

The Euler Substitutions

Phil Lucht

Rimrock Digital Technology, Salt Lake City, Utah 84103

last update: Nov 6, 2016

rimrock@xmission.com

The material in this document is copyrighted by the author.

1. Introduction.....	1
2. Comments about $R = a + bx + cx^2$	4
3. A substitution that does not work.....	5
4. Substitution A.....	7
5. Substitution B.....	10
6. Substitution C.....	14
7. More forms for the integral of $R^{-1/2}$	16
Appendix: About Euler's original paper.....	21
References.....	22

1. Introduction

There are many techniques for the exact analytic evaluation of indefinite integrals. For example, Zwillinger provides a solid list of methods in Section III of his *Handbook of Integration*. In any such list, the very first method is usually "change of variables" by means of a "substitution". To take a trivial example which establishes the variable names we use below, consider how the substitution $t = \sin x$ helps in evaluating a trigonometric integral,

$$\begin{aligned} t(x) &= \sin x & dt &= \cos x \, dx \\ \int^x \sin x \cos x \, dx &= \int^{t(x)} t \, dt = (1/2) t^2|^{t(x)} = (1/2) \sin^2 x . \end{aligned} \quad (1.1)$$

The phrase "Euler substitutions" refers to three substitutions used for evaluating certain integrals involving powers of x along with powers of the radical $\sqrt{a+bx+cx^2}$. These substitutions are briefly reviewed in Section 2.25 (p 92) of Gradshteyn and Ryzhik [GR7]. Wiki suggests that these substitutions appears in many Russian calculus texts, and we have found them mentioned in a book by Piskunov.

Our very elementary purpose in this document is to flesh out the details of using these Euler substitutions and to state the results in a systematic manner.

As an illustration, $\int dx \, 1/\sqrt{a+bx+cx^2}$ is evaluated using each of the substitutions. The results are then transformed into other forms, and all the results are collected in (7.15). This integral is of particular interest to the author because it is used in Goldstein's *Classical Mechanics* to derive equations for

planetary orbits, and there seems to be a benign sign error in that development as noted below. Physicists spend a lot of time worrying about signs of things. Sometimes such errors are related to confusion about how branch cuts are "taken off" for functions of a complex variable, the choice of Riemann sheets, and other esoteric matters, but more often than not the sign error is caused by a trivial mistake in 7th grade algebra.

A certain class of functions of t are called rational functions and have the form

$$\mathcal{R}(t) = \frac{\text{poly}_1(t)}{\text{poly}_2(t)} \quad (1.2)$$

where the numerator and denominator are just polynomials of t with non-negative powers. In fact negative powers are allowed and can be cleared by multiplying top and bottom by some t^n . Any function having the form $\mathcal{R}(t)$ can easily be integrated using the method of partial fractions, and this topic is clearly outlined in Section 2.10 of GR7. In the discussion below, the Euler substitutions result in integrands of the form $\mathcal{R}(t)$ and then we know that the integration from that point on is just turning a crank.

Rational functions of two variables x and y are defined analogously to the above,

$$\mathcal{R}(x,y) = \frac{\text{poly}_1(x,y)}{\text{poly}_2(x,y)} \quad (1.3)$$

where now the numerator and denominator are polynomials in x and y , such as $3x^2 - 2xy^9 + 4 - y$.

The Euler substitutions apply to a class of functions of a single variable x which have this form

$$f(x) = \mathcal{R}(x, \sqrt{a+bx+cx^2}) . \quad (1.4)$$

For example, a typical such function might be

$$\mathcal{R}(x, \sqrt{a+bx+cx^2}) = \frac{Ax\sqrt{a+bx+cx^2} + Bx^7(\sqrt{a+bx+cx^2})^3 + x^3}{2x^3\sqrt{a+bx+cx^2} + x^2} . \quad (1.5)$$

Since the numerator terms can be treated separately, one can limit one's interest to the following form,

$$\mathcal{R}(x, \sqrt{a+bx+cx^2}) = \frac{x^m(\sqrt{a+bx+cx^2})^n}{\text{poly}(x, \sqrt{a+bx+cx^2})} . \quad (1.6)$$

As noted, the specific example we shall study is the following, shown here in one of its forms (4.9),

$$\int dx \frac{1}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}} \ln [2cx + b + 2\sqrt{c} \sqrt{a+bx+cx^2}] + \text{constant} \quad c > 0 . \quad (1.7)$$

The above example serves as a good model to look at while reading the comments below concerning the general integrand shown in (1.4) .

Comments:

- In general, any indefinite integral should really be written $\int^x dx f(x) = F(x) + \text{constant}$ where the constant is arbitrary. One can always test a candidate $F(x)$ by computing dF/dx to see if the result is $f(x)$. This task is made easy using Maple or another computer calculus program.

- When there are parameters such as a, b, c shown above, that integration constant can be any function $g(a, b, c)$ which is of course not a function of x .

- We shall use the symbol $A \doteq B$ to mean $A = B + \text{constant}$ in the above sense. Thus we can write

$$\begin{aligned} \int dx \frac{1}{\sqrt{a+bx+cx^2}} &\doteq \frac{1}{\sqrt{c}} \ln [2cx + b + 2\sqrt{c} \sqrt{a+bx+cx^2}] \\ &\doteq \frac{1}{\sqrt{c}} \ln \left[\frac{2cx + b + 2\sqrt{c} \sqrt{a+bx+cx^2}}{\sqrt{4ac-b^2}} \right] . \end{aligned} \quad (1.8)$$

On the second line we have added a denominator which in effect creates the additive constant $g(a, b, c) = - (1/\sqrt{c}) \ln(\sqrt{4ac-b^2})$. Both forms shown above are "correct" and well-defined when $c > 0$ and $4ac-b^2 > 0$.

- If one thinks of x having units of distance L , then $\dim(x) = L$, $\dim(a) = L^2$, $\dim(b) = L$ and $\dim(c) = 1$ make the integral be dimensionless. Then the second form above involves the log of a dimensionless ratio, whereas the first form does not, somewhat clarifying the dimensionless nature of the integral.

- As discussed more below, $\sqrt{a+bx+cx^2}$ is real for certain ranges of a, b, c, x and one can imagine that the integral being evaluated is over a range of x where $\sqrt{a+bx+cx^2}$ is real. One normally thinks of a, b, c as real parameters.

- Once an integral has been evaluated for "reasonable" values of the parameters like a, b, c , one can extend one or more of these parameters to the complex plane allowing one to analytically continue both sides of an integral evaluation. We shall give an example below in Section 7.

Having stated these general comments, we now look specifically at the object $\sqrt{a+bx+cx^2}$.

2. Comments about $R = a + bx + cx^2$

The letters a, b, c are defined consistently with GR7. It is probably more standard to write ax^2+bx+c , in which case one has the familiar rote solution for the roots $[-b \pm \sqrt{b^2-4ac}]/(2a)$, so in our current context one must remember that in fact the roots of $a+bx+cx^2=0$ are given by

$$\alpha_{\pm} \equiv [-b \pm \sqrt{b^2-4ac}]/(2c) . \quad (2.1)$$

The quantity b^2-4ac is often called "the discriminant". GR7 define Δ to be the negative of this discriminant,

$$\Delta \equiv 4ac - b^2 . \quad (2.2)$$

Also consistent with GR7 we define R by

$$R \equiv a+bx+cx^2 = c [x^2 + (b/c)x + (a/c)] = c(x-\alpha_+)(x-\alpha_-) \quad (2.3)$$

and it is for this reason that we have used $\mathcal{R}$ above for the ratio of polynomials. Note that $\alpha_{\pm}$ are the roots of R for $c > 0$, for $c = |c|e^{i\theta}$, and for $c < 0$.

