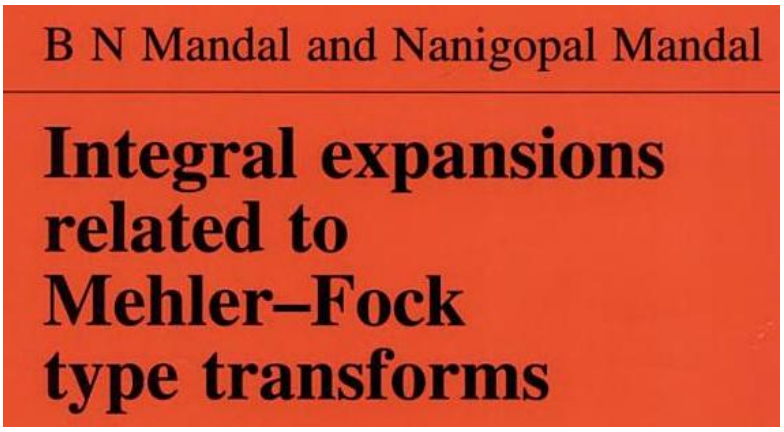
Web Search for Mehler integrals PhL 1.28.11

Searching on "integrals involving conical", I basically get one reference to a certain Felson, but it is in J. Math and Phys which only BYU has (discontinued journal).

Nothing in "integrals involving Mehler". But that led to this book:

Reference 1



which certainly seems on topic, and a whole book! Here are some clips, where our friend Felson appears right away:

3.1 Integral expansions in (−1, 1)

3.1.1 Integral expansion of a function in terms of associated Legendre functions $P^{-\mu}_{-\frac{1}{2}+i\tau}(\pm x)$ $(-1 < x < 1)$ in which the superscript appears as an integration variable while the subscript $-\frac{1}{2} + i\tau$ (τ real) remains fixed, was first introduced by Felsen (1958). He formally developed such expansion formulae from a unique δ-function representation as mentioned in Chapter 1 (cf. equation (1.5)). He considered the Green's function $G(\theta, \theta'; \tau; \lambda)$ satisfying

$$\left[-\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \left(\frac{1}{4}+\tau^2\right)\sin\theta + \frac{\lambda}{\sin\theta}\right] G(\theta,\theta';\tau;\lambda) = \delta(\theta-\theta'),\ \theta_1 \le \theta \le \theta_2, \tag{3.1.1}$$

with the boundary conditions at the end points θ_1, θ_2 as yet unspecified and τ is a specified real parameter. Here θ is the latitudinal coordinate of the spherical coordi-

nate system used in connection with the study of some diffraction problems involving conical boundaries. A δ-function representation is obtained from the relation (1.4), as given by

$$\delta(\theta-\theta')\sin\theta' = \frac{1}{2\pi i}\oint_{\Gamma} G(\theta,\theta';\tau;\lambda)\,d\lambda, \tag{3.1.2}$$

where the contour Γ encloses all the singularities of G in the anticlockwise sense. An integral expansion for a suitable function $f(\theta)$ in the domain $\theta_1 \le \theta \le \theta_2$ can be obtained formally by multiplying both sides of the relation (3.1.2) by $\frac{f(\theta')}{\sin\theta'}$ and integrating over θ' between θ_1 and θ_2. For various intervals in the θ-domain, different types of integral expansion formulae, related to Mehler–Fock type transforms, can be developed formally (see equation (1.5)).

We give below expressions for the Green's functions and corresponding δ-function representations for various intervals as well as boundary conditions.

