More study of the 1/sqrt(R) integral PhL 11.3.16

Here I summarize work done in two other docs, and derive new forms as well. It was a royal pain, but I think everything is down pat. Ultimately you just differentiate to test a result, perhaps with Maple help. But I wanted to derive results, not test results handed to me.

1. Derive the arcsinh formula **1**
2. Obtain the arcsin formula **3**
3. Obtain the arccos formula **5**
4. Summary of formulas derived above **5**
5. Goldstein page 77 Application **6**
6. Other forms of the integral **7**

1. Derive the arcsinh formula

Here is a statement of the arcsinh formula of interest from GR7 :

2.26 Forms containing $\sqrt{a+bx+cx^2}$ and integral powers of x

Notation: $R = a + bx + cx^2, \quad \Delta = 4ac - b^2$

2.261[11] For $n = -1$

$$\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{c}} \ln\left(\frac{2\sqrt{cR} + 2cx + b}{\sqrt{\Delta}}\right) \qquad [c > 0] \qquad \text{TI (127)}$$

$$= \frac{1}{\sqrt{c}} \operatorname{arcsinh}\left(\frac{2cx + b}{\sqrt{\Delta}}\right) \qquad [c > 0, \quad \Delta > 0] \qquad \text{DW}$$

I have derived the first form involving the log. I understand that the denominator in the log result is not significant and the basic fact is this:

$$I = \int^{x} dx/\sqrt{a+bx+cx^2} \;=\; (1/\sqrt{c}\,)\;\ln\,[\,2cx + b + 2\sqrt{cR}\,]\;\; + \text{constant}$$

where the constant can depend on a,b,c but not on x.

Question: How can one use this last result to derive the sh^{-1} version of the result stated above, namely,

$$I = \int^{x} dx/\sqrt{a+bx+cx^2} \;=\; (1/\sqrt{c}\,)\;sh^{-1}\,[\,(2cx+b)/\sqrt{4ac-b^2}\,]\;\; + \text{constant}$$

It must be possible to show that

$$sh^{-1}\,[\,(2cx+b)/\sqrt{4ac-b^2}\,] \;=\; \ln\,[\,2cx + b + 2\sqrt{cR}\,]\;\; +\; \text{constant that does not depend on x}$$

In this case, differentiating both functions should give $1/\sqrt{a+bx+cx^2}$. The sh^{-1} form also appears in Schaum p 72. One thing I know is, from Schaum p 29,

$sh^{-1}y = \ln(y + \sqrt{y^2+1})$ // valid for all real y

Using this, I can obtain a "result to show" that involves only logs.

$y = (2cx + b)/\sqrt{4ac-b^2}$

$y^2+1 = (2cx+b)^2/(4ac-b^2) + 1 = [\,(2cx+b)^2 + (4ac-b^2)\,] / (4ac-b^2)$

$= [\,4c^2x^2 + b^2 + 4cbx + 4ac - b^2\,] / (4ac-b^2)$

$= [\,4c^2x^2 + 4cbx + 4ac\,] / (4ac-b^2)$

$= 4c[\,cx^2 + bx + a\,] / (4ac-b^2)$

$= 4cR/(4ac-b^2)$

$\sqrt{y^2+1} = 2\sqrt{c}\,\sqrt{R} \,/\, \sqrt{4ac-b^2}$.

Then

$sh^{-1}[\,(2cx+b)/\sqrt{4ac-b^2}\,] = sh^{-1}y = \ln(y + \sqrt{y^2+1})$

$= \ln[\,(2cx+b)/\sqrt{4ac-b^2} + 2\sqrt{c}\,\sqrt{R}\,/\sqrt{4ac-b^2}\,]$

$= \ln[\,2cx + b + 2\sqrt{c}\,\sqrt{R}\,] - \ln(\sqrt{4ac-b^2})$

and there you have it, case closed. Thus I have proven both these results,

$I = \int^{x} dx/\sqrt{a+bx+cx^2} = (1/\sqrt{c})\,\ln[\,2cx + b + 2\sqrt{cR}\,] + \text{constant}$

$I = \int^{x} dx/\sqrt{a+bx+cx^2} = (1/\sqrt{c})\,sh^{-1}[\,(2cx+b)/\sqrt{4ac-b^2}\,] + \text{constant}$

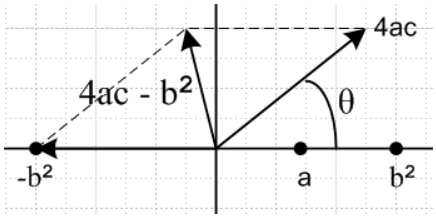
This last form assumes that $4ac - b^2 > 0$ and $c > 0$. Let's summarize this as

$I = (1/\sqrt{c})\,sh^{-1}[\,(2cx+b)/\sqrt{4ac-b^2}\,]$ $\quad 4ac - b^2 > 0$ and $c > 0$ [$a > 0$ is forced]

This situation implies that $4ac > b^2$ and so $a > b^2/(4c)$ so it is implied that $a > 0$.