Special case:

$$b^2 = 4ac \ (\Delta = 0) \Rightarrow \alpha_{\pm} = -b/(2c) \Rightarrow R = c(x-\alpha_+)^2 \Rightarrow \sqrt{R} = \sqrt{c} [x + b/(2c)] \quad (2.4)$$

Geometry

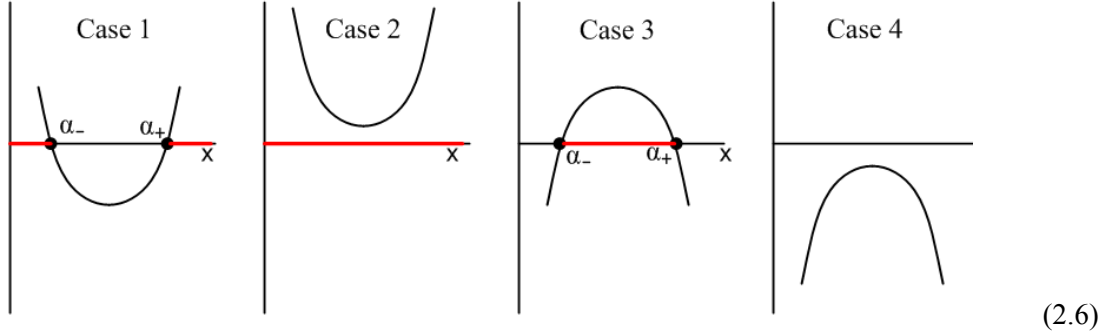
If $b^2 - 4ac > 0$ then the roots in (2.1) are real. This means that the graphed function $f(x) = a + bx + cx^2$ has two intersections with the x axis.

If $c > 0$, the parabola cups up. Therefore to the right of the upper root, $x > \alpha_+$, one has $a + bx + cx^2 > 0$ and so $\sqrt{a+bx+cx^2}$ is real and well defined. It is also real for $x < \alpha_-$.

If $c < 0$, the parabola cups down. In this case $\sqrt{a+bx+cx^2}$ is real and well defined for $\alpha_- < x < \alpha_+$. By well defined we simply mean that the square root implies a positive real number and we don't worry about branches of the square root function.

When talking about integrals involving $\sqrt{a+bx+cx^2}$, it seems best to start with an integral over a range of x where $a + bx + cx^2$ is positive, so then $\sqrt{a+bx+cx^2}$ is real and positive. There are four cases of interest:

- Case 1: $c > 0$, $b^2 - 4ac > 0$ (real roots), cups up, $x > \alpha_+$ or $x < \alpha_-$ to have real $\sqrt{a+bx+cx^2}$
- Case 2: $c > 0$, $b^2 - 4ac < 0$ (imag roots), cups up, all real values of x give real $\sqrt{a+bx+cx^2}$
- Case 3: $c < 0$, $b^2 - 4ac > 0$ (real roots), cups down, must have $\alpha_- < x < \alpha_+$ to have real $\sqrt{a+bx+cx^2}$
- Case 4: $c < 0$, $b^2 - 4ac < 0$ (imag roots), cups down, *no* real value of x gives real $\sqrt{a+bx+cx^2}$ (2.5)



The red bars show the range of x for which $\sqrt{a+bx+cx^2}$ is real.

We shall initially work in Case 2 below. This means $c > 0$ and $b^2 - 4ac < 0$.

3. A substitution that does not work

One's first inclination in evaluating integrals including $\sqrt{a+bx+cx^2}$ might be to make the substitution

$$t(x) = \sqrt{a+bx+cx^2} \quad (3.1)$$

so that

$$\mathcal{R}(x, \sqrt{a+bx+cx^2}) \rightarrow \mathcal{R}(x(t), t) . \quad (3.2)$$

But,

$$\begin{aligned} t^2 &= a+bx+cx^2 \quad \Rightarrow \quad cx^2 + bx + (a-t^2) = 0 \\ \Rightarrow \quad x(t) &= \frac{-b \pm \sqrt{b^2 - 4c(a-t^2)}}{2c} . \end{aligned} \quad (3.3)$$

The result then is,

$$\mathcal{R}(x, \sqrt{a+bx+cx^2}) \rightarrow \mathcal{R}\left(\frac{-b \pm \sqrt{b^2 - 4c(a-t^2)}}{2c}, t\right) . \quad (3.4)$$

The goal is to *remove* square roots from the integrand, but this substitution just replaces one square root with another square root and is thus not very useful, so we reject this substitution.

We are then led to three substitutions which are credited to Leonhard (brave lion) Euler (1707-1783) and are now known as "the Euler substitutions". There is some disagreement about how these are numbered 1,2 and 3 so we instead call them substitutions A,B and C favoring the ordering of Piskunov :

		<u>Piskunov</u>	<u>GR7</u>	<u>Boyadzhiev</u>	
A	$t + \sqrt{c} x = \sqrt{a+bx+cx^2}$	1	2	1	
B	$xt + \sqrt{a} = \sqrt{a+bx+cx^2}$	2	1	3	
C	$t(x-\alpha) = \sqrt{a+bx+cx^2}$	3	3	2	(3.5)

Euler below is squinting at $e^{i\pi} + 1 = 0$ written on his blackboard and is wondering what it all means.



https://en.wikipedia.org/wiki/Leonhard_Euler

(3.6)

Gradshteyn and Ryzhik

Another of Zwillinger's "methods" for doing an indefinite integral is "looking it up" in a table of integrals. The astounding *Table of Integrals, Series, and Products* associated with Gradshteyn and Ryzhik contains, as a small fraction of its content, about 200 pages of indefinite integrals of elementary functions which have accumulated over two centuries. Currently in the editorial hands of Dan Zwillinger and Victor Moll, the book is in its 8th edition, though we continue to use the 7th edition. During the period of each edition, new integrals and errata for old integrals are collected to be incorporated into the next edition. The first edition was published by Russian mathematician Ryzhik in 1941, and he was joined by Gradshteyn in 1951 for the 3rd edition, see wiki.