Case I. $0 \le \theta \le \pi$

In this case, the end points $\theta = 0$ and $\theta = \pi$ are singular points of the differential equation (3.1.1). If we impose the condition that G is to be finite at the end points, then we find that the appropriate expression for G is given by

$$G(\theta,\theta';\tau;\lambda) = \frac{\pi}{2}\frac{\Gamma\left(\frac{1}{2}+i\tau+\mu\right)}{\Gamma\left(\frac{1}{2}+i\tau-\mu\right)}\,\frac{P^{-\mu}_{-\frac{1}{2}+i\tau}(\cos\theta)\;P^{-\mu}_{-\frac{1}{2}+i\tau}(-\cos\theta')}{\cos[\pi(i\tau-\mu)]},\quad (0<\theta<\theta'<\pi), \tag{3.1.3}$$

where $\mu = \sqrt{\lambda}$, $\mathrm{Re}\,\sqrt{\lambda} > 0$. For $0 < \theta' < \theta < \pi$, the Green's function can be obtained from (3.1.3) by interchanging θ and θ'. It is obvious that G has only a branch point singularity at $\lambda = 0$ in the complex λ-plane and so a branch cut is chosen along the negative real axis of the λ-plane. As there are no other singularities, Γ can be taken as a circle at infinity not crossing the branch in the complex λ-plane. By Cauchy's theorem, the integral over the circle is equal to the integral over a contour which starts at infinity, runs parallel to but just above the negative real axis up to a point near to the point right of the origin, then runs parallel to but just below the negative real axis up to infinity. In the complex μ-plane, this contour transforms to a line

At this point we hit a missing page, but he is doing μ plane stuff, not so important to me. There is nothing else in this Google book view for this book. Here is the contents

1 Introduction 1

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1.2 Some properties of associated Legendre functions 8
1.3 Some properties of generalized associated Legendre functions 16

2 Integral expansions related to Mehler–Fock transforms 21

2.1 Integral expansions involving Legendre functions 21
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3 Integral expansions related to Mehler–Fock type transforms involving associated Legendre functions 65

3.1 Integral expansions in $(-1, 1)$ 65
3.2 Integral expansions in (x_1, x_2) where $-1 \leq x_1 < x_2 \leq 1$ 88
3.3 Integral expansions in $(1, \infty)$ 99
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4 Integral expansions related to Mehler–Fock type transforms involving generalized associated Legendre functions 110

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5.1 Some properties of the non-periodic Legendre functions 118
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5.4 Integral expansions involving non-periodic associated Legendre functions 125

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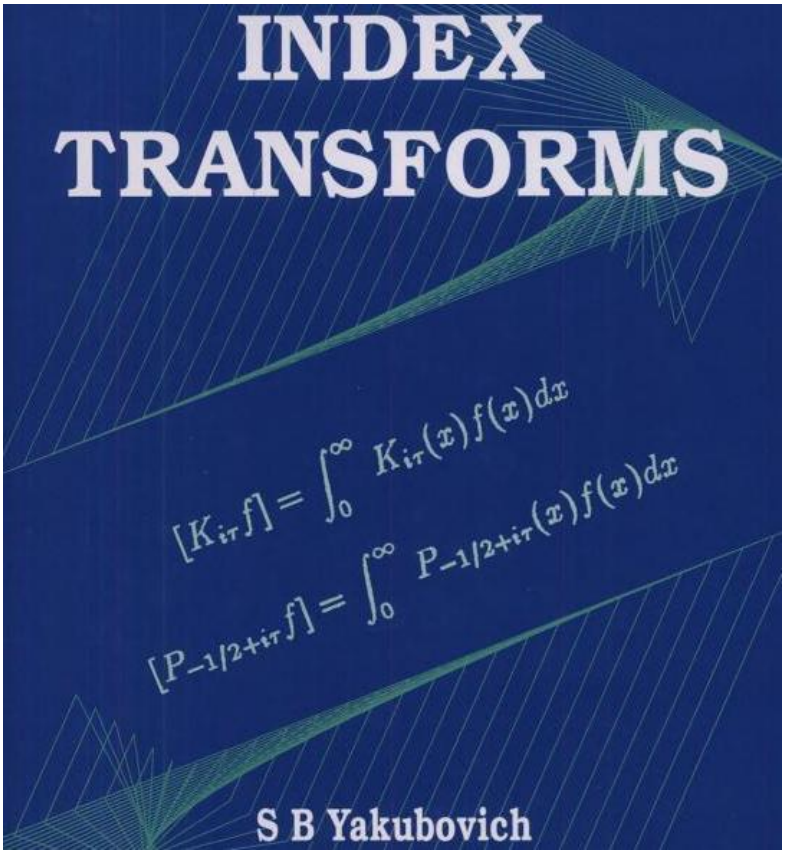
and perhaps Chapter 4 might have what I want. This book is not at Marriott.

