A certain algebraic integral PhL 3.26.05

Note that page numbering is turned on in this template.

$$\int_{\varphi}^{\theta_0} d\theta \, \sin\theta/\sqrt{\cos\varphi-\cos\theta} * \cos^{-1}[\sqrt{1+\cos\theta_0}\,/\sqrt{1+\cos\theta}\,] = \sqrt{2}\,\pi\,[\cos(\varphi/2) - \cos(\theta_0/2)\,] \qquad (G.22)$$

$$\sqrt{\cos\varphi-\cos\theta} = \sqrt{2}\,\sqrt{\cos^2(\varphi/2)-\cos^2(\theta/2)} \qquad ***$$

$$\cos^{-1}[\sqrt{1+\cos\theta_0}\,/\sqrt{1+\cos\theta}\,] = \cos^{-1}[\cos(\theta_0/2)/\cos(\theta/2)]$$

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2) \qquad ***$$

so

$$\text{LHS} = \sqrt{2} \int_{\varphi}^{\theta_0} d\theta \, \sin(\theta/2)\cos(\theta/2)/\sqrt{\cos^2(\varphi/2)-\cos^2(\theta/2)} * \cos^{-1}[\cos(\theta_0/2)/\cos(\theta/2)]$$

Let $x = \cos(\theta/2)$ and $a = \cos(\varphi/2)$ and $b = \cos(\theta_0/2)$.

Then $dx = -(1/2)\sin(\theta/2)\,d\theta$ so get ***

$$\text{LHS} = \sqrt{2} \int_{b}^{a} 2\,dx\,x / \sqrt{a^2-x^2} * \cos^{-1}(b/x)$$

$$= (2\sqrt{2}) \int_{b}^{a} dx\,x / \sqrt{a^2-x^2} * \sec^{-1}(x/b)$$

Now write

$$x\,(a^2-x^2)^{-1/2} = 2\,(-1/2)\,\partial_x\,(a^2-x^2)^{1/2} = -\,\partial_x\,(a^2-x^2)^{1/2}$$ Maple agrees

so now

$$\text{LHS} = -(2\sqrt{2}) \int_{b}^{a} dx\,\partial_x\,(a^2-x^2)^{1/2}\,\sec^{-1}(x/b)$$

$$= -(2\sqrt{2})\,[\,(a^2-x^2)^{1/2}\,\sec^{-1}(x/b)]|_{b}^{a} + (2\sqrt{2}) \int_{b}^{a} dx(a^2-x^2)^{1/2}\,\partial_x\,\sec^{-1}(x/b)$$

$$= (2\sqrt{2}) \int_{b}^{a} dx\,(a^2-x^2)^{1/2}\,\partial_x\,\sec^{-1}(x/b)$$

But we know that

$\partial_x \sec^{-1}(x/b) = (b/x)/\sqrt{(x/b)^2-1} * (1/b) = (1/x)/\sqrt{(x/b)^2-1} = (b/x)/\sqrt{x^2-b^2}$

So we then have

LHS = $(2\sqrt{2}\ b) \int_b^a dx\ x^{-1} (a^2-x^2)^{1/2} (x^2-b^2)^{-1/2}$

Now let $y = x^2$ so $dy = 2xdx$ and get $dx/x = dy/(2x^2) = dy/(2y)$. Let $a^2 = \alpha$ and $b^2 = \beta$. ***

LHS = $(2\sqrt{2}\ b) \int_\beta^\alpha dy/(2y) (\alpha-y)^{1/2} (y-\beta)^{-1/2}$

$= (\sqrt{2}\ b) \int_\beta^\alpha dy/(y) (\alpha-y)^{1/2} (y-\beta)^{-1/2}$

Now we want to look up this integral

$$\int_\beta^\alpha \frac{\sqrt{\alpha-y}}{y\sqrt{y-\beta}}\,dy$$

Definite integral:

$$\int_b^a \frac{\sqrt{a-y}}{y\sqrt{y-b}}\,dy = \pi\left(\sqrt{\frac{a}{b}}-1\right)$$

I think this has to get reduced to an F function. I cannot find it in GR7. Maple can't do it. But Wolfram can do it and gets

