 - (\sqrt{a}+\sqrt{R})^2\,] / (x\sqrt{c})^2$$

$$\operatorname{sech}\{(\sqrt{c}/2)\,I\} = [\,cx^2 - (\sqrt{a}+\sqrt{R})^2\,]^{1/2} / (x\sqrt{c})$$

$$(\sqrt{c}/2)\,I = \operatorname{sech}^{-1}\,[\,cx^2 - (\sqrt{a}+\sqrt{R})^2\,]^{1/2} / (x\sqrt{c})$$

$$I = (2/\sqrt{c})\ \text{sech}^{-1}\ \{ [\, cx^2 - (\sqrt{a} + \sqrt{R})^2 \,]^{1/2} / (x\sqrt{c}) \}$$

$$= (2/\sqrt{c})\ \text{sech}^{-1}\ \{ [\, -2a - bx - 2\sqrt{a}\sqrt{R} \,]^{1/2} / (x\sqrt{c}) \}$$

$$= (2/\sqrt{c})\ \text{sech}^{-1} \left(\frac{\sqrt{c}\sqrt{-2a - bx - 2\sqrt{a}\sqrt{R}}}{x} \right)$$

$$= (2/\sqrt{c})\ \text{sech}^{-1} \left(\frac{\sqrt{c}\sqrt{cx^2 - (\sqrt{a} + \sqrt{R})^2}}{x} \right)$$

I have not verified this result by differentiation, it is just a waypoint. But since later result is verified, this must be valid as well.

Now

$$\cosh^2 \{(\sqrt{c}/2)\, I\} = \{ \text{sech}^2 \{(\sqrt{c}/2)\, I\}^{-1}$$

$$= (x\sqrt{c})^2 / [\, cx^2 - (\sqrt{a} + \sqrt{R})^2 \,]$$

$$\cosh\{(\sqrt{c}/2)\, I\} = (x\sqrt{c}) / [\, cx^2 - (\sqrt{a} + \sqrt{R})^2 \,]^{1/2}$$

$$(\sqrt{c}/2)\, I = \cosh^{-1} \{ (x\sqrt{c}) / [\, cx^2 - (\sqrt{a} + \sqrt{R})^2 \,]^{1/2} \}$$

$$I = (2/\sqrt{c})\cosh^{-1} \{ (x\sqrt{c}) / [\, cx^2 - (\sqrt{a} + \sqrt{R})^2 \,]^{1/2} \}$$

$$= (2/\sqrt{c})\cosh^{-1} \{ (x\sqrt{c}) / [\, cx^2 - a - 2\sqrt{a}\sqrt{R} - R \,]^{1/2} \}$$

$$= (2/\sqrt{c})\cosh^{-1} \{ (x\sqrt{c}) / [\, -2a - bx - 2\sqrt{a}\sqrt{R} \,]^{1/2} \}$$

I enter the first form into Maple as J3 because I want Maple to do what seems to me is a messy derivative,

```
J3 := (2/sqrt(c))* arccosh( (x*sqrt(c))/ sqrt(c*x^2 - (sqrt(a)+sqrt(R))^2));
```

$$J3 := 2\,\frac{\operatorname{arccosh}\left(\dfrac{x\sqrt{c}}{\sqrt{c\,x^2 - (\sqrt{a}+\sqrt{R})^2}}\right)}{\sqrt{c}}$$

I replace $R = a + bx + cx^2$ and *then* I differentiate to get

```
Q := diff(J3,x): Q := simplify(%);
```

$$Q = \frac{4\sqrt{a+bx+cx^2}\,a + 4a^{\left(\frac{3}{2}\right)} + 3x\sqrt{a}\,b + 2\sqrt{a}\,cx^2 + x\sqrt{a+bx+cx^2}\,b}{(2a + 2\sqrt{a}\sqrt{a+bx+cx^2} + bx)\sqrt{-2a - 2\sqrt{a}\sqrt{a+bx+cx^2} - bx}\,\sqrt{a+bx+cx^2}\,\sqrt{\dfrac{x\sqrt{c} - \sqrt{-2a - 2\sqrt{a}\sqrt{a+bx+cx^2} - bx}}{\sqrt{-2a - 2\sqrt{a}\sqrt{a+bx+cx^2} - bx}}}\,\sqrt{\dfrac{x\sqrt{c} + \sqrt{-2a - 2\sqrt{a}\sqrt{a+bx+cx^2} - bx}}{\sqrt{-2a - 2\sqrt{a}\sqrt{a+bx+cx^2} - bx}}}}$$