Reference 2



Certainly another on topic book!

§15. Integral transforms with the generalized associated Legendre functions

It is known that the method of the integral transforms is one of the most effective modern method for solving differential and integral equations, for the boundary value problems of mathematical physics and theory of elasticity, for solving various homogeneous and nonhomogeneous problems, etc. [7, 8, 12, 18, 22, 29, 38, 39, 41, 42, 45, 47, 50, 53, 72-77, 80, 82, 83, 85, 87, 88, 97-103, 108, 112, 114, 120, 122-124, 127, 137, 138, 144, 145, 147].

The operational calculus for corresponding integral transform allows us to look at the various branches of mathematical physics with indivisible point of view.

The most familiar transforms are the Fourier, the Laplace and the Mellin integral transforms. However, other integral transforms (the Meijer, the Kontorovich-Lebedev, the Mehler-Fock, the Wimp-Yakubovich transforms etc. [18, 22, 25, 26, 38, 39, 42, 47, 52, 53, 72, 80, 83, 88, 102, 122, 137, 138, 144, 145]) are not less important.

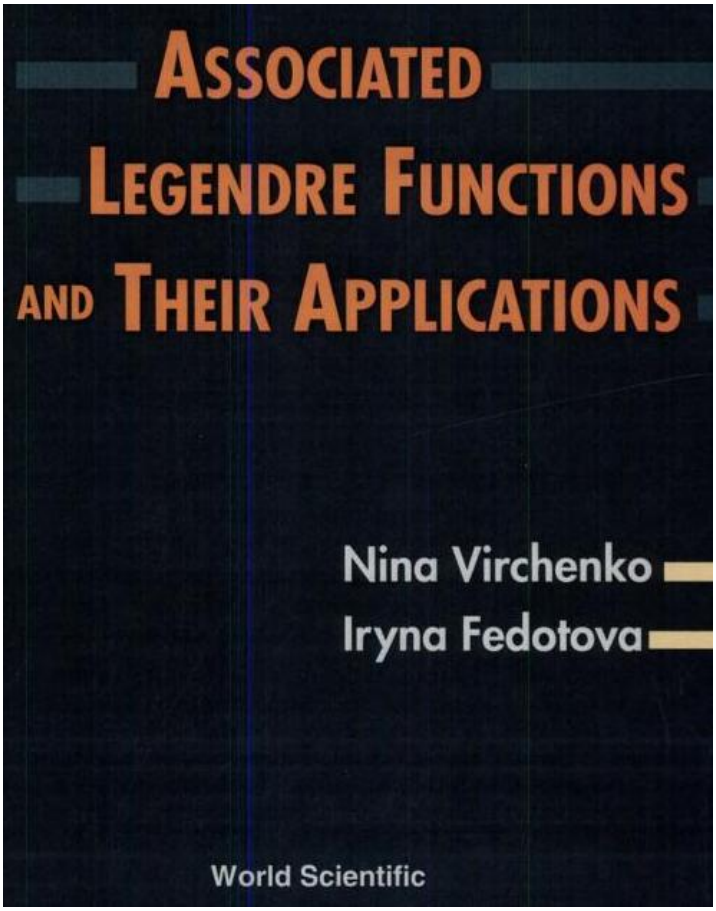
For the first time, the integral transform with the Legendre function as a kernel was introduced by Mehler (1881) [88]. Later on V. Fock [39] and N. Lebedev [72-76] substantiated and developed some theoretical aspects of this transform, gave interesting applications of the theory to solving differential and integral equations.

The book then goes on to prove technical Mehler related theorems and that fills out this chapter

3 The Mehler-Fock Transform 75
3.1 Definition. Inversion in L_p-space 75
3.2 The composition theorem of inversion 90
3.3 The generalized Mehler-Fock transform 92
3.4 Parseval's equality 95
3.5 An index-convolution transform related to the Mehler-Fock integral 101

Really this is not what I am looking for right now, convergence theorems and such.