2. Obtain the arcsin formula

Let us now assume that a > 0 before continuing. Then $4ac-b^2$ is going to go negative for c < 0. Consider this picture



At θ = 0, the vector $4ac-b^2$ lies on the real axis and points to the right assuming $4ac-b^2 > 0$. But as we swing the vector c over to θ = +π, we end up with 4ac pointing to the left and $4ac-b^2$ also points even more to the left, so here is the status of things

angle(c) = 0	⇒	angle($4ac-b^2$) = 0	// before swing
angle(c) = π	⇒	angle($4ac-b^2$) = π	// after swing

We end up with $4ac - b^2 = -4a|c| - b^2$. Since we have assumed a > 0, we have $4ac - b^2 < 0$ after doing this swing operation, so this quantity changed sign. After the swing we have

$$c = |c|\, e^{i\pi} = (-c)e^{i\pi}$$

$$(4ac-b^2) = |4ac-b^2|\, e^{i\pi} = |b^2-4ac|\, e^{i\pi} = (b^2-4ac)e^{i\pi}$$

Therefore (note that the phases are determined here, you don't pick separate ± i for each one)

$$\sqrt{c} = \sqrt{-c}\, i$$

$$\sqrt{4ac-b^2} = \sqrt{b^2-4ac}\; i$$

We can then modify our integral I above using these rules to get

$$I = (1/\sqrt{c})\; sh^{-1}\left[(2cx+b)/\sqrt{4ac-b^2} \right]$$

$$= (1/\sqrt{-c}\, i)\; sh^{-1}\left[(2cx+b)/(\sqrt{b^2-4ac}\; i) \right]$$

$$= (-i)(1/\sqrt{-c})\; sh^{-1}\left[-i(2cx+b)/\sqrt{b^2-4ac} \right]$$

$$= (-i)^2(1/\sqrt{-c})\; \sin^{-1}\left[(2cx+b)/\sqrt{b^2-4ac} \right]$$

$$= -(1/\sqrt{-c})\; \sin^{-1}\left[(2cx+b)/\sqrt{b^2-4ac} \right]$$

Therefore we find

$I = -(1/\sqrt{-c})\ \sin^{-1}[(2cx+b)/\sqrt{b^2-4ac}]$ $\quad 4ac - b^2 < 0$ and $c < 0$ [$a > 0$ assumed]

and this is our first statement of the $\sin^{-1}$ evaluation of the I integral. Now recall from GR7

2.26 Forms containing $\sqrt{a+bx+cx^2}$ and integral powers of x

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$$= \frac{1}{\sqrt{c}} \operatorname{arcsinh}\left(\frac{2cx+b}{\sqrt{\Delta}}\right) \qquad [c > 0, \quad \Delta > 0] \qquad \text{DW}$$

$$= \frac{1}{\sqrt{c}} \ln(2cx + b) \qquad [c > 0, \quad \Delta = 0] \qquad \text{DW}$$

$$= \frac{-1}{\sqrt{-c}} \arcsin\left(\frac{2cx+b}{\sqrt{-\Delta}}\right) \qquad [c < 0, \quad \Delta < 0] \qquad \text{TI (128)}$$

Recall our two results from above:

$I = (1/\sqrt{c})\ \text{sh}^{-1}[(2cx+b)/\sqrt{4ac-b^2}]$ $\quad 4ac - b^2 > 0$ and $c > 0$ [$a > 0$ is forced]

$I = -(1/\sqrt{-c})\ \sin^{-1}[(2cx+b)/\sqrt{b^2-4ac}]$ $\quad 4ac - b^2 < 0$ and $c < 0$ [$a > 0$ assumed]

This agrees exactly with the lines 2 and 4 of the quotes above. I have already derived the other two lines in a separate doc, so all four forms are now derived.