4. Substitution A

Instead of using the substitution (3.1), consider

$$t(x) = \sqrt{a+bx+cx^2} - \sqrt{c} x \quad \text{or} \quad t + \sqrt{c} x = \sqrt{a+bx+cx^2}. \quad (4.1)$$

When we finish this section we can replace $\sqrt{c} \rightarrow -\sqrt{c}$ everywhere and thereby generate an alternate result. One now has,

$$a + bx + \textcolor{red}{cx}^2 = t^2 + 2\sqrt{c} t x + \textcolor{red}{cx}^2. \quad (4.2)$$

The key idea is that the two cx^2 terms cancel out, giving

$$\begin{aligned} a + bx &= t^2 + 2\sqrt{c} t x \quad \Rightarrow \quad (b - 2\sqrt{c} t)x = (t^2 - a) \quad \Rightarrow \\ x(t) &= \frac{t^2 - a}{b - 2\sqrt{c} t}, \end{aligned} \quad (4.3)$$

an expression with no messy square roots, unlike (3.3). Then,

$$\begin{aligned} \sqrt{a+bx+cx^2} &= t + \sqrt{c} x = t + \sqrt{c} \frac{t^2 - a}{b - 2\sqrt{c} t} = \frac{tb - 2\sqrt{c} t^2}{b - 2\sqrt{c} t} + \frac{\sqrt{c} t^2 - \sqrt{c} a}{b - 2\sqrt{c} t} \\ &= \frac{-\sqrt{c} t^2 + bt - \sqrt{c} a}{b - 2\sqrt{c} t}. \end{aligned} \quad (4.4)$$

Then our general replacement becomes

$$\mathcal{R}(x, \sqrt{a+bx+cx^2}) \rightarrow \mathcal{R}\left(\frac{t^2 - a}{b - 2\sqrt{c} t}, \frac{-\sqrt{c} t^2 + bt - \sqrt{c} a}{b - 2\sqrt{c} t}\right). \quad (4.5)$$

One may then compute

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt} \left(\frac{t^2 - a}{b - 2\sqrt{c} t} \right) = \frac{(b - 2\sqrt{c} t)(2t) - (t^2 - a)(-2\sqrt{c})}{(b - 2\sqrt{c} t)^2} = \frac{2bt - 4\sqrt{c} t^2 + 2\sqrt{c} t^2 - 2a\sqrt{c}}{(b - 2\sqrt{c} t)^2} \\ &= 2 \frac{bt - \sqrt{c} t^2 - a\sqrt{c}}{(b - 2\sqrt{c} t)^2} \end{aligned}$$

so that

$$dx = 2 \frac{-\sqrt{c} t^2 + bt - a\sqrt{c}}{(b - 2\sqrt{c} t)^2} dt. \quad (4.6)$$

Our integral evaluation then becomes

$$\int^x dx \mathcal{R}(x, \sqrt{a+bx+cx^2}) = \int^{t(x)} dt \frac{-\sqrt{c} t^2 + bt - a\sqrt{c}}{(b - 2\sqrt{c} t)^2} * \mathcal{R}\left(\frac{t^2 - a}{b - 2\sqrt{c} t}, \frac{-\sqrt{c} t^2 + bt - \sqrt{c} a}{b - 2\sqrt{c} t}\right)$$

where $t(x) = \sqrt{a+bx+cx^2} - \sqrt{c} x$. (4.7)

Notice that there are no messy square roots anywhere in the dt integrand. The integrand is now a rational function in the variable t : integrand = poly₁(t)/poly₂(t).

As noted earlier, one may replace $\sqrt{c} \rightarrow -\sqrt{c}$ (and $c \rightarrow c$) to obtain the following alternative form,

$$\int^x dx \mathcal{R}(x, \sqrt{a+bx+cx^2}) = \int^{t(x)} dt \frac{\sqrt{c} t^2 + bt + a\sqrt{c}}{(b + 2\sqrt{c} t)^2} * \mathcal{R}\left(\frac{t^2 - a}{b + 2\sqrt{c} t}, \frac{\sqrt{c} t^2 + bt + \sqrt{c} a}{b + 2\sqrt{c} t}\right)$$

where $t(x) = \sqrt{a+bx+cx^2} + \sqrt{c} x$. (4.8)

Example: Let $\mathcal{R}(x, \sqrt{a+bx+cx^2}) = 1/\sqrt{a+bx+cx^2}$. Then using (4.8),

$$\begin{aligned} \int^x dx \frac{1}{\sqrt{a+bx+cx^2}} &= \int^{t(x)} dt \frac{\sqrt{c} t^2 + bt + a\sqrt{c}}{(b + 2\sqrt{c} t)^2} * \frac{b + 2\sqrt{c} t}{\sqrt{c} t^2 + bt + \sqrt{c} a} \\ &= 2 \int^{t(x)} dt \frac{1}{b + 2\sqrt{c} t} = \frac{1}{\sqrt{c}} \int^{t(x)} dt \frac{1}{t + [b/2\sqrt{c}]} \\ &= \frac{1}{\sqrt{c}} \ln (t + [b/2\sqrt{c}]) \Big|^{t(x)} \\ &= \frac{1}{\sqrt{c}} \ln (\sqrt{a+bx+cx^2} + \sqrt{c} x + [b/2\sqrt{c}]) = \frac{1}{\sqrt{c}} \ln (\sqrt{R} + \sqrt{c} x + [b/2\sqrt{c}]) \\ &\doteq \frac{1}{\sqrt{c}} \ln [2\sqrt{c}\sqrt{R} + 2cx + b] \end{aligned} \quad (4.9)$$

in agreement with (1.7) stated earlier without proof. To get the last line, we multiplied by $2\sqrt{c}$ top and bottom inside the log, then dropped the constant term $-(1/\sqrt{c}) \ln (2\sqrt{c})$, hence the $\doteq$ sign .

From (4.7) we would have gotten instead

$$\int^x dx \frac{1}{\sqrt{a+bx+cx^2}} = -\frac{1}{\sqrt{c}} \ln [-2\sqrt{c}\sqrt{R} + 2cx + b] . \quad (4.10)$$

These seemingly different results are both valid since they differ by a constant independent of x :

$$\begin{aligned}
& \frac{1}{\sqrt{c}} \ln [2 \sqrt{c} \sqrt{R} + 2cx + b] - \left\{ -\frac{1}{\sqrt{c}} \ln [-2 \sqrt{c} \sqrt{R} + 2cx + b] \right\} \\
&= \frac{1}{\sqrt{c}} \ln [(2 \sqrt{c} \sqrt{R} + 2cx + b)(-2 \sqrt{c} \sqrt{R} + 2cx + b)] = \frac{1}{\sqrt{c}} \ln [(2cx+b)^2 - 4cR] \\
&= \frac{1}{\sqrt{c}} \ln [(2cx+b)^2 - 4c(a+bx+cx^2)] = \frac{1}{\sqrt{c}} \ln [4c^2x^2 + 4cbx + b^2 - 4ca + 4cbx - 4c^2x^2] \\
&= \frac{1}{\sqrt{c}} \ln [b^2 - 4ca].
\end{aligned} \tag{4.11}$$

We then write,

$$\int^x dx \frac{1}{\sqrt{a+bx+cx^2}} \doteq \frac{1}{\sqrt{c}} \ln [2 \sqrt{c} \sqrt{R} + 2cx + b] \doteq -\frac{1}{\sqrt{c}} \ln [-2 \sqrt{c} \sqrt{R} + 2cx + b]. \tag{4.12}$$

Just for the record, Maple comes up with the first of these forms (again apart from a constant),

```

assume(c>0);
Int(1/sqrt(a + b*x + c*x^2), x);

```

$$\int \frac{1}{\sqrt{a+bx+cx^2}} dx$$

```

value(%);

```

$$\frac{\ln\left(\frac{\frac{1}{2}b+cx}{\sqrt{c}} + \sqrt{a+bx+cx^2}\right)}{\sqrt{c}}$$

(4.13)

which is

$$\frac{1}{\sqrt{c}} \ln \left[\frac{b+2cx+2\sqrt{c}\sqrt{R}}{2\sqrt{c}} \right] = \frac{1}{\sqrt{c}} \ln [2 \sqrt{c} \sqrt{R} + 2cx + b] - \frac{1}{\sqrt{c}} \ln (2\sqrt{c}).$$

In the special base that $4ac = b^2$ we know from (2.4) that $\sqrt{R} = \sqrt{c} [x + b/(2c)]$ so (4.9) becomes

$$\begin{aligned}
\int^x dx \frac{1}{\sqrt{R}} &\doteq \frac{1}{\sqrt{c}} \ln (2 \sqrt{c} \sqrt{R} + 2cx + b) \\
&= \frac{1}{\sqrt{c}} \ln (2 \sqrt{c} [\sqrt{c} (x + b/(2c)) + 2cx + b]) \\
&= \frac{1}{\sqrt{c}} \ln (2cx + b + 2cx + b) = \frac{1}{\sqrt{c}} \ln (4cx + 2b) \\
&\doteq \frac{1}{\sqrt{c}} \ln (2cx + b). \qquad c > 0, 4ac = b^2
\end{aligned} \tag{4.14}$$

5. Substitution B

Here we mimic the previous section as closely as possible, using matching equation numbers.