This is a big mess, and Maple refuses to simplify the denominator in the obvious manner that a person would do, and I don't know how to beat this problem. So my next step is to break Q = Qnum/Qden and replace $\sqrt{a+bx+cx^2}$ by "r". This then gives for the denominator

$$Qden := (2a+2\sqrt{a}\sqrt{r}+bx)\sqrt{-2a-2\sqrt{a}\sqrt{r}-bx}\sqrt{r}\sqrt{\frac{x\sqrt{c}-\sqrt{-2a-2\sqrt{a}\sqrt{r}-bx}}{\sqrt{-2a-2\sqrt{a}\sqrt{r}-bx}}}\sqrt{\frac{x\sqrt{c}+\sqrt{-2a-2\sqrt{a}\sqrt{r}-bx}}{\sqrt{-2a-2\sqrt{a}\sqrt{r}-bx}}}$$

I *manually* simplify that to get

$$Qden = (2a+2\sqrt{a}\sqrt{r}+bx)\,F^{1/2}\sqrt{r}\,[x^2c-(-2a-2\sqrt{a}\sqrt{r}-bx)]^{1/2}\,F^{-1/2}$$

$$= (2a+2\sqrt{a}\sqrt{r}+bx)\,\sqrt{r}\,[x^2c+2a+2\sqrt{a}\sqrt{r}+bx]^{1/2}$$

In the square root I replace $x^2c = r - a - bx$ to get

$$[x^2c+2a+2\sqrt{a}\sqrt{r}+bx]^{1/2} = [r-a-bx+2a+2\sqrt{a}\sqrt{r}+bx]^{1/2} = [r+a+2\sqrt{a}\sqrt{r}]^{1/2}$$

$$= [(\sqrt{r}+\sqrt{a})^2]^{1/2}a = (\sqrt{r}+\sqrt{a})$$

which is a crucial step. I then have this greatly simplified denominator,

$$Qden = (2a+2\sqrt{a}\sqrt{r}+bx)\,\sqrt{r}\,\,(\sqrt{r}+\sqrt{a})$$

I then manually enter this into Maple to override the previous expression for Qden.,

```
Qden := (2*a+2*sqrt(a)*sqrt(r)+b*x)*sqrt(r)*(sqrt(a)+sqrt(r));
```

$$Qden := (2a+2\sqrt{a}\sqrt{r}+bx)\sqrt{r}\,(\sqrt{a}+\sqrt{r})$$

I want now to show that Qnum/Qden = $1/\sqrt{r}$ and then I will have verified the above integral I. So define

$$\delta = Qden/\sqrt{r} - Qnum$$

and the goal then is to show that $\delta = 0$:

```
Qnum := subs(sqrt(a+b*x+c*x^2) = sqrt(r),Qnum);
```

$$Qnum := 4\sqrt{r}\,a + 4\,a^{\left(\frac{3}{2}\right)} + 3\,x\sqrt{a}\,b + 2\sqrt{a}\,c\,x^2 + x\sqrt{r}\,b$$

```
Qnum - Qden/sqrt(r): delta := expand(%);
```

$$\delta := 2\,a^{\left(\frac{3}{2}\right)} + 2\,x\sqrt{a}\,b + 2\sqrt{a}\,c\,x^2 - 2\sqrt{a}\,r$$

```
r := a+b*x+c*x^2;
```

$$r := a + b\,x + c\,x^2$$

```
delta: simplify(%);
```

$$0$$

This took me about 1.5 days to get right! So at this point I know have several forms for my integral

$$I \equiv \int^{x} dx/\sqrt{R}$$

Here is what I have so far, where each form has been verified by differentiation. The first two come using **Method A** above,

$I = -(1/\sqrt{c})\ \ln[\,-2cx - b + 2\sqrt{cR}\,] + \text{const}$ // this is PL(x) from above (1)

$I = (1/\sqrt{c})\ \ln[\,+2cx + b + 2\sqrt{cR}\,] + \text{const}$ // this is GL7(x) from above (Maple too) (2)

Form (2) agrees with GR7 and with Maple's evaluation of this integral. I showed that form (1) is valid if form (2) is valid. Form (2) also appears in Schaum p 72 top.