Now in the first case $a > 0$ is forced, while in the second it was only assumed. In the second case we need

$4ac < b^2$

$-4a|c| < b^2$

$4a|c| > -b^2$

$a > -b^2/(4|c|)$

This in fact allows a range of negative a without impacting the $\sin^{-1}$ formula shown above. In the first case we have

$4ac > b^2$

$a > b^2/(4c)$

So let's restate the above two evaluations this way

$$I = (1/\sqrt{c})\ \mathrm{sh}^{-1}\left[(2cx+b)/\sqrt{4ac-b^2}\right] \qquad 4ac - b^2 > 0 \text{ and } c > 0 \quad [\Rightarrow a > b^2/(4c)] \qquad (1)$$

$$I = -(1/\sqrt{-c})\ \sin^{-1}\left[(2cx+b)/\sqrt{b^2-4ac}\right] \qquad 4ac - b^2 < 0 \text{ and } c < 0 \quad [\Rightarrow a > -b^2/(4|c|)] \qquad (2)$$

3. Obtain the arccos formula

It is a fact that

$$\sin^{-1}(x) + \cos^{-1}(x) = \pi/2$$

where x is real, this is a very obvious fact when stated in English text. There is no corresponding equation for hyperbolic functions!!!!

So we can shift the constant and write the second result above as

$$I = +(1/\sqrt{-c})\ \cos^{-1}\left[(2cx+b)/\sqrt{b^2-4ac}\right] \qquad 4ac - b^2 < 0 \text{ and } c < 0 \quad [\Rightarrow a > -b^2/(4|c|)] \qquad (3)$$

There is no simple $\cosh^{-1}$ formula. However, I did derive a certain $\cosh^{-1}$ formula with a completely different argument than that shown above. I will gather up my obscure formulas later on.

There is another version of the $\cos^{-1}$ which follows from (2),

$$I = (1/\sqrt{-c})\ \sin^{-1}\left[-(2cx+b)/\sqrt{b^2-4ac}\right] \qquad 4ac - b^2 < 0 \text{ and } c < 0 \quad [\Rightarrow a > -b^2/(4|c|)] \qquad (4)$$

Then

$$I = -(1/\sqrt{-c})\ \cos^{-1}\left[-(2cx+b)/\sqrt{b^2-4ac}\right] \qquad 4ac - b^2 < 0 \text{ and } c < 0 \quad [\Rightarrow a > -b^2/(4|c|)] \qquad (5)$$

You get this same result going directly from the previous cos result, using Schaum p 18.

4. Summary of formulas derived above

$$I = (1/\sqrt{c})\ \mathrm{sh}^{-1}\left[(2cx+b)/\sqrt{4ac-b^2}\right] \qquad 4ac - b^2 > 0 \text{ and } c > 0 \quad [\Rightarrow a > b^2/(4c)] \qquad (1)$$

$$I = -(1/\sqrt{-c})\ \sin^{-1}\left[(2cx+b)/\sqrt{b^2-4ac}\right] \qquad 4ac - b^2 < 0 \text{ and } c < 0 \quad [\Rightarrow a > -b^2/(4|c|)] \qquad (2)$$

$$I = +(1/\sqrt{-c})\ \cos^{-1}\left[(2cx+b)/\sqrt{b^2-4ac}\right] \qquad 4ac - b^2 < 0 \text{ and } c < 0 \quad [\Rightarrow a > -b^2/(4|c|)] \qquad (3)$$

$$I = (1/\sqrt{-c})\ \sin^{-1}\left[-(2cx+b)/\sqrt{b^2-4ac}\right] \qquad 4ac - b^2 < 0 \text{ and } c < 0 \quad [\Rightarrow a > -b^2/(4|c|)] \qquad (4)$$

$$I = -(1/\sqrt{-c})\ \cos^{-1}[\,-(2cx+b)/\sqrt{b^2-4ac}\,] \quad 4ac - b^2 < 0 \text{ and } c < 0 \quad [\Rightarrow\ a > -b^2/(4|c|)] \qquad (5)$$

5. Goldstein page 77 Application

I claim that Goldstein page (3-45) is item (5) above but with an overall sign error. This was my conclusion a long time ago, though I did not document it at that time (it was marked in pencil). Let us now see if this sign error affects anything of significance.