Instead of using the substitution (3.1), consider

$$t(x) = (\sqrt{a+bx+cx^2} - \sqrt{a}) / x \quad \text{or} \quad xt + \sqrt{a} = \sqrt{a+bx+cx^2}. \quad (5.1)$$

When we finish this section we can replace $\sqrt{a} \rightarrow -\sqrt{a}$ everywhere and thereby generate an alternate result. One now has,

$$a + bx + cx^2 = x^2 t^2 + 2xt\sqrt{a} + a. \quad (5.2)$$

The key idea is that the two a terms cancel out, giving

$$\begin{aligned} bx + cx^2 &= x^2 t^2 + 2\sqrt{a} x t \Rightarrow b + cx = xt^2 + 2\sqrt{a} t \Rightarrow x(c-t^2) = 2\sqrt{a} t - b \Rightarrow \\ x(t) &= \frac{2\sqrt{a} t - b}{c-t^2}, \end{aligned} \quad (5.3)$$

an expression with no messy square roots, unlike (3.3). Then,

$$\begin{aligned} \sqrt{a+bx+cx^2} &= xt + \sqrt{a} = \frac{2\sqrt{a} t - b}{c-t^2} t + \sqrt{a} = \frac{(2\sqrt{a} t - b)t}{c-t^2} + \frac{\sqrt{a} (c-t^2)}{c-t^2} \\ &= \frac{\sqrt{a} t^2 - bt + \sqrt{a} c}{c-t^2}. \end{aligned} \quad (5.4)$$

Then our general replacement becomes

$$\mathcal{R}(x, \sqrt{a+bx+cx^2}) \rightarrow \mathcal{R}\left(\frac{2\sqrt{a} t - b}{c-t^2}, \frac{\sqrt{a} t^2 - bt + \sqrt{a} c}{c-t^2}\right). \quad (5.5)$$

One may then compute

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt} \left(\frac{2\sqrt{a} t - b}{c-t^2} \right) = \frac{(c-t^2)2\sqrt{a} - (2\sqrt{a} t - b)(-2t)}{(c-t^2)^2} = \frac{2c\sqrt{a} - 2t^2\sqrt{a} + 4t^2\sqrt{a} - 2bt}{(c-t^2)^2} \\ &= 2 \frac{\sqrt{a} t^2 - bt + \sqrt{a} c}{(c-t^2)^2} \end{aligned}$$

so that

$$dx = 2 \frac{\sqrt{a} t^2 - bt + \sqrt{a} c}{(c-t^2)^2} dt . \quad (5.6)$$

Our integral evaluation then becomes

$$\int^x dx \mathcal{R}(x, \sqrt{a+bx+cx^2}) = \int^{t(x)} dt 2 \frac{\sqrt{a} t^2 - bt + \sqrt{a} c}{(c-t^2)^2} * \mathcal{R}\left(\frac{2\sqrt{a} t - b}{c-t^2}, \frac{\sqrt{a} t^2 - bt + \sqrt{a} c}{c-t^2}\right)$$

where $t(x) = (\sqrt{a+bx+cx^2} - \sqrt{a}) / x$.

(5.7)

Notice that there are no messy square roots anywhere in the dt integrand. The integrand is now a rational function in the variable t : integrand = poly₁(t)/poly₂(t).

As noted earlier, one may replace $\sqrt{a} \rightarrow -\sqrt{a}$ (and $a \rightarrow a$) to obtain the following alternative form.

$$\int^x dx \mathcal{R}(x, \sqrt{a+bx+cx^2}) = \int^{t(x)} dt 2 \frac{-\sqrt{a} t^2 - bt - \sqrt{a} c}{(c-t^2)^2} * \mathcal{R}\left(\frac{-2\sqrt{a} t - b}{c-t^2}, \frac{-\sqrt{a} t^2 - bt - \sqrt{a} c}{c-t^2}\right)$$

where $t(x) = (\sqrt{a+bx+cx^2} + \sqrt{a}) / x$.

(5.8)

Example: Let $\mathcal{R}(x, \sqrt{a+bx+cx^2}) = 1/\sqrt{a+bx+cx^2}$. Then using (5.8),

$$\begin{aligned} \int^x dx \frac{1}{\sqrt{a+bx+cx^2}} &= \int^{t(x)} dt 2 \frac{-\sqrt{a} t^2 - bt - \sqrt{a} c}{(c-t^2)^2} * \frac{c-t^2}{-\sqrt{a} t^2 - bt - \sqrt{a} c} \\ &= 2 \int^{t(x)} dt \frac{1}{c-t^2} = -2 \int^{t(x)} dt \frac{1}{t^2-c} . \end{aligned}$$

Maple kindly computes this integral,

Int (1/(t^2-c), t);

$$\int \frac{1}{t^2-c} dt$$

value(%);

$$-\frac{\operatorname{arctanh}\left(\frac{t}{\sqrt{c}}\right)}{\sqrt{c}}$$

(5.8a)

so we continue,

$$\begin{aligned} \int^x dx \frac{1}{\sqrt{a+bx+cx^2}} &= + \frac{2}{\sqrt{c}} \tanh^{-1}\left(\frac{t}{\sqrt{c}}\right) \Big|^{t=t(x)} \\ &= \frac{2}{\sqrt{c}} \tanh^{-1}\left[\frac{(\sqrt{a+bx+cx^2} + \sqrt{a}) / x}{\sqrt{c}}\right] \end{aligned}$$

$$= \frac{2}{\sqrt{c}} \tanh^{-1} \left(\frac{\sqrt{R} + \sqrt{a}}{x\sqrt{c}} \right). \quad (5.9)$$

From (5.7) we would have instead found

$$\int^x dx \frac{1}{\sqrt{a+bx+cx^2}} = \frac{2}{\sqrt{c}} \tanh^{-1} \left(\frac{\sqrt{R} - \sqrt{a}}{x\sqrt{c}} \right). \quad (5.10)$$