Reference 3



The contents are concealed, rather amazing. They do reel off some integral representations

$$P_k(\mathrm{ch}\alpha)=\frac{2}{\pi}\int_0^{\alpha}\mathrm{ch}\left[\left(k+\frac{1}{2}\right)\varphi\right]\frac{d\varphi}{\sqrt{2\mathrm{ch}\alpha-2\mathrm{ch}\varphi}}, \tag{1.6}$$

$$P_k(\mathrm{ch}\alpha)=\frac{1}{\pi}\int_0^{\pi}\frac{d\varphi}{(\mathrm{ch}\alpha+\mathrm{sh}\alpha\cos\varphi)^{k+1}}, \tag{1.7}$$

$$P_k(\mathrm{ch}\alpha)=\frac{2}{\pi}\cos\left[\left(k+\frac{1}{2}\right)\pi\right]\int_0^{\infty}\mathrm{ch}\left[\left(k+\frac{1}{2}\right)\varphi\right]\frac{d\varphi}{\sqrt{2\mathrm{ch}\alpha+2\mathrm{ch}\varphi}}, \tag{1.8}$$

$$(\alpha\geq 0,\quad -1<\mathrm{Re}\,k<0),$$

$$P_k(\mathrm{ch}\alpha)=\frac{2}{\pi}\mathrm{ctg}\left[\left(k+\frac{1}{2}\right)\pi\right]\int_{\alpha}^{\infty}\mathrm{ch}\left[\left(k+\frac{1}{2}\right)\varphi\right]\frac{d\varphi}{\sqrt{2\mathrm{ch}\varphi-2\mathrm{ch}\alpha}}, \tag{1.9}$$

$$(\alpha\geq 0,\quad -1<\mathrm{Re}\,k<0).$$

$$P_k(\cos\alpha)=\frac{2}{\pi}\int_0^{\alpha}\cos\left[\left(k+\frac{1}{2}\right)\varphi\right]\frac{d\varphi}{\sqrt{2\cos\varphi-2\cos\alpha}}, \tag{1.10}$$

and then versions of the same with μ present. Then some interesting claims

For $|\nu|\to\infty$ with $|\arg\nu|<\frac{\pi}{2}$ we have [8]

$$P^{m}_{\nu-1/2}(\mathrm{ch}\,\alpha)=\frac{\nu^{m-1/2}}{\sqrt{2\pi\,\mathrm{sh}\,\alpha}}\left\{\frac{\left[1+\sqrt{\mathrm{ch}\,\alpha}O\left(|\nu|^{-1}\right)\right]}{e^{-\nu\alpha}}\pm\frac{(-1)^m i\left[1+O\left(|\nu|^{-1}\right)\right]}{e^{\nu\alpha}}\right\}, \tag{1.59}$$

$$Q^{m}_{\nu-1/2}(\mathrm{ch}\,\alpha)=(-1)^m\nu^{m-1/2}\left(\frac{\pi}{2\,\mathrm{sh}\,\alpha}\right)^{1/2}e^{-\nu\alpha}\left[1+O\left(|\nu|^{-1}\right)\right]. \tag{1.60}$$

If $|\mu|\to\infty$, then we have asymptotic expansions [76]

$$P^{-\mu}_{-1/2+i\tau}(x)=\left(\frac{1+x}{1-x}\right)^{-\mu/2}\frac{1}{\Gamma(1+\mu)}\left[1+O\left(|\mu|^{-1}\right)\right], \tag{1.61}$$

$$P^{-\mu}_{-1/2+i\tau}(-x)=\left(\frac{1-x}{1+x}\right)^{-\mu/2}\frac{1}{\Gamma(1+\mu)}\left[1+O\left(|\mu|^{-1}\right)\right], \tag{1.62}$$

And here is a little discussion I was about to write, but they have done it for me

2. Let us give example of argumentation which can be used for construction of an integral transform [45].

We consider the differential equation of the second order

$$-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right]+[q(x)-\lambda r(x)]u=0, \quad a<x<b, \tag{15.6}$$

where $p(x)$, $q(x)$ and $r(x)$ are continuous functions on (a,b), $u(x)$ satisfies to some conditions at points of this interval. The Green's function $G(x,\xi,\lambda)$ satisfies the non-homogeneous equation

$$\left\{-\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]+q(x)-\lambda r(x)\right\}G(x,\xi,\lambda)=\delta(x-\xi). \tag{15.7}$$