$\pi\ (\sqrt{\alpha/\beta}\ - 1)$

So our result is

LHS = $(\sqrt{2}\ b)\ \pi\ (\ a/b - 1)$

$= (\sqrt{2}\ \pi) * (a - b) = \sqrt{2}\ \pi\ (\cos(\varphi/2) - \cos(\theta_0/2))$

So how did Wolfram do this integral? It can do the indefinite integral! Well, I have an idea

Let $z = (a-y)/(y-b)$ so the integration range is then 0 to ∞

Maple shows that $dy/y = dz\ (b-a)/[(z+1)(zb+a)]$. The integral in blue above becomes

$(a-b) \int_0^\infty dz\ \sqrt{z}\ /\ [(z+1)(zb+a)] = (a-b)/b^* \int_0^\infty dz\ \sqrt{z}\ /\ [(z+1)(z+a/b)]$

$$= ([a/b] - 1) \int_0^\infty dz \sqrt{z} / [(z+1)(z+[a/b])]$$

Let c = a/b so integral is then

$$(c-1) \int_0^\infty dz \sqrt{z} / [(z+1)(z+c)]$$

Now write

$$1/[(z+1)(z+c) = (c-1)^{-1} [\ 1/(z+1) - 1/(z+c)\ ]$$

Then our integral becomes

$$\int_0^\infty dz \sqrt{z} / [\ 1/(z+1) - 1/(z+c)\ ] = \int_0^\infty dz\sqrt{z} /(z+1) - \int_0^\infty dz\sqrt{z} /(z+c)$$

But each term diverges! Suppose we regulate with a cutoff? Maple says

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Int(sqrt(z)/(z+c),z);
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$$\int \frac{\sqrt{z}}{z+c}\,dz$$

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int(sqrt(z)/(z+c),z);
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$$2\sqrt{z} - 2\sqrt{c}\ \arctan\left(\frac{\sqrt{z}}{\sqrt{c}}\right)$$

Thus, the zero endpoint gives nothing and we have

$$\int_0^\Lambda dz\sqrt{z} /(z+c) = 2\sqrt{\Lambda} - 2\sqrt{c}\ \tan^{-1}(\sqrt{\Lambda}/\sqrt{c}\ )$$

Our difference will then be

$$\int_0^\Lambda dz\sqrt{z} /(z+c) - \int_0^\Lambda dz\sqrt{z} /(z+1)$$

$$= [2\sqrt{\Lambda} - 2\sqrt{c}\ \tan^{-1}(\sqrt{\Lambda}/\sqrt{c}\ )\ ] - [2\sqrt{\Lambda} - 2\tan^{-1}(\sqrt{\Lambda}\ )$$

$$= 2\tan^{-1}(\sqrt{\Lambda}\ ) - 2\sqrt{c}\ \tan^{-1}(\sqrt{\Lambda}/\sqrt{c}\ )$$

Then we take $\Lambda\to\infty$ and $\tan^{-1}\infty = \pi/2$ and we have

$$= \pi - \pi\sqrt{c} \quad = \pi(1-\sqrt{c}\ )$$

This we have shown that

$$\int_{\beta}^{\alpha} dy/(y)\,(\alpha-y)^{1/2}\,(y-\beta)^{-1/2} = (c-1)\int_{0}^{\infty} dz\,\sqrt{z}\,/\,[(z+1)(z+c)]$$

$$= \text{regulate}\ \{\ \int_{0}^{\infty} dz\sqrt{z}\,/(z+1)\ -\ \int_{0}^{\infty} dz\sqrt{z}\,/(z+c)\ \}$$

$$= \pi(\sqrt{c}\,-1)$$

So here is how this all goes:

Start with

$$R = \int_{\varphi}^{\theta_0} d\theta\,\sin\theta/\sqrt{\cos\varphi-\cos\theta}\ *\ \cos^{-1}[\sqrt{1+\cos\theta_0}\,/\sqrt{1+\cos\theta}\,]$$

Let $x = \cos(\theta/2)$ and $a = \cos(\varphi/2)$ and $b = \cos(\theta_0/2)$ to get

$$R = (2\sqrt{2}\,)\int_{b}^{a} dx\,x\,/\,\sqrt{a^2-x^2}\ *\ \sec^{-1}(x/b)$$

Write $x\,(a^2-x^2)^{-1/2} = -\,\partial_x\,(a^2-x^2)^{1/2}$ and do parts (parts vanish) to get

$$R = (2\sqrt{2}\,)\int_{b}^{a} dx\,(a^2-x^2)^{1/2}\,\partial_x\,\sec^{-1}(x/b)$$