One might reasonably wonder how both these results can be correct since there is a sign difference. The answer again is that the two forms differ by a constant independent of x . To show this, one can use

$$\tanh^{-1} u = \frac{1}{2} \ln \left(\frac{1+u}{1-u} \right) \quad |u| < 1 \quad // \text{ Spiegel 8.57}$$

with

$$u = \frac{\sqrt{R} - \sqrt{a}}{x\sqrt{c}} \quad \Rightarrow \quad 1 \pm u = \frac{x\sqrt{c}}{x\sqrt{c}} \pm \frac{\sqrt{R} - \sqrt{a}}{x\sqrt{c}} = \frac{x\sqrt{c} \pm (\sqrt{R} - \sqrt{a})}{x\sqrt{c}}$$

so that

$$\begin{aligned} \tanh^{-1} \left(\frac{\sqrt{R} - \sqrt{a}}{x\sqrt{c}} \right) &= \frac{1}{2} \ln \left[\frac{x\sqrt{c} + \sqrt{R} - \sqrt{a}}{x\sqrt{c} - \sqrt{R} + \sqrt{a}} \right] \\ \tanh^{-1} \left(\frac{\sqrt{R} + \sqrt{a}}{x\sqrt{c}} \right) &= \frac{1}{2} \ln \left[\frac{x\sqrt{c} + \sqrt{R} + \sqrt{a}}{x\sqrt{c} - \sqrt{R} - \sqrt{a}} \right]. \end{aligned}$$

The difference between these two arctangents is then

$$\begin{aligned} \tanh^{-1} \left(\frac{\sqrt{R} + \sqrt{a}}{x\sqrt{c}} \right) - \tanh^{-1} \left(\frac{\sqrt{R} - \sqrt{a}}{x\sqrt{c}} \right) &= \frac{1}{2} \ln \left[\frac{x\sqrt{c} + \sqrt{R} + \sqrt{a}}{x\sqrt{c} - \sqrt{R} - \sqrt{a}} \right] - \frac{1}{2} \ln \left[\frac{x\sqrt{c} + \sqrt{R} - \sqrt{a}}{x\sqrt{c} - \sqrt{R} + \sqrt{a}} \right] \\ &= \frac{1}{2} \ln \left[\frac{x\sqrt{c} + \sqrt{R} + \sqrt{a}}{x\sqrt{c} - \sqrt{R} - \sqrt{a}} * \frac{x\sqrt{c} - \sqrt{R} + \sqrt{a}}{x\sqrt{c} + \sqrt{R} - \sqrt{a}} \right] = \frac{1}{2} \ln \left[\frac{(x\sqrt{c} + \sqrt{a})^2 - R}{(x\sqrt{c} - \sqrt{a})^2 - R} \right] \\ &= \frac{1}{2} \ln \left[\frac{x^2 c + \sqrt{a} \sqrt{c} x + a - (a + bx + cx^2)}{x^2 c - \sqrt{a} \sqrt{c} x + a - (a + bx + cx^2)} \right] = \frac{1}{2} \ln \left[\frac{\sqrt{a} \sqrt{c} x - bx}{-\sqrt{a} \sqrt{c} x - bx} \right] \\ &= \frac{1}{2} \ln \left[\frac{\sqrt{a} \sqrt{c} - b}{-\sqrt{a} \sqrt{c} - b} \right]. \end{aligned}$$

Therefore

$$\frac{2}{\sqrt{c}} \tanh^{-1}\left(\frac{\sqrt{R}+\sqrt{a}}{x\sqrt{c}}\right) - \frac{2}{\sqrt{c}} \tanh^{-1}\left(\frac{\sqrt{R}-\sqrt{a}}{x\sqrt{c}}\right) = \frac{2}{\sqrt{c}} \frac{1}{2} \ln \left[\frac{\sqrt{a}\sqrt{c} - b}{-\sqrt{a}\sqrt{c} - b} \right] \quad (5.11)$$

which is a constant independent of x . Therefore we write

$$\int^x dx \frac{1}{\sqrt{a+bx+cx^2}} \doteq \frac{2}{\sqrt{c}} \tanh^{-1}\left(\frac{\sqrt{R}+\sqrt{a}}{x\sqrt{c}}\right) \doteq \frac{2}{\sqrt{c}} \tanh^{-1}\left(\frac{\sqrt{R}-\sqrt{a}}{x\sqrt{c}}\right) \quad (5.12)$$

and we have now accumulated two more forms for this integral. Both forms can be verified by direct differentiation as Maple shows,

```
R := a + b*x + c*x^2;
```

$$R = a + bx + cx^2$$

```
J1 := (2/sqrt(c))* arctanh((sqrt(a) + sqrt(R))/(x*sqrt(c)));
```

$$J1 = 2 \frac{\operatorname{arctanh}\left(\frac{\sqrt{a} + \sqrt{a+bx+cx^2}}{x\sqrt{c}}\right)}{\sqrt{c}}$$

```
diff(J1,x): simplify(%);
```

$$J2 := (2/sqrt(c))* arctanh((-sqrt(a) + sqrt(R))/(x*sqrt(c)));$$

$$J2 = 2 \frac{\operatorname{arctanh}\left(\frac{-\sqrt{a} + \sqrt{a+bx+cx^2}}{x\sqrt{c}}\right)}{\sqrt{c}}$$

```
diff(J2,x);
```

$$2 \frac{\frac{1}{2} \frac{b+2cx}{\sqrt{a+bx+cx^2} x \sqrt{c}} - \frac{-\sqrt{a} + \sqrt{a+bx+cx^2}}{x^2 \sqrt{c}}}{\sqrt{c} \left(1 - \frac{(-\sqrt{a} + \sqrt{a+bx+cx^2})^2}{x^2 c} \right)}$$

```
simplify(%);
```

$$\frac{1}{\sqrt{a+bx+cx^2}} \quad (5.13)$$

In the Maple language, a colon suppresses output from a command, while symbol % refers to the last computed quantity. In the diff(J1,x) line we suppress output and simplify to get $1/\sqrt{R}$, but for J2 we show the typically messy expression Maple generates, followed by the simplified result.

6. Substitution C

Rename the roots of $R = 0$ to be $\alpha = \alpha_-$ and $\beta = \alpha_+$. Recall that

$$a+bx+cx^2 = c(x-\alpha)(x-\beta) . \quad (2.3)$$

Now, instead of using the substitution (3.1), consider

$$t(x) = \sqrt{a+bx+cx^2} / (x-\alpha) = \sqrt{c(x-\alpha)(x-\beta)} / (x-\alpha) = \sqrt{c} \frac{\sqrt{x-\beta}}{\sqrt{x-\alpha}} . \quad (6.1)$$

When we finish this section we can do $\alpha \leftrightarrow \beta$ everywhere and thereby generate an alternate result.