$G(x,\xi,\lambda)$ is expresses in terms of two linearly independent solutions $u_1(x)$ and $u_2(x)$ of the equation (15.6) by the formula

$$G(x,\xi,\lambda)=\begin{cases}\dfrac{u_1(\xi)u_2(x)}{pW}, & x<\xi,\\ \dfrac{u_1(x)u_2(\xi)}{pW}, & x>\xi,\end{cases} \tag{15.8}$$

where $W(x)$ is the Wronskian of the equation (15.6) and is of the form

$$W(x)=u_1(x)u_2'(x)-u_1'(x)u_2(x). \tag{15.9}$$

By using the Dirac integral representation for δ-function we have

$$\frac{\delta(x-\xi)}{r(\xi)}=-\frac{1}{2\pi i}\oint_{C+}G(x,\xi,\lambda)d\lambda, \tag{15.10}$$

where $C+$ is a closed contour surrounding all singular points of the function $G(x,\xi,\lambda)$ on the complex plane. The function $f(x)$ which is defined on (a,b) can be represented in the form

$$f(x)=\int_a^b\delta(x-\xi)f(\xi)d\xi=\frac{1}{2\pi i}\oint_{C^+}\frac{u_1(x)}{pW}\left[\int_a^b r(\xi)f(\xi)u_2(\xi)d\xi\right]d\lambda. \tag{15.11}$$

But nothing else useful to me in this preview. $73 for the PDF!

Reference 4.

Here is a paper I can probably get from Jstor

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 11, November 1997, Pages 3243–3253
S 0002-9939(97)03957-9

MEHLER-FOCK TRANSFORMS OF GENERALIZED FUNCTIONS VIA THE METHOD OF ADJOINTS

BENITO J. GONZÁLEZ AND EMILIO R. NEGRIN

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In this paper we analyse the Mehler-Fock transform of generalized functions via the method of adjoints. For a distribution of compact support, we prove that its Mehler-Fock transform agrees with its transform via the kernel method. A Paley-Wiener type theorem is established.

1. INTRODUCTION

The Mehler-Fock transform of a distribution f of compact support on I (from now on $I = (0, \infty)$) was defined by the kernel method in [10] as:

$$F(\tau) = \langle f(x), P_{-\frac{1}{2}+i\tau}(\cosh x)\rangle, \quad \tau > 0, \tag{1.1}$$

where, as it is usual, $P_{-\frac{1}{2}+i\tau}$ is the Legendre function of first kind and order zero.

Related studies on Mehler-Fock transforms of generalized functions have been carried out in [3], [4], [5], [8], amongst others.

In this paper, we study the Mehler-Fock transform by using the method of adjoints. For this purpose, it will be required to introduce two Fréchet spaces V and W satisfying the condition that the linear operator $\mathcal{L} : W \longrightarrow V$ given by

$$(\mathcal{L}\psi)(x) = \int_0^\infty P_{-\frac{1}{2}+i\tau}(\cosh x)\psi(\tau)d\tau, \quad x > 0, \tag{1.2}$$

be continuous.

Specifically, we denote by V the vector space consisting of all complex-valued functions ϕ which possess derivatives of all orders on I and such that, for all $k \in \mathbb{N} \cup \{0\}$,

$$\gamma_k(\phi) = \sup_{x \in I} \left|(\sinh x) A_x^k \phi(x)\right| < \infty,$$

where A_x is the differential operator

but it is not likely to have what I need.

