Study of a certain Integral PhL 11.1.16

Motivation: This appears on page 77 of Goldstein.

The integral is the following indefinite integral whose results I quote from GR7 page 94

2.26 Forms containing $\sqrt{a+bx+cx^2}$ and integral powers of x

Notation: $R = a + bx + cx^2, \quad \Delta = 4ac - b^2$

2.261[11] For $n = -1$

$$\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{c}} \ln\left(\frac{2\sqrt{cR} + 2cx + b}{\sqrt{\Delta}}\right) \qquad [c > 0] \qquad \text{TI (127)}$$

$$= \frac{1}{\sqrt{c}} \operatorname{arcsinh}\left(\frac{2cx + b}{\sqrt{\Delta}}\right) \qquad [c > 0, \quad \Delta > 0] \qquad \text{DW}$$

$$= \frac{1}{\sqrt{c}} \ln(2cx + b) \qquad [c > 0, \quad \Delta = 0] \qquad \text{DW}$$

$$= \frac{-1}{\sqrt{-c}} \arcsin\left(\frac{2cx + b}{\sqrt{-\Delta}}\right) \qquad [c < 0, \quad \Delta < 0] \qquad \text{TI (128)}$$

Here TI is a Russian source I don't have

Press, London, 1948.

TI Timofeyev, A. F. *Integrirovaniye funktsiy* (Integration of functions), part I. GTTI, Moscow and Leningrad, 1933.

The second DW result seems odd since a does not appear, but you are allowed to drop "constants" in expressing an indefinite integral, so maybe that is where it went!

Question: How do you actually manually do such an integral?

GR7 gives us a hint as follows, involving **Euler substitutions**,

2.251 It is possible to rationalize the integrand in integrals of the form $\int R\left(x, \sqrt{a+bx+cx^2}\right) dx$ by using one or more of the following three substitutions, known as the "Euler substitutions":

1. $\sqrt{a+bx+cx^2} = xt \pm \sqrt{a}$ for $a > 0$;
2. $\sqrt{a+bx+cx^2} = t \pm x\sqrt{c}$ for $c > 0$;
3. $\sqrt{c(x-x_1)(x-x_2)} = t(x-x_1)$ when x_1 and x_2 are real roots of the equation $a+bx+cx^2 = 0$.

In my case we have c = -1, and a and b could have either sign, so none of these directly applies, but maybe something similar does. Wiki on these subs refers to a Zwillinger book "The Handbook of Integration" 1992. I have two of Z's books but not this one. I am taking a web look for this book, it does exist and is about integration methods, something I would like to have,

http://en.booksee.org/book/1423095 claims to have it, download is slow, we shall see. Yes, I have it!

Wiki mentions that above Euler things as well but has different expressions!!! Here they are from wiki

1. $\sqrt{R} = \pm x\sqrt{a} + t$

2. $\sqrt{R} = xt \pm \sqrt{c}$

3. $\sqrt{R} = (x-\alpha)t$ where α is a root of R.

But wiki have a↔c from GR7, so I change wiki to read

1. $\sqrt{R} = \pm x\sqrt{c} + t$

2. $\sqrt{R} = xt \pm \sqrt{a}$

3. $\sqrt{R} = (x-\alpha)t$ where α is a root of R.

These are numbered differently from GR7, but they are the same 3 subs! Relation is 1↔2.

References [edit]

1. ^ N. Piskunov, *Diferentsiaal- ja integraalarvutus kõrgematele tehnilistele õppeasutustele. Viies, täiendatud trukk. Kirjastus Valgus*, Tallinn (1965). Note: Euler substitutions can be found in most Russian calculus textbooks.
2. ^ Zwillinger, Daniel. *The Handbook of Integration*. 1992: Jones and Bartlett. pp. 145–146. ISBN 978-0867202939.

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and they give a good example. Maybe Z will have nothing more than those subs.. // It is a good book, Z knows a lot of stuff. Z born 1957 so is age 43+15 = 58, has some miles to go, I have had some emails with him. The book however does not mention this kind of integral, index does not include Euler subs.

Try an Euler substitution

Consider:

$R = (a + bx + cx^2)$

$\sqrt{R} = xt \pm \sqrt{a}$ (1) $t(x) = (\sqrt{R} \mp \sqrt{a})/x$

I want this to have the form

$x = f(t)$ and then we can do $dx = f'(t)\,dt$

So square (1)

$$R = (a + bx + cx^2) = (xt \pm \sqrt{a})^2 = x^2t^2 \pm 2\sqrt{a}\, xt + a$$

Notice that the a cancels on both sides, so left with

$$bx + cx^2 = x^2t^2 \pm 2\sqrt{a}\, xt$$

or

$$b + cx = xt^2 \pm 2\sqrt{a}\, t$$

or

$$x(c-t^2) = \pm 2\sqrt{a}\, t - b$$

$$x = (\pm 2\sqrt{a}\, t - b)(c-t^2)^{-1}$$

Maple confirms where I select the + sign

```
restart;
R := a + b*x + c*x^2;
```

$$R := a + b\,x + c\,x^2$$

```
eq1 := R = (x*t+sqrt(a))^2;
```

$$eq1 := a + b\,x + c\,x^2 = (x\,t + \sqrt{a})^2$$

```
S := solve(eq1,x);
```

$$S := 0, -\frac{b - 2\,t\,\sqrt{a}}{c - t^2}$$

```
x := S[2];
```

$$x := -\frac{b - 2\,t\,\sqrt{a}}{c - t^2}$$

Now want to compute dx :

```
diff(f,t): df := simplify(%);
```

$$df := 2\,\frac{\sqrt{a}\,c + \sqrt{a}\,t^2 - t\,b}{(c - t^2)^2}$$

So at this point I have (still with just the + sign)