If $b^2 = 4ac$, then item 5a. above tells us that $\sqrt{R} = \sqrt{c}\,(x + [b/2c])$. In this case we get

$$2cx + b + 2\sqrt{cR} = 2cx + b + 2\sqrt{c}\sqrt{c}\,(x + [b/2c]) = 2cx + b + 2cx + b = 4cx + 2b$$

Then form (2) says

$$I(\Delta=0) = (1/\sqrt{c})\ \ln[4cx + 2b] = (1/\sqrt{c})\ \ln[2cx + b]$$

in agreement with GR7. Alternatively, we can do the integral directly for $\Delta = 0$ to get,

$$\int^{x} dx/\sqrt{R} = \int^{x} dx\ \frac{1}{\sqrt{c}\,(x + [b/2c])} = (1/\sqrt{c})\ \int^{x} dx\ \frac{1}{(x + [b/2c])}$$

$$= (1/\sqrt{c})\ \ln(x + [b/2c]) = (1/\sqrt{c})\ \ln\left(\frac{2cx+b}{2c}\right) = (1/\sqrt{c})\ \ln(2cx + b)$$

From **Method B** I get two more diff-verified forms

$$I \equiv \int^{x} dx/\sqrt{R} = (2/\sqrt{c})\tanh^{-1}\left[\frac{(-\sqrt{a}+\sqrt{R})}{x\sqrt{c}}\right] + \text{const} \qquad (3)$$

$$I \equiv \int^{x} dx/\sqrt{R} = (2/\sqrt{c})\tanh^{-1}\left[\frac{(+\sqrt{a}+\sqrt{R})}{x\sqrt{c}}\right] + \text{const} \qquad (4)$$

I then convert this to some other forms. First I get,

$$I = (2/\sqrt{c})\ \text{sech}^{-1}\left(\frac{\sqrt{c}\sqrt{cx^2 - (\sqrt{a}+\sqrt{R})^2}}{x}\right) + \text{const} \qquad (5)$$

$$= (2/\sqrt{c})\ \text{sech}^{-1}\left(\frac{\sqrt{c}\sqrt{-2a - bx - 2\sqrt{a}\sqrt{R}}}{x}\right) + \text{const} \qquad (6)$$

Then comes the most recent result which is diff-verified,

$$I = (2/\sqrt{c})\cosh^{-1}\left(\frac{x\sqrt{c}}{\sqrt{cx^2 - (\sqrt{a}+\sqrt{R})^2}}\right) + \text{const}$$

11. Try to convert this to $\sin^{-1}$ form (did not work)

$$\cosh\{(\sqrt{c}/2)\,I\} = \frac{x\sqrt{c}}{\sqrt{cx^2 - (\sqrt{a}+\sqrt{R})^2}}$$

$= \cosh\{(-i\sqrt{-c}/2)\,I\}$ $\qquad\qquad$ $\sqrt{-c} = i\sqrt{c}$ = my choice of sign

$= \cosh\{(i\sqrt{-c}/2)\,I\} = \cos\{(\sqrt{-c}/2)\,I\}$

So at this point I have

$$\cos\{(\sqrt{-c}/2)\,I\} = \frac{x\sqrt{c}}{\sqrt{cx^2 - (\sqrt{a}+\sqrt{R})^2}}$$

Now use a half-angle formula to write

$$\cos(2x) = 2\cos^2(x) - 1$$

or

$$\cos(x) = 2\cos^2(x/2) - 1$$

Then have

$$\cos\{(\sqrt{-c})\,I\} = 2\left[\frac{x\sqrt{c}}{\sqrt{cx^2 - (\sqrt{a}+\sqrt{R})^2}}\right]^2 - 1 = 2\frac{cx^2}{cx^2 - (\sqrt{a}+\sqrt{R})^2} - 1$$

$$= [\ 2cx^2\ - cx^2\ + (\sqrt{a}\ + \sqrt{R}\ )^2\ ]\ /\ [cx^2\ - (\sqrt{a}\ + \sqrt{R}\ )^2]$$

$$= [\ cx^2\ + (\sqrt{a}\ + \sqrt{R}\ )^2\ ]\ /\ [cx^2\ - (\sqrt{a}\ + \sqrt{R}\ )^2]$$

So my latest result is then this

$$I\ = (1/\sqrt{-c}\ )\ \cos^{-1}\left(\frac{cx^2\ + (\sqrt{a}\ + \sqrt{R}\ )^2}{cx^2\ - (\sqrt{a}\ + \sqrt{R}\ )^2}\right)$$

Now since $\cos^{-1} = \pi/2 - \sin^{-1}$, this implies that

$$I\ = -\ (1/\sqrt{-c}\ )\ \sin^{-1}\left(\frac{cx^2\ + (\sqrt{a}\ + \sqrt{R}\ )^2}{cx^2\ - (\sqrt{a}\ + \sqrt{R}\ )^2}\right)\ + K_1$$