Solving for x one finds (there is no equation (6.2) because we are matching the previous sections),

$$\begin{aligned} t^2 = c (x-\beta)/(x-\alpha) &\Rightarrow t^2(x-\alpha) = c(x-\beta) \Rightarrow x(t^2-c) = (\alpha t^2 - c\beta) \Rightarrow \\ x(t) &= \frac{\alpha t^2 - c\beta}{t^2 - c} , \end{aligned} \quad (6.3)$$

an expression with no messy square roots, unlike (3.3). Then,

$$\begin{aligned} \sqrt{a+bx+cx^2} &= (x-\alpha)t = \left(\frac{\alpha t^2 - c\beta}{t^2 - c} - \alpha \right) t = \frac{t(\alpha t^2 - c\beta)}{t^2 - c} - \frac{\alpha t(t^2 - c)}{t^2 - c} = \frac{\alpha t^3 - c\beta t - \alpha t^3 + \alpha c t}{t^2 - c} \\ &= \frac{c(\alpha - \beta)t}{t^2 - c} . \end{aligned} \quad (6.4)$$

Then our general replacement becomes

$$\mathcal{R}(x, \sqrt{a+bx+cx^2}) \rightarrow \mathcal{R}\left(\frac{\alpha t^2 - c\beta}{t^2 - c}, \frac{c(\alpha - \beta)t}{t^2 - c}\right) . \quad (6.5)$$

One may then compute

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt} \left(\frac{\alpha t^2 - c\beta}{t^2 - c} \right) = \frac{(t^2 - c)2\alpha t - (\alpha t^2 - c\beta)2t}{(t^2 - c)^2} = \frac{2\alpha t^3 - 2\alpha c t - 2\alpha t^3 + 2c\beta t}{(t^2 - c)^2} = 2 \frac{-\alpha c t + c\beta t}{(t^2 - c)^2} \\ &= 2 \frac{c(\beta - \alpha)t}{(t^2 - c)^2} \end{aligned}$$

so that

$$dx = 2 \frac{c(\beta - \alpha)t}{(t^2 - c)^2} dt . \quad (6.6)$$

Our integral evaluation then becomes

$$\int^x dx \mathcal{R}(x, \sqrt{a+bx+cx^2}) = \int^{t(x)} dt 2 \frac{c(\beta-\alpha)t}{(t^2-c)^2} * \mathcal{R}\left(\frac{\alpha t^2 - c\beta}{t^2-c}, \frac{c(\alpha-\beta)t}{t^2-c}\right)$$

where $t(x) = \sqrt{a+bx+cx^2} / (x-\alpha) = \sqrt{R} / (x-\alpha)$.

(6.7)

Notice that there are no messy square roots anywhere in the dt integrand. The integrand is now a rational function in the variable t : integrand = poly₁(t)/poly₂(t).

As noted earlier, one may swap $\alpha \leftrightarrow \beta$ to obtain the following alternative form,

$$\int^x dx \mathcal{R}(x, \sqrt{a+bx+cx^2}) = \int^{t(x)} dt 2 \frac{c(\alpha-\beta)t}{(t^2-c)^2} * \mathcal{R}\left(\frac{\beta t^2 - c\alpha}{t^2-c}, \frac{c(\beta-\alpha)t}{t^2-c}\right)$$

where $t(x) = \sqrt{a+bx+cx^2} / (x-\beta) = \sqrt{R} / (x-\beta)$.

(6.8)

Example: Let $\mathcal{R}(x, \sqrt{a+bx+cx^2}) = 1/\sqrt{a+bx+cx^2}$. Then using (6.7),

$$\begin{aligned} \int^x dx \frac{1}{\sqrt{a+bx+cx^2}} &= \int^{t(x)} dt 2 \frac{c(\beta-\alpha)t}{(t^2-c)^2} * \frac{t^2-c}{c(\alpha-\beta)t} = -2 \int^{t(x)} dt \frac{1}{t^2-c} \\ &= +\frac{2}{\sqrt{c}} \tanh^{-1}\left(\frac{t}{\sqrt{c}}\right) \Big|^{t(x)} \quad // \text{ using (5.8a)} \\ &= \frac{2}{\sqrt{c}} \tanh^{-1}\left(\frac{\sqrt{x-\beta}}{\sqrt{x-\alpha}}\right). \end{aligned}$$
(6.9)

Thus we arrive at yet another form for our $\int dx/\sqrt{R}$ integral. Maple verifies it as follows,

```
J3 := (2/sqrt(c))*arctanh(sqrt(x-beta)/sqrt(x-alpha));
```

$$J3 = 2 \frac{\operatorname{arctanh}\left(\frac{\sqrt{x-\beta}}{\sqrt{x-\alpha}}\right)}{\sqrt{c}}$$

```
diff(J3,x): simplify(%);
```

$$\frac{1}{\sqrt{c} \sqrt{x-\beta} \sqrt{x-\alpha}}$$

which is just $1/\sqrt{a+bx+cx^2}$. The result is clearly symmetric under $\alpha \leftrightarrow \beta$, so one has

$$\int^x dx \frac{1}{\sqrt{a+bx+cx^2}} \doteq \frac{2}{\sqrt{c}} \tanh^{-1}\left(\frac{\sqrt{x-\beta}}{\sqrt{x-\alpha}}\right) \doteq \frac{2}{\sqrt{c}} \tanh^{-1}\left(\frac{\sqrt{x-\alpha}}{\sqrt{x-\beta}}\right).$$
(6.10)

We leave it to the reader to find the constant by which these two forms differ from each other and from those forms presented earlier.

7. More forms for the integral of $R^{-1/2}$

Define

$$y \equiv \frac{2cx + b}{\sqrt{4ac - b^2}} . \quad (7.1)$$

We wish to use the following identity with the above y ,

$$\sinh^{-1} y = \ln(y + \sqrt{y^2 + 1}) \quad |y| < \infty \quad // \text{ Spiegel 8.55} \quad (7.2)$$

so we need to evaluate

$$\begin{aligned} y^2 + 1 &= \left(\frac{2cx + b}{\sqrt{4ac - b^2}} \right)^2 + 1 = \frac{(2cx + b)^2}{4ac - b^2} + 1 = \frac{(2cx + b)^2 + 4ac - b^2}{4ac - b^2} = \frac{4c^2x^2 + 4cbx + b^2 + 4ac - b^2}{4ac - b^2} \\ &= \frac{4c(cx^2 + bx + a)}{4ac - b^2} = \frac{4cR}{4ac - b^2} . \end{aligned} \quad (7.3)$$

Then

$$y + \sqrt{y^2 + 1} = \frac{2cx + b}{\sqrt{4ac - b^2}} + \frac{2\sqrt{c}\sqrt{R}}{\sqrt{4ac - b^2}} = \frac{2cx + b + 2\sqrt{c}\sqrt{R}}{\sqrt{4ac - b^2}} .$$

Then from (7.2),

$$\sinh^{-1} \left(\frac{2cx + b}{\sqrt{4ac - b^2}} \right) = \ln \left(\frac{2cx + b + 2\sqrt{c}\sqrt{R}}{\sqrt{4ac - b^2}} \right) \doteq \ln (2cx + b + 2\sqrt{c}\sqrt{R}) \quad (7.4)$$

where as usual we have thrown out a constant $g(a, b, c)$. Comparing this result to (4.9),

$$\int^x dx \frac{1}{\sqrt{a + bx + cx^2}} \doteq \frac{1}{\sqrt{c}} \ln [2\sqrt{c}\sqrt{R} + 2cx + b] , \quad (4.9)$$

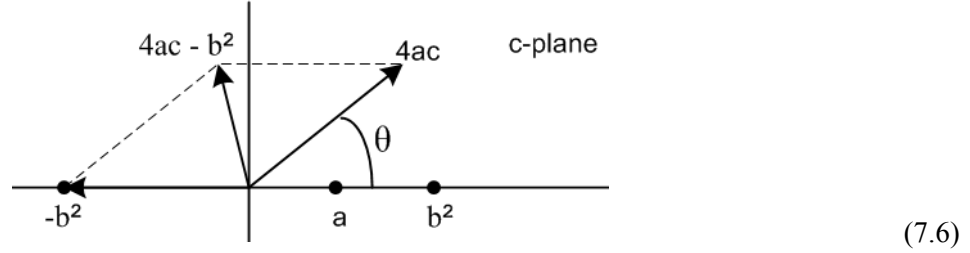
we may conclude that

$$\int^x dx \frac{1}{\sqrt{a + bx + cx^2}} \doteq \frac{1}{\sqrt{c}} \sinh^{-1} \left(\frac{2cx + b}{\sqrt{4ac - b^2}} \right) \quad (7.5)$$

giving a commonly appearing form for the integral valid for $c > 0$ and $4ac - b^2 > 0$.

