This is the Title PhL 1.11.15

Analytic Continuation

Consider this generic problem.

Let

$f(z) = \sqrt{z-z_0}$ with $z_0 > 0$

For complex z write

$z = |z|e^{i\theta}$

$(z-z_0) = |z-z_0|\, e^{i\varphi}$

We start with real z = a where $a > z_0$ so $(a-z_0) > 0$. In this case

$a = |a|e^{i0}$ so $\theta = 0$

$(a-z_0) = |a-z_0|e^{i0}$ so $\varphi = 0$

Now let z = -a which we define to be -a+iε just above the cut shown below. Then

$-a = |a|e^{i\pi} = a\, e^{i\pi}$

$(-a-z_0) = |-a-z_0|\, e^{i\pi} = |a+z_0|e^{i\pi} = (a+z_0)e^{i\pi}$

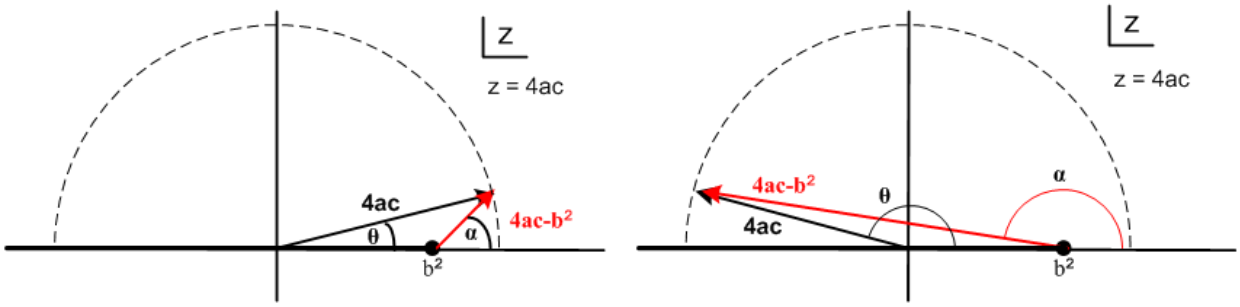
Then

$\sqrt{-a} = \sqrt{a}\, e^{i\pi/2} = i\sqrt{a}$

$\sqrt{-a-z_0} = \sqrt{a+z_0}\, e^{i\pi/2} = i\sqrt{a+z_0}$

Start with Case 2 of (2.5) where $4ac - b^2 > 0$ and $c > 0$. Assume further (for the moment) that $a > 0$. Then define $z \equiv 4ac$. We shall treat z as a complex variable and analytically continue $\sqrt{z-b^2}$ from $z > 0$ to $z<0$. The function $\sqrt{z-b^2}$ has a branch point at $z = b^2$ whose cut we take off to the left. We shall associate the

quantity $z > 0$ with $(4ac)e^{i0}$ and $z < 0$ with $(4ac)e^{i\pi}$, a point above the cut. We then smoothly continue from $z > 0$ to $z < 0$ as shown in this drawing,



For $z > 0$ ($c > 0$) we have $\theta = 0$ and $\alpha = 0$.

For $z < 0$ ($c < 0$) we have $\theta = \pi$ and $\alpha = \pi$.

$c = |c|\, e^{i\pi}$ $\qquad\qquad$ $\sqrt{c} = \sqrt{|c|}\, e^{i\pi/2}$

$(4ac\text{-}b^2) = |4ac\text{-}b^2|\, e^{i\pi}$