Reference 5. This is from Encyclopedia of Mathematics by Hazewinkel

MEHLER–FOCK TRANSFORM, *Mehler–Fok transform* - The **integral transform**

$$F(x) = \int_0^\infty P_{i\tau-1/2}(x) f(\tau)\, d\tau, \quad 1 \leqslant x < \infty, \tag{1}$$

where $P_\nu(x)$ is the Legendre function of the first kind (cf. **Legendre functions**). If $f \in L[0, \infty)$, the function $|f'(\tau)|$ is locally integrable on $[0, \infty)$ and $f(0)=0$, then the following inversion formula is valid:

$$f(\tau) = \tau \tanh \pi\tau \int_1^\infty P_{i\tau-1/2}(x) F(x)\, dx. \tag{2}$$

The *Parseval identity*. Consider the Mehler−Fock transform and its inverse defined by the equalities

$$G(\tau) = \int_1^\infty \sqrt{\tau \tanh \pi\tau}\, P_{i\tau-1/2}(x) g(x)\, dx,$$

$$g(x) = \int_0^\infty \sqrt{\tau \tanh \pi\tau}\, P_{i\tau-1/2}(x) G(\tau)\, d\tau.$$

If $g_i(x)$, $i=1, 2$, are arbitrary real-valued functions satisfying the conditions

$$g_i(x) x^{-1/2} \ln(1+x) \in L(1, \infty), \quad g_i(x) \in L_2(1, \infty),$$

then

$$\int_0^\infty G_1(\tau) G_2(\tau)\, d\tau = \int_1^\infty g_1(x) g_2(x)\, dx.$$

The generalized Mehler−Fock transform and the corresponding inversion formula are:

$$F(x) = \int_0^\infty P^{(k)}_{i\tau-1/2}(x) f(\tau)\, d\tau, \tag{3}$$

and

$$f(\tau) = \frac{1}{\pi} \tau \sinh \pi\tau \Gamma\left(\frac{1}{2} - k + i\tau\right) \Gamma\left(\frac{1}{2} - k - i\tau\right) \times \tag{4}$$

$$\times \int_1^\infty P^{(k)}_{i\tau-1/2}(x) F(x)\, dx,$$

where $P^{(k)}_\nu(x)$ are the associated Legendre functions of the first kind. For $k=0$ formulas (3) and (4) reduce to (1) and (2); for $k=1/2$, $y=\cosh\alpha$, formulas (3) and (4) lead to the **Fourier cosine transform**, and for $k=-1/2$, $y=\cosh\alpha$ to the **Fourier sine transform**. The transforms (1) and (2) were introduced by F.G. Mehler [1]. The basic theorems were proved by V.A. Fock [V.A. Fok].

References

[1] MEHLER, F.G.: 'Ueber eine mit den Kugel- und Cylinderfunctionen verwandte Function und ihre Anwendung in der Theorie der Electricitätsvertheilung', *Math. Ann.* **18** (1881), 161-194.

[2] FOK, V.A.: 'On the representation of an arbitrary function by an integral involving Legendre functions with complex index', *Dokl. Akad. Nauk SSSR* **39** (1943), 253-256 (in Russian).

[3] DITKIN, V.A. and PRUDNIKOV, A.P.: 'Operational calculus', *Progress in Math.* **1** (1968), 1-75. (*Itogi Nauk. Mat. Anal. 1966* (1967), 7-82)

Yu.A. Brychkov
A.P. Prudnikov

Editorial comments.

References

[A1] SNEDDON, I.N.: *The use of integral transforms*, McGraw-Hill, 1972.

AMS 1980 Subject Classification: 44A15

There is good old Sneddon again! That book might have something for me and is just sitting there

Sneddon, Ian Naismith.:
The use of integral transforms[by] Ian N. Sneddon.
New York : McGraw-Hill, [c1972]