Analytic Continuation

We wish now to analytically continue this integral to $c < 0$. For the moment we assume $a > 0$ and draw this picture



where vector $c = |c| e^{i\theta}$ is aligned with the vector $4ac$ shown in the figure. One can see that as the $4ac$ vector is swung counterclockwise to the point $\theta = +\pi$, the angle of the vector $4ac - b^2$ also moves to $+\pi$:

$$\begin{aligned} \text{angle}(c) = \theta = 0 & \Rightarrow \text{angle}(4ac - b^2) = 0 & // \text{ before swing, } c > 0 \\ \text{angle}(c) = \theta = \pi & \Rightarrow \text{angle}(4ac - b^2) = \pi & // \text{ after swing, } c < 0 \end{aligned} \quad (7.7)$$

We then write after the swing,

$$\begin{aligned} c &= |c| e^{i\pi} = (-c) e^{i\pi} \\ (4ac - b^2) &= |4ac - b^2| e^{i\pi} = |b^2 - 4ac| e^{i\pi} = (b^2 - 4ac) e^{i\pi} \end{aligned} \quad (7.8)$$

Therefore (the phases are correlated here, you *don't* pick separate $\pm i$ for each square root),

$$\begin{aligned} \sqrt{c} &= \sqrt{-c} i \\ \sqrt{4ac - b^2} &= \sqrt{b^2 - 4ac} i \end{aligned} \quad (7.9)$$

We can then modify our integral above using these rules to get

$$\begin{aligned} \int^x dx (1/\sqrt{R}) &= (1/\sqrt{c}) \sinh^{-1} [(2cx + b)/\sqrt{4ac - b^2}] & // (7.5) \\ &= (1/[\sqrt{-c} i]) \sinh^{-1} [(2cx + b)/(\sqrt{b^2 - 4ac} i)] \\ &= (-i)(1/\sqrt{-c}) \sinh^{-1} [-i(2cx + b)/\sqrt{b^2 - 4ac}] \\ &= -(-i)(1/\sqrt{-c}) \sinh^{-1} [i(2cx + b)/\sqrt{b^2 - 4ac}] & // \text{ Spiegel 8.64} \\ &= -i(-i)(1/\sqrt{-c}) \sin^{-1} [(2cx + b)/\sqrt{b^2 - 4ac}] & // \text{ Spiegel 8.93} \\ &= -\frac{1}{\sqrt{-c}} \sin^{-1} \left[\frac{2cx + b}{\sqrt{b^2 - 4ac}} \right] = +\frac{1}{\sqrt{-c}} \sin^{-1} \left[\frac{-2cx - b}{\sqrt{b^2 - 4ac}} \right], \end{aligned} \quad (7.10)$$

giving forms valid for $c < 0$ and $b^2 - 4ac > 0$. Next we use this relation

$$\sin^{-1}(z) = -\cos^{-1}(z) + \pi/2 \quad // \text{ Spiegel 5.74}$$

$$\stackrel{\bullet}{=} -\cos^{-1}(z) \quad (7.11)$$

to obtain two more forms,

$$\int^x dx (1/\sqrt{R}) \stackrel{\bullet}{=} + \frac{1}{\sqrt{-c}} \cos^{-1} \left[\frac{2cx+b}{\sqrt{b^2-4ac}} \right] = - \frac{1}{\sqrt{-c}} \cos^{-1} \left[\frac{-2cx-b}{\sqrt{b^2-4ac}} \right]. \quad (7.12)$$

Trust but verify,

$$\begin{aligned} \text{J4} &:= - (1/\text{sqrt}(-c)) * \arccos((-2*c*x - b)/\text{sqrt}(b^2-4*a*c)); \\ &\quad \arccos\left(\frac{-2cx-b}{\sqrt{b^2-4ac}}\right) \\ \text{J4} &:= - \frac{\arccos\left(\frac{-2cx-b}{\sqrt{b^2-4ac}}\right)}{\sqrt{-c}} \\ \text{diff(J4,x): simplify(\%);} \\ &\quad \frac{1}{\sqrt{a+cx^2+xb}} \end{aligned} \quad (7.13)$$

In these last integrals, we started with $a > 0$, but the results can be continued to a part of the range $a < 0$ where we have

$$\begin{aligned} b^2 - 4ac > 0 &\Rightarrow b^2 + 4a|c| > 0 \Rightarrow 4a|c| > -b^2 \Rightarrow \\ a &> -(b^2/|c|). \end{aligned} \quad (7.14)$$

Goldstein Classical Mechanics Typo

In the discussion of orbits with an inverse-square force law, Goldstein (1950) on page 77 writes the second integral in (7.12) omitting the leading minus sign. This error is repeated on page 93 of the later 2001 third edition of the book (Goldstein, Poole and Safko, all deceased), from which we quote, where $a, b, c = \alpha, \beta, \gamma$,

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \arccos - \frac{\beta + 2\gamma x}{\sqrt{q}}, \quad (3.51) \quad q = \beta^2 - 4\alpha\gamma.$$

This results in another sign error in (3.54), but as it turns out, this error makes no difference in the key final result (3.55) due to the fact that $\cos(\theta - \theta') = \cos(\theta' - \theta)$. That final result is this.

$$\frac{1}{r} = \frac{mk}{l^2} \left(1 + \sqrt{1 + \frac{2El^2}{mk^2} \cos(\theta - \theta')} \right). \quad (3.55)$$

The inverse-square force law is $F = -k/r^2$, a particle has mass m , energy E , and angular momentum ℓ . This last result shows that the orbits are conic sections expressed in polar coordinates r, θ where the radical is the orbit eccentricity ϵ . For a sun-planet system, m, r, θ refer to an equivalent one-body problem where m is the reduced mass, and r, θ are relative to the center of mass.

$$\text{A summary of forms appearing in this document} \quad R = a + bx + cx^2 \quad (7.15)$$

$$\int^x dx \frac{1}{\sqrt{R}} \quad \doteq \frac{1}{\sqrt{c}} \ln (2cx + b + 2\sqrt{c}\sqrt{R}) \quad // (4.9) \quad c > 0$$

$$\doteq -\frac{1}{\sqrt{c}} \ln (2cx + b - 2\sqrt{c}\sqrt{R}) \quad // (4.10) \quad c > 0$$

$$\doteq \frac{1}{\sqrt{c}} \ln (2cx + b) \quad // (4.14) \quad c > 0, b^2 - 4ac = 0$$

$$\doteq \frac{1}{\sqrt{c}} \ln \left[\frac{2cx + b + 2\sqrt{c}\sqrt{a+bx+cx^2}}{\sqrt{4ac-b^2}} \right] \quad // (1.8) \quad c > 0, b^2 - 4ac < 0$$

$$\doteq \frac{2}{\sqrt{c}} \tanh^{-1} \left(\frac{\sqrt{R} + \sqrt{a}}{x\sqrt{c}} \right) \quad // (5.9) \quad c > 0, a > 0$$