$$dx = 2\,[\sqrt{a}\, c - bt + \sqrt{a}\, t^2\,](t^2-c)^{-2}\, dt$$

OK, now the next step is to get R as a function of t!

```
R: R := simplify(%);
```

$$R := \frac{a c^2 + 2 a c t^2 + a t^4 + b^2 t^2 - 2 b t \sqrt{a}\, c - 2 b t^3 \sqrt{a}}{(c - t^2)^2}$$

```
N := numer(R);
```

$$N := a c^2 + 2 a c t^2 + a t^4 + b^2 t^2 - 2 b t \sqrt{a}\, c - 2 b t^3 \sqrt{a}$$

This does not look good because we want to talk about $\sqrt{R}$. Write it out

$N = a\, t^4 - 2b\sqrt{a}\, t^3 + (b^2 + 2ac)t^2 - 2b\sqrt{a}\, c\, t + ac^2$

How on earth is this going to "simplify" our integral? We will have square root of the above in the denominator. Maple says this does not factor. Write out 6 terms like so

$N = a\, t^4 - 2b\sqrt{a}\, t^3 + b^2t^2 + 2act^2 - 2b\sqrt{a}\, c\, t + ac^2$

Replace a by α^2 and $\sqrt{a} = \alpha$:

$N = \alpha^2\, t^4 - 2b\alpha t^3 + b^2t^2 + 2\alpha^2ct^2 - 2b\alpha c\, t + \alpha^2c^2$

This was the trick, Here is what Maple says,

```
N1 := subs({sqrt(a) = alpha, a = alpha^2}, N);
```

$$N1 := \alpha^2 c^2 + 2 \alpha^2 c t^2 + \alpha^2 t^4 + b^2 t^2 - 2 b t \alpha c - 2 b t^3 \alpha$$

```
factor(N1);
```

$$(t b - \alpha c - t^2 \alpha)^2$$

Then we can say

$R = (\alpha t^2 - bt + \alpha c)^2 / (c - t^2)^2$

$\sqrt{R} = \pm (\alpha t^2 - bt + \alpha c) / (c - t^2)$

From above we have

$dx = 2\,[\sqrt{a}\, c - bt + \sqrt{a}\, t^2\,](t^2 - c)^{-2}\, dt$

$= 2\,[\alpha c - bt + \alpha t^2\,](t^2 - c)^{-2}\, dt$

$= 2(\alpha t^2 - bt + \alpha c)(t^2 - c)^{-2}\, dt$

Then we get the payoff

$dx/\sqrt{R} = \pm 2(\alpha t^2 - bt + \alpha c)(t^2-c)^{-2}\, dt \;//\; (\alpha t^2-bt+\alpha c)/(c-t^2)$

$= \pm 2(t^2-c)^{-2}\, dt \;//\; 1/(c-t^2)$

$= \pm 2(t^2-c)^{-2}\, dt \;*\; (c-t^2)$

$= \mp 2(t^2-c)^{-2}\, dt \;*\; (t^2-c)$

$= \mp 2(t^2-c)^{-1}\, dt$

Once again

$\int^{x} dx/\sqrt{R} = \mp 2 \int^{t(x)} dt/(t^2-c)$ where $t(x) = (\sqrt{R} \mp \sqrt{a})/x$

and I do admit that is a significant simplification. Let's now just look up that integral, Schaum p 64 says

14.125 $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

14.144 $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)$ or $\displaystyle -\frac{1}{a}\coth^{-1}\frac{x}{a}$

So lets take the second one with the log and see what we get with $a^2 = c$

$\int^{x} dx/\sqrt{R} = \mp 2 \int^{t(x)} dt/(t^2-c) = \mp 2\,[\,(1/2\sqrt{c})\ln\frac{t(x)-\sqrt{c}}{t(x)+\sqrt{c}}\,]$

$t(x) = (\sqrt{R} \mp \sqrt{a})/x$

$xt(x) = \sqrt{R} \mp \sqrt{a}$

So mult up and down by x to get

$\int^{x} dx/\sqrt{R} = \mp\ (1/\sqrt{c})\ \ln\frac{(\sqrt{R} \mp \sqrt{a}) - \sqrt{c}}{(\sqrt{R} \mp \sqrt{a}) + \sqrt{c}}$

I am trying to end up with something like this,

$$= \frac{1}{\sqrt{c}}\ln(2cx+b)$$

Pause at 1:30. I have at least made some progress using one of the Euler substitutions, but I have not been able to replicate any of the results that GR7 claims. This is the first time I have ever tried this in my life!