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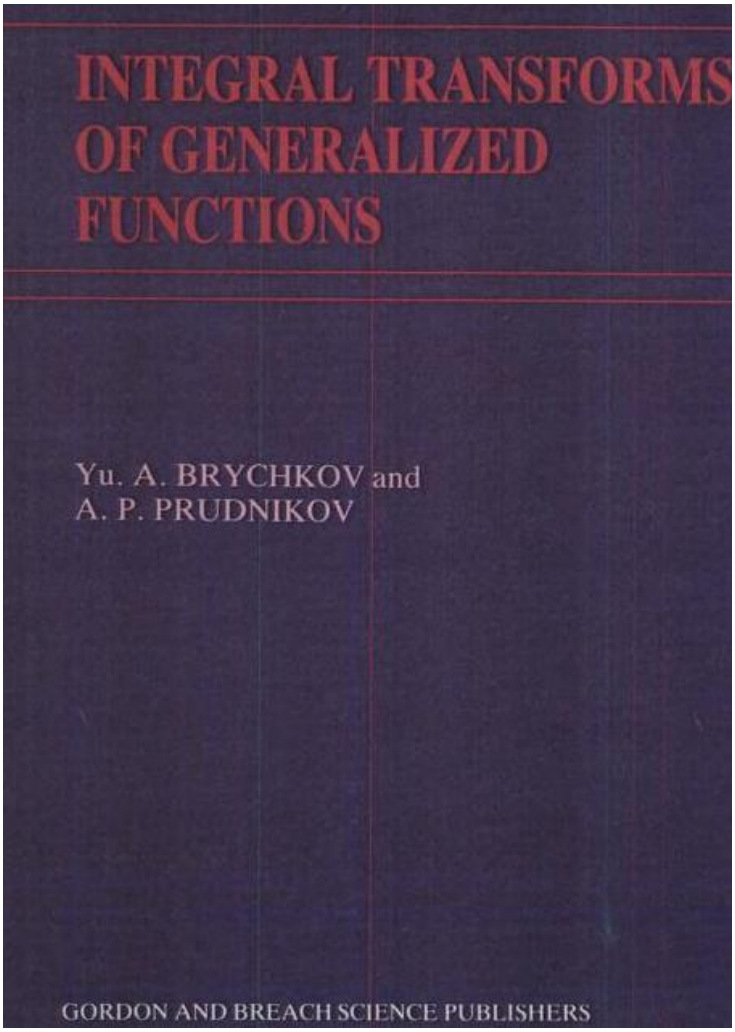
Request	Marriott Library	Widstoe Bldg: Math Library	QA432 .S64		60 Days	On Shelf		31114014178620		

Reference 6 These are the guys who did the above encyclopedia entry:



7.9 THE MEHLER-FOCK TRANSFORMATION

1. *The Mehler-Fock transform* of a suitably restricted ordinary function $f(t)$ is defined by

$$F(x)=\int_1^\infty P^{\mu}_{ix-1/2}(t)f(t)\mathrm{d}t, \qquad x\geqslant 0, \tag{1}$$

where $P^{\mu}_{\nu}(z)$ is the Legendre function of the first kind.

2. Buggle [1] has given an extension of (1) to generalized functions for $\mu=-n=0,-1,-2,\ldots$ Set $0<\alpha\leqslant 1$, $I=(1,\infty)$,

$$\lambda_{\alpha}(t)=\begin{cases}1, & 1<t\leqslant 2,\\ (t-q)^{(1-\alpha)/2}, & 2<t<\infty,\end{cases}$$

The section then gets very technical, reviewing lots of people's results. Nothing on how to do integrals!

Reference 7 Lectures by Sneddon specifically on M-F

Accession Number : AD0634364

Title : LECTURES ON MEHLER-FOCK TRANSFORMS AND SOME OF THEIR APPLICATIONS,

Corporate Author : NORTH CAROLINA STATE UNIV RALEIGH APPLIED MATHEMATICS RESEARCH GROUP

Personal Author(s) : Sneddon, Ian N.

Report Date : 20 APR 1966

Pagination or Media Count : 76

Abstract : The notes are based upon the content of four lectures given at North Carolina State University in the spring of 1966 and may be regarded as a continuation of the notes issued in the two preceding years. Their aim is to provide basic results in the theory of Mehler-Fock transforms that can be used to study certain boundary value problems in the classical theory of elasticity.

Descriptors : (*INTEGRAL TRANSFORMS, THEORY), (*BOUNDARY VALUE PROBLEMS, INTEGRAL TRANSFORMS), (*ELASTIC PROPERTIES, BOUNDARY VALUE PROBLEMS)

Subject Categories : NUMERICAL MATHEMATICS
MECHANICS

Distribution Statement : APPROVED FOR PUBLIC RELEASE

I don't know how to narrow down to finding a table of integrals or instructions how to do them! My web search has not really produced anything useful.