He has $a = 2mE/\ell^2$ and $b = 2mk/\ell^2$ and $c = -1$. Going back to his (3.44) I would say

$$\theta = \theta' - I = \theta' - \{ -(1/\sqrt{-c})\ \cos^{-1}[\,-(2cx+b)/\sqrt{b^2-4ac}\,] + K\}$$

$$= \theta' + (1/\sqrt{-c})\cos^{-1}[\,-(2cx+b)/\sqrt{b^2-4ac}\,] + K$$

Now consider

$$b^2-4ac = [2mk/\ell^2]^2 + 4[\,2mE/\ell^2] = [2mk/\ell^2]^2\ \{\,1 + 4[\,2mE/\ell^2]/[2mk/\ell^2]^2\,\}$$

$$= [2mk/\ell^2]^2\ \{\,1 + 4[\,2mE/\ell^2]/[4m^2k^2/\ell^4]\}$$

$$= [2mk/\ell^2]^2\ \{\,1 + [\,2E]/[mk^2/\ell^2]\}$$

$$= [2mk/\ell^2]^2\ \{\,1 + 2E\ell^2/mk^2\} \qquad = q \qquad \text{// agrees with page 77 } \alpha$$

and also

$$-2cx - b = +2u - [2mk/\ell^2] = [2mk/\ell^2]\ \{\ 2u/[2mk/\ell^2] - 1\ \}$$

$$= [2mk/\ell^2]\ \{\ u\ell^2/mk - 1\ \}$$

The above is then

$$\theta = \theta' + K + \cos^{-1}[\ \ [2mk/\ell^2]\ \{\ u\ell^2/mk - 1\ \}/\ [2mk/\ell^2]\ \sqrt{1 + 2E\ell^2/mk^2}\ \ ]$$

$$= \theta' + K + \cos^{-1}[\,(u\ell^2/mk - 1)/\sqrt{1 + 2E\ell^2/mk^2}\,]$$

$$= \theta' + \cos^{-1}[\,(u\ell^2/mk - 1)/\sqrt{1 + 2E\ell^2/mk^2}\,] \qquad \text{// set } K = 0$$

So I claim he has a sign error as well in equation β. Continuing.

$$\theta - \theta' = \cos^{-1}[\,(u\ell^2/mk - 1)/\sqrt{1 + 2E\ell^2/mk^2}\,]$$

$$\cos(\theta - \theta') = (u\ell^2/mk - 1)/\sqrt{1 + 2E\ell^2/mk^2}$$

Notice that this last equation is exactly the same with or without the sign error, because cos is even !! Continue along then

$$\sqrt{1 + 2E\ell^2/mk^2}\cos(\theta-\theta') = (u\ell^2/mk - 1)$$

$$u(\ell^2/mk) = 1 + \sqrt{1 + 2E\ell^2/mk^2}\cos(\theta-\theta')$$

$$1/r = (mk/\ell^2)\left[1 + \sqrt{1 + 2E\ell^2/mk^2}\cos(\theta-\theta')\right] \qquad \text{// agrees with (3-46)}$$

So Goldstein has a sign wrong in (3-45) and in β, but this does not affect (3-46) !!

6. Other forms of the integral

These other proven-valid forms I found by doing Euler substitutions,

$$I = GR7(x) = + (1/\sqrt{c})\ln\left[+2cx + b + 2\sqrt{cR}\right]$$

$$I = PL(x) = - (1/\sqrt{c})\ln\left[-2cx - b + 2\sqrt{cR}\right]$$

$$I = (2/\sqrt{c})\tanh^{-1}\left[\frac{(-\sqrt{a} + \sqrt{R})}{x\sqrt{c}}\right]$$

$$I = (2/\sqrt{c})\tanh^{-1}\left[\frac{(+\sqrt{a} + \sqrt{R})}{x\sqrt{c}}\right]$$

$$I = (2/\sqrt{c})\,\mathrm{sech}^{-1}\left(\frac{\sqrt{c}\sqrt{cx^2 - (\sqrt{a} + \sqrt{R})^2}}{x}\right) = (2/\sqrt{c})\,\mathrm{sech}^{-1}\left(\frac{\sqrt{c}\sqrt{-2a - bx - 2\sqrt{a}\sqrt{R}}}{x}\right)$$

$$I = (2/\sqrt{c})\cosh^{-1}\left(\frac{x\sqrt{c}}{\sqrt{cx^2 - (\sqrt{a} + \sqrt{R})^2}}\right)$$

In general these are unpleasant formulas because they involve $\sqrt{R}$ and worse.