How do you get from this result to this GR7 result

$$= \frac{-1}{\sqrt{-c}} \arcsin\left(\frac{2cx+b}{\sqrt{-\Delta}}\right) + K_2$$

You would have to show that

$$-\ (1/\sqrt{-c}\ )\ \sin^{-1}\left(\frac{cx^2\ + (\sqrt{a}\ + \sqrt{R}\ )^2}{cx^2\ - (\sqrt{a}\ + \sqrt{R}\ )^2}\right)\ + K_1 \quad = -\ (1/\sqrt{-c}\ )\ \sin^{-1}\left(\frac{2cx+b}{\sqrt{b^2-4ac}}\right)\ + K_2$$

or

$$-\ (1/\sqrt{-c}\ )\ \sin^{-1}\left(\frac{cx^2\ + (\sqrt{a}\ + \sqrt{R}\ )^2}{cx^2\ - (\sqrt{a}\ + \sqrt{R}\ )^2}\right)\ = -\ (1/\sqrt{-c}\ )\ \sin^{-1}\left(\frac{2cx+b}{\sqrt{b^2-4ac}}\right)\ + K_3$$

or

$$\sin^{-1}\left(\frac{cx^2\ + (\sqrt{a}\ + \sqrt{R}\ )^2}{cx^2\ - (\sqrt{a}\ + \sqrt{R}\ )^2}\right)\ = \sin^{-1}\left(\frac{2cx+b}{\sqrt{b^2-4ac}}\right)\ -\ \sqrt{-c}\ K_3$$

That looks like a lot of work to prove or disprove.

You might have hoped that

$$\frac{cx^2\ + (\sqrt{a}\ + \sqrt{R}\ )^2}{cx^2\ - (\sqrt{a}\ + \sqrt{R}\ )^2} \quad = \frac{2cx+b}{\sqrt{b^2-4ac}} \quad = \frac{N}{D}$$

or

$$N\sqrt{b^2-4ac}\ \ = D\ (2cx+b)$$

or

$$N^2(b^2-4ac) = D^2(2cx+b)^2$$

or

T1 = T2

Try this in Maple. // It does not work, they are not equal.

12. Go back to the log form which is perhaps the simplest and most flexible,

$I = (1/\sqrt{c})\ \ln [\ 2cx + b + 2\sqrt{cR}\ ]$ // this is GL7(x)

I am allowed to replace this with

$I = (1/\sqrt{c})\ \ln [\ (2cx + b + 2\sqrt{cR}\ )\ f(a,b,c)]$

where f is an arbitrary function!

Can I get the argument of ln to have this form

$y + \sqrt{y^2+1} = (2cx + b + 2\sqrt{cR}\ )$

$\sqrt{y^2+1} = (2cx + b + 2\sqrt{cR}\ ) - y$

$(y^2+1) = (2cx + b + 2\sqrt{cR} - y)^2 f^2 = y^2 - 2(2cx + b + 2\sqrt{cR}\ )y + (2cx + b + 2\sqrt{cR}\ )^2$

so

$1 = -2(2cx + b + 2\sqrt{cR}\ )y + (2cx + b + 2\sqrt{cR}\ )^2$

$2(2cx + b + 2\sqrt{cR}\ )y = (2cx + b + 2\sqrt{cR}\ )^2 - 1$

$$y = \frac{(2cx + b + 2\sqrt{cR}\ )^2 - 1}{2(2cx + b + 2\sqrt{cR}\ )}$$

OK, then according to Schaum p 29 I can claim that

$I = (1/\sqrt{c})\ \ln [\ 2cx + b + 2\sqrt{cR}\ ] = (1/\sqrt{c})\ \ln [y + \sqrt{y^2+1}\ ]$

$= (1/\sqrt{c})\ \sinh^{-1} y$

$$= (1/\sqrt{c})\ \sinh^{-1} \left[\frac{(2cx + b + 2\sqrt{cR}\ )^2 - 1}{2(2cx + b + 2\sqrt{cR}\ )}\right]$$