Compute $\partial_x\,\sec^{-1}(x/b) = (b/x)\,/\sqrt{x^2-b^2}$ to get

$$R = (2\sqrt{2}\,b)\int_{b}^{a} dx\,x^{-1}\,(a^2-x^2)^{1/2}\,(x^2-b^2)^{-1/2}$$

Let $y = x^2$ to get

$$R = (\sqrt{2}\,b)\int_{\beta}^{\alpha} dy/(y)\,(\alpha-y)^{1/2}\,(y-\beta)^{-1/2} \qquad a^2 = \alpha \text{ and } b^2 = \beta$$

Let $z = (a-y)/(y-b)$ to get

$$R = ([a/b]-1)\int_{0}^{\infty} dz\,\sqrt{z}\,/\,[(z+1)(z+[a/b])]$$

Let $c = a/b$ etc etc. OK, this is not really any faster than my original method.

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or

$$R = \int_{\cos\theta_0}^{\cos\varphi} dz \,/\sqrt{\cos\varphi - z}\,) * \cos^{-1}[\sqrt{1+\cos\theta_0}\,/\sqrt{1+z}\,]$$

$$= \int_{a}^{b} dz\,(b-z)^{-1/2} \cos^{-1}[\sqrt{1+a}\,/\sqrt{1+z}\,] \qquad a = \cos\theta_0 \qquad b = \cos\varphi$$

where the reader is asked to forget about the previous a and b. In this form we can do parts integration to get rid of the painful arc trig function. The "parts" vanish and we are left with

$$R(\varphi) = 2 \int_{a}^{b} dz\,(b-z)^{1/2}\,\partial_z \sec^{-1}[\sqrt{1+z}\,/\sqrt{1+a}\,]$$

$$= \sqrt{1+a} \int_{a}^{b} dz\,(b-z)^{1/2} / [(1+z)\sqrt{z-a}\,] \qquad \text{let } z = y+a:$$

$$= \sqrt{1+a} \int_{0}^{b-a} dy\,([b-a]-y)^{1/2}\,(y+1+a)^{-1}\,y^{-1/2} \qquad \text{let } s = y^{-1};$$

$$= (b-a)^{1/2}\,(1+a)^{-1/2} \int_{1/(b-a)}^{\infty} ds\, s^{-1}\,(\,s - [b-a]^{-1})^{1/2}(s + [1+a]^{-1})^{-1}$$

$$= (\pi/2)\,(b-a)\,(b+1)^{1/2}\,(a+1)^{-1}\,\{\,F(1,3/2;\,2;\,-(b-a)/(1+a))\} \qquad \text{// see below}$$

$$= (\pi/2)\,(b-a)\,(b+1)^{1/2}\,(a+1)^{-1}\,\{\;2(a+1)\,[\sqrt{b+1} - \sqrt{a+1}\;]\,/\,[\sqrt{b+1}\,(b-a)\,]\,\}$$

$$= \pi\;\;[\sqrt{b+1} - \sqrt{a+1}\;]$$

$$= \sqrt{2}\,\pi\;[\cos(\varphi/2) - \cos(\theta_0/2)\,] \qquad \text{(G.20)}$$

where we used corrected GR7 page 317 3.197.2,

$$\int_{u}^{\infty} ds\, s^{-\lambda}\,(s-u)^{\mu-1}\,(s+\beta)^{\nu} = u^{-\lambda}\,(\beta+u)^{\mu+\nu}\;B(\lambda-\mu-\nu,\mu)\,F(\lambda,\mu;\,\lambda-\nu;\,-\beta/u) \qquad \text{(G.21)}$$

So we have shown that

$$\int_{\varphi}^{\theta_0} d\theta\,\sin\theta/\sqrt{\cos\varphi-\cos\theta} * \cos^{-1}[\sqrt{1+\cos\theta_0}\,/\sqrt{1+\cos\theta}\,] = \sqrt{2}\,\pi\;[\cos(\varphi/2) - \cos(\theta_0/2)\,] \qquad \text{(G.22)}$$