$$\doteq \frac{2}{\sqrt{c}} \tanh^{-1} \left(\frac{\sqrt{R} - \sqrt{a}}{x\sqrt{c}} \right) \quad // (5.10) \quad c > 0, a > 0$$

$$\doteq \frac{2}{\sqrt{c}} \tanh^{-1} \left(\frac{\sqrt{x-\beta}}{\sqrt{x-\alpha}} \right) \quad // (6.9) \quad c > 0, \alpha, \beta \text{ roots of } R$$

$$\doteq \frac{1}{\sqrt{c}} \sinh^{-1} \left(\frac{2cx + b}{\sqrt{4ac-b^2}} \right) \quad // (7.5) \quad c > 0, b^2 - 4ac < 0$$

$$\doteq -\frac{1}{\sqrt{-c}} \sin^{-1} \left(\frac{2cx + b}{\sqrt{b^2 - 4ac}} \right) \quad // (7.10) \quad c < 0, b^2 - 4ac > 0$$

$$\doteq +\frac{1}{\sqrt{-c}} \cos^{-1} \left(\frac{2cx + b}{\sqrt{b^2 - 4ac}} \right) \quad // (7.12) \quad c < 0, b^2 - 4ac > 0$$

$$\doteq -\frac{1}{\sqrt{-c}} \cos^{-1} \left(\frac{-2cx - b}{\sqrt{b^2 - 4ac}} \right) \quad // (7.12) \quad c < 0, b^2 - 4ac > 0$$

Four of these results appear in GR7 page 94 :

2.26 Forms containing $\sqrt{a + bx + cx^2}$ and integral powers of x

$$\text{Notation: } R = a + bx + cx^2, \quad \Delta = 4ac - b^2 \quad (7.16)$$

$$\begin{aligned} \mathbf{2.261}^{11} \quad \text{For } n = -1 \quad \int \frac{dx}{\sqrt{R}} &= \frac{1}{\sqrt{c}} \ln \left(\frac{2\sqrt{cR} + 2cx + b}{\sqrt{\Delta}} \right) & [c > 0] & \quad \text{TI (127)} \\ &= \frac{1}{\sqrt{c}} \operatorname{arcsinh} \left(\frac{2cx + b}{\sqrt{\Delta}} \right) & [c > 0, \quad \Delta > 0] & \quad \text{DW} \\ &= \frac{1}{\sqrt{c}} \ln(2cx + b) & [c > 0, \quad \Delta = 0] & \quad \text{DW} \\ &= \frac{-1}{\sqrt{-c}} \arcsin \left(\frac{2cx + b}{\sqrt{-\Delta}} \right) & [c < 0, \quad \Delta < 0] & \quad \text{TI (128)} \end{aligned}$$

DW Dwight, H. B., *Tables of Integrals and Other Mathematical Data*, Macmillan, New York, 1934.

TI Timofeyev, A. F. *Integrirovaniye funktsiy* (Integration of functions), part I. GTTI, Moscow and Leningrad, 1933.

See also Spiegel 14.280 which, however, uses $R = ax^2 + bx + c$. Both Spiegel and GR7 present many integrals of the form $x^m (\sqrt{R})^n$ for m and odd n being various positive and negative integers.

Footnote concerning TI above. Adrian (Fedorovich) Timofeev (1882-1954) led a complicated life in Russia and wrote a few non-mathematical books about it (e.g., *My Prison Diary*).

<http://adriantimofeev1.blogspot.com/2012/07/this-is-photos-from-life-in-1890-1915.html>

Appendix: About Euler's original paper

The Euler substitution idea first appeared in 1786 in the following publication,

L. Euler, "Speculationes super formula integrali $\int (x^n dx) / \sqrt{(aa-2bx+cxx)}$, ubi simul egregiae observationes circa fractiones continuas occurrunt", *Acta Academiae Scientiarum Imperialis Petropolitinae* 1782, 1786, pp. 62-84.

or

L. Euler, "Speculations concerning the integral formula $\int (x^n dx) / \sqrt{(aa-2bx+cxx)}$, where at once occur exceptional observations about continued fractions", *Transactions of the Imperial Academy of Sciences in St. Petersburg* 1782, 1786, pp. 62-84.

Euler wrote this paper in 1775 (along with about 60 other papers and letters), presented it in 1782, and had it published in 1786.

The original paper and a German translation with better typesetting can be viewed in the Euler archive

<http://eulerarchive.maa.org//>

by looking up Subject / Mathematics / Integration / index number 606.

Here are two clips of Euler's original text concerning his substitutions,

§. 4. Nunc ad sequentem casum progressuri, consideremus formulam $s = \sqrt{(aa - 2bx + cxx)} - a$, ut scilicet evanescat facto $x = 0$, et quoniam est

$$ds = \frac{-b dx + c x dx}{\sqrt{aa - 2bx + cxx}}$$

§. 5. Sumamus porro $s = x \sqrt{(aa - 2bx + cxx)}$ fiet $ds = \frac{a x dx - b x dx + c x x dx}{\sqrt{(aa - 2bx + cxx)}}$, vnde vicissim integrando

Papers were often written in Latin at the time and presumably everything was typeset by hand which must have been exceedingly painful, especially for equations. Note that a^2 and cx^2 are written aa and cxx. His presentation is, not surprisingly, quite different from ours. His logical thread differs so his substitutions do not match those shown in (3.5), but the end results are the same.

Euler was Swiss but moved to St. Petersburg, later to Berlin, and then back to St. Petersburg where he published the above paper. He had a prolific and eventful life in eventful times.

References

Links were last checked on 6 Nov 2016.

https://en.wikipedia.org/wiki/Euler_substitution

<http://planetmath.org/eulerssubstitutionsforintegration>

K.N. Boyadzhiev, "Euler Substitutions" (Ohio Northern University, 2006), 5p. Has examples.

http://www2.onu.edu/~m-caragiu.1/bonus_files/EULER-SU.pdf

H. Goldstein, *Classical Mechanics* (Addison-Wesley, Boston, 1950).

H. Goldstein, C.P. Poole Jr. and J.L. Safko, *Classical Mechanics, 3rd Ed.* (Addison-Wesley, New York, 2001) now rebranded by Pearson, London. Authors deceased 2005, 2015 and 2016.

[GR7] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products, 7th Ed.* (Academic Press, New York, 2007). The 8th edition came out Oct 2014. Editor Dan Zwillinger has errata for editions 6,7 and 8 at <http://www.mathtable.com/errata> .

N.S. Piskunov, *Differential and Integral Calculus* (Mir, Moscow, 1969). This 895 page text was translated from the Russian by G. Yankovsky (search the web). Section 12 pp 372-375 mentions the Euler substitutions with a few examples.

M.R. Spiegel, *Schaum's Outlines: Mathematical Handbook of Formulas and Tables* (McGraw-Hill, New York, 1968). Our page references are to this edition. The current version of this book is given below.

M.R. Spiegel, S. Lipschutz, M. and J. Liu, *Schaum's Outlines: Mathematical Handbook of Formulas and Tables, 4th Ed.* (McGraw-Hill, New York, 2012). John Liu was added for the 1999 2nd Ed, and Seymour Lipschutz joined for the 2008 3rd Ed. Not to be confused with a watered-down "Easy Outline" version. This low-cost paperback is an excellent fast reference for well-known mathematical facts.

D. Zwillinger, *Handbook of Integration* (Jones and Bartlett, Boston, 1992).