That was pretty fast. Now Schaum p 72 claims this should be

$$= (1/\sqrt{c})\ \sinh^{-1} \frac{2cx+b}{\sqrt{4ac-b^2}}$$

I don't see how that can happen even accounting for a generic constant. You would need to have this be true,

$$(1/\sqrt{c})\ \sinh^{-1}\left[\frac{(2cx + b + 2\sqrt{cR})^2 - 1}{2(2cx + b + 2\sqrt{cR})}\right] = (1/\sqrt{c})\ \sinh^{-1}\frac{2cx+b}{\sqrt{4ac-b^2}} + f(a,b,c)$$

or

$$\sinh^{-1}\left[\frac{(2cx + b + 2\sqrt{cR})^2 - 1}{2(2cx + b + 2\sqrt{cR})}\right] - \sinh^{-1}\frac{2cx+b}{\sqrt{4ac-b^2}} = g(a,b,c)$$

or

$$sh^{-1}X - sh^{-1}Y = g$$

or

$$\alpha - \beta = g \qquad sh\alpha = X \qquad sh\beta = Y$$

OK, apply sh to get

$$sh(\alpha-\beta) = sh(g)$$

$$sh\alpha\ ch\beta - ch\alpha sh\beta = sh(g)$$

$$sh\alpha\sqrt{1 + sh^2\alpha} - sh\beta\sqrt{1 + sh^2\beta} = sh(g)$$

$$X\sqrt{1+X^2} - Y\sqrt{1+Y^2} = sh(g)$$

Now we have

$$X = \frac{(2cx + b + 2\sqrt{cR})^2 - 1}{2(2cx + b + 2\sqrt{cR})} \qquad Y = \frac{2cx+b}{\sqrt{4ac-b^2}}$$

I can ask Maple now whether it is possible that $X\sqrt{1+X^2} - Y\sqrt{1+Y^2}$ = a function not of x ! It seems that this has to be true for both forms for I to be correct. But this is too much mess for Maple, needs to be spoon fed. So let

$$A = (2cx+b+2\sqrt{cR}) \qquad X = (A^2-1)/2A \qquad 1+X^2 = 1 + (A^2-1)/2A = [2A+A^2-1]/(2A)$$

$$B = \sqrt{4ac-b^2} \qquad Y = (2cx+b)/B \qquad 1+Y^2 = 1 + (2cx+b)/B = [B+2cx+b]/B$$

Then

$$\sqrt{1+X^2} = \sqrt{A^2+2A-1})\ /\sqrt{2A}$$

$$\sqrt{1+Y^2} = \sqrt{B+2cx+b}\ /\sqrt{B}$$

$$X\sqrt{1+X^2} - Y\sqrt{1+Y^2} = (A^2-1)/2A * \sqrt{A^2+2A-1})\ /\sqrt{2A} - (2cx+b)/B * \sqrt{B+2cx+b}\ /\sqrt{B}$$

$$= (A^2-1)\sqrt{A^2+2A-1})\,(2A)^{-3/2} - (2cx+b)\sqrt{B+2cx+b}\,B^{-1/2}$$

$$(2A)^{3/2}\,B^{1/2}\,[\,X\sqrt{1+X^2} - Y\sqrt{1+Y^2}\,] = (A^2-1)\sqrt{A^2+2A-1})\,B^{1/2} - (2cx+b)\sqrt{B+2cx+b}\,(2A)^{3/2}$$

This is something Maple could evaluate. I could then want to show that when evaluated, you get

$$(A^2-1)\sqrt{A^2+2A-1})\,B^{1/2} - (2cx+b)\sqrt{B+2cx+b}\,(2A)^{3/2} = (2A)^{3/2}\,B^{1/2}\,\mathrm{sh}(g)$$

where g does not depend on x. I would say this has a 1% chance at most. // Did not work.

STOP. I need a more direct way to get the sh^{-1} form of the integral.

$$\mathrm{sh}\,(\mathrm{sh}^{-1}X - \mathrm{sh}^{-1}Y) = \mathrm{shg}$$

or

$$\mathrm{sh}\,(\mathrm{sh}^{-1}X)\mathrm{ch}(\mathrm{sh}^{-1}Y) - \mathrm{ch}(\mathrm{sh}^{-1}X)\,\mathrm{sh}(\mathrm{sh}^{-1}Y) = \mathrm{shg}$$

or

$$X\,\mathrm{ch}(\mathrm{sh}^{-1}Y) - \mathrm{ch}(\mathrm{sh}^{-1}X)\,Y = \mathrm{shg}$$

13. Try adding a factor f(a,b,c)

Suppose I insert my arbitrary function by replacing $(2cx + b + 2\sqrt{cR}) \rightarrow (2cx + b + 2\sqrt{cR})\,f$. Then

$$= (1/\sqrt{c})\ \sinh^{-1}\left[\frac{(2cx + b + 2\sqrt{cR})^2f^2 - 1}{2(2cx + b + 2\sqrt{cR})f}\right]$$

Can I find a function f(a,b,c) such that

$$\frac{(2cx + b + 2\sqrt{cR})^2f^2 - 1}{2(2cx + b + 2\sqrt{cR})f} = \frac{2cx+b}{\sqrt{4ac-b^2}}$$

Define $g \equiv (2cx + b + 2\sqrt{cR})$. Then we have

$$(g^2f^2-1)/2gf = \frac{2cx+b}{\sqrt{4ac-b^2}}$$

$$(g^2f^2-1) = \frac{2cx+b}{\sqrt{4ac-b^2}}\,2gf$$

$$g^2f^2 - \frac{2cx+b}{\sqrt{4ac-b^2}}\,2gf - 1 = 0$$

or

$(gf)^2 + B(gf) - 1 = 0$ $\qquad B = -\dfrac{2cx+b}{\sqrt{4ac-b^2}}$

This says

$$fg = [\ -B \pm \sqrt{B^2+4}\ ]/2$$

Now take a look see:

$$B^2+4 = (2cx+b)^2/(4ac-b^2) + 4(4ac-b^2)/(4ac-b^2)$$

$$= [\ (2cx+b)^2 + 4(4ac-b^2)] / (4ac-b^2)$$

$$= [\ 4c^2x^2 + b^2 + 4cxb + 16ac - 4b^2] / (4ac-b^2)$$

$$= [\ 4c^2x^2 - 3b^2 + 4cxb + 16ac\ ] / (4ac-b^2)$$

Then you get this ugly result

$$fg = [\ \frac{2cx+b}{\sqrt{4ac-b^2}} \pm \sqrt{4c^2x^2 - 3b^2 + 4cxb + 16ac}\ / \sqrt{4ac-b^2}\ ]\ /2$$

or

$$2fg = [\ 2cx+b \pm \sqrt{4c^2x^2 - 3b^2 + 4cxb + 16ac}\ ]\ / \sqrt{4ac-b^2} \qquad g \equiv (2cx + b + 2\sqrt{cR}\ )$$

This is leading nowhere.

Status: I am unable to derive the $\sin^{-1}$ or $\sinh^{-1}$ forms.

14. Start once again

Assume *the exact form* given in GR7 so that

$$I = (1/\sqrt{c}\ )\ \sinh^{-1}(y) \qquad y = [2cx+b+2\sqrt{cR}\ ]\ /\ \sqrt{4ac-b^2}$$

Then another form for the solution *should be*

$$I = (1/\sqrt{c}\ )\ \ln [\ y + \sqrt{y^2+1}\ ]$$

To help Maple, define

$$A = (2cx+b+2\sqrt{cR}\ )$$

$$B = \sqrt{4ac-b^2}$$

$y = A/B$

Then another form for I *should be*

$$I = (1/\sqrt{c}\,)\ln\left[\,(A/B) + \sqrt{(A/B)^2+1}\,\right]$$

$$= (1/\sqrt{c}\,)\ln\left[\,(A/B) + (1/B)\sqrt{(A^2+B^2)}\,\right]$$

$$= (1/\sqrt{c}\,)\ln\left[A + \sqrt{A^2+B^2}\,\right] \qquad \text{// dropping ln(B)}$$

What we really want is to find that

$$A + \sqrt{A^2+B^2} = f(a,b,c)\,A$$

and then you end up with the GR7 ln form for I. So what you need is

$$\sqrt{A^2+B^2} = g\,A$$

$$(A^2+B^2) = g^2A^2 \qquad\qquad A = (2cx+b+2\sqrt{cR}\,)$$

This seems very unlikely. Maple says,

```
(A^2+B^2): expand(%);
```

$$8c^2x^2+8cbx+8\sqrt{ac+cbx+c^2x^2}\,cx+4\sqrt{ac+cbx+c^2x^2}\,b+8ac$$

```
A^2: expand(%);
```

$$8c^2x^2+8cbx+8\sqrt{ac+cbx+c^2x^2}\,cx+b^2+4\sqrt{ac+cbx+c^2x^2}\,b+4ac$$

Do it by hand

$$(2cx+b+2\sqrt{cR}\,)^2 + (4ac-b^2) = g^2\,(2cx+b+2\sqrt{cR}\,)^2 \qquad ??$$

This DOES NOT FLY, so I don't thing to two forms are linked in this manner.