Integral: cos(nx) over sqrt(a-bcosx) PhL 1.18.10

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New 0. New summary (now that things have come to light finally)

From our "transforms.doc" we can write the cosine Fourier series this way for f(θ) even in θ:

$$f(\theta) = a_0/2 + \Sigma_{n=1}^{\infty} a_n \cos(n\theta) = (1/2) \Sigma_{n=0}^{\infty} \varepsilon_n \, a_n \cos(n\theta) \quad \text{// expansion}$$

$$a_n = (1/\pi) \int_{-\pi}^{\pi} d\theta \, f(\theta) \cos(n\theta) \quad \text{where } \varepsilon_n = 2-\delta_{n,0} \quad \text{// projection}$$

$$\int_0^{\pi} d\theta \cos(n\theta)\cos(n'\theta) = (\pi/\varepsilon_n) \delta_{nn'} \quad \text{// orthogonality}$$

$$\Sigma_{n=0}^{+\infty} \varepsilon_n \cos[n(\theta-\theta')] = 2\pi \, \delta(\theta-\theta') \quad \text{// completeness}$$

and I will translate the first two lines to say

$$f(x) = (1/2) \Sigma_{n=0}^{\infty} \varepsilon_n \, A_n \cos(nx) \quad \text{// expansion}$$

$$A_n = (2/\pi) \int_0^{\pi} dx \, f(x) \cos(nx) \quad \text{// projection}$$

We wish now to apply this to the function

$$f(x) = 1/\sqrt{a - b\cos(x)}$$

and we find that

$$A_n = (2/\pi) \int_0^{\pi} dx\, f(x)\cos(nx) = (2/\pi) \int_0^{\pi} dx\, \cos(nx)/\sqrt{a - b\cos(x)}$$

The integral shown here is the one of interest in this document! We show in Section 9 below that this integral is given by this amazingly simple result

$$\int_0^{\pi} dx\, \cos(nx)/\sqrt{a - b\cos(x)} = \sqrt{2/b}\; Q_{n-1/2}(a/b)$$

so that we then have

$$A_n = (2/\pi)\sqrt{2/b}\; Q_{n-1/2}(a/b)$$

which is a MUCH simpler result that the finite sum of hypergeometric functions I was getting before I realized this simple form for the result. Therefore, we know that

$$1/\sqrt{a - b\cos(x)} = (1/\pi)\sqrt{2/b}\; \Sigma_{n=0}^{\infty}\, \varepsilon_n\; Q_{n-1/2}(a/b)\cos(nx) \qquad \text{// expansion}$$

$$\int_0^{\pi} dx\, \cos(nx)/\sqrt{a - b\cos(x)} = \sqrt{2/b}\; Q_{n-1/2}(a/b) \qquad \text{// projection}$$

New 1. An immediate application: free space Green's Function in Cylindrical Coordinates

If we have two points $\mathbf{r} = (\rho,\varphi,z)$ and $\mathbf{r}' = (\rho',\varphi',z')$ in cylindrical coordinates, it is very easy first of all to show that [using $x = \rho\cos\varphi$, $y = \rho\sin\varphi$]

$$1/R \equiv 1/|\mathbf{r}-\mathbf{r}'| = 1/\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi') + (z-z')^2}$$

We can then define

$$a = \rho^2 + \rho'^2 + (z-z')^2 \qquad a/b = (\rho^2 + \rho'^2 + (z-z')^2)/(2\rho\rho')$$
$$b = 2\rho\rho' \qquad \sqrt{2/b} = 1/\sqrt{\rho\rho'}$$
$$x = \varphi-\varphi'$$

so that

$$1/R = 1/\sqrt{a - b\cos(x)} = (1/\pi)\sqrt{2/b}\; \Sigma_{n=0}^{\infty}\, \varepsilon_n\; Q_{n-1/2}(a/b)\cos(nx)$$

$$= (1/\pi\sqrt{\rho\rho'})\, \Sigma_{n=0}^{\infty}\, \varepsilon_n\; \cos[n(\varphi-\varphi')]\, Q_{n-1/2}[(\rho^2 + \rho'^2 + (z-z')^2)/(2\rho\rho')]$$

This then is a form of the "free space Green's function" (aka the "fundamental solution" by Stak) in cylindrical coordinates. This equation appears as (5) in the 2007 Selvaggi paper "toroidal electrostatics.pdf" which I have today been reading. [5.2.10] It also appears in the 1953 famous monograph of Chester Snow in this form

$$(6)\qquad \frac{1}{R} = \frac{2}{\pi\sqrt{\rho\rho_1}} \sum_{m=0}^{\infty} \epsilon_m Q_{m-\frac{1}{2}}\left(1 + \frac{D^2}{2\rho\rho_1}\right)\cos m(\phi-\phi_1), \quad \begin{cases} \epsilon_0 = \frac{1}{2} \\ \epsilon_m = 1, \text{ if } m \neq 0. \end{cases}$$

$$D^2=(x-x_1)^2+(\rho-\rho_1)^2,$$

where Chester defines his ε'_n so that $2\varepsilon'_n = \varepsilon_n$ and he uses x instead of z and 1 instead of prime. I think this came from a website called the Hathi Trust

http://catalog.hathitrust.org/Record/001687950

The above equation comes from page 229 of " Hypergeometric and Legendre functions with applications ...a" which has full view at the Hathi site, but you cannot download anything more than a page at a time. The book on Cap and Ind calculations is there, but you cannot view it!

We should compare this 1/R result to Jackson page 62 in sphericals

$1/R = \Sigma_{n=0}^{\infty} (r_<^n / r_>^{n+1})\ P_n[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')]$

There is something that should be said about this comparison, but I don't yet know what it is. In this case outside the P_n we have a non-oscillatory factor, with two oscillatory appearing in the P argument. But in the Q form above, we have an outside oscillatory factor and inside we have the other two. I think you can always write the 1/R in several different ways for a given coordinate system, I have seen this where we had 1/R ~ JJ one way and ~ IK a different way. An addition theorem is involved in all these things. And this brings in the group theory stuff as usual.

New 2. Evaluation of $\int_0^{2\pi} d\psi\ \cos(n\psi) / \sqrt{r^2 + r'^2 - 2rr'\cos(\psi-\psi')}$

Consider the following integral

$$I_n = \int_0^{2\pi} d\psi\ \cos(n\psi) / \sqrt{r^2 + r'^2 - 2rr'\cos(\psi-\psi')}$$

We can change the integration variable to $x = \psi-\psi'$ to get (and also shift the range)

$$I_n = \int_{-\pi}^{\pi} dx\ \cos(n[x+\psi']) / \sqrt{r^2 + r'^2 - 2rr'\cos x}$$

We then expand

$\cos(n[x+\psi']) = \cos(nx)\cos(n\psi') - \sin(nx)\sin(n\psi')$

The 2nd term results in an even integral with an odd integrand and so makes no contribution. We then conclude that

$$I_n = \cos(n\psi') \int_{-\pi}^{\pi} dx \cos(nx) / \sqrt{r^2 + r'^2 - 2rr'\cos x}$$

$$= 2\cos(n\psi') \int_{0}^{\pi} dx \cos(nx) / \sqrt{r^2 + r'^2 - 2rr'\cos x}$$

But above we claimed that

$$\int_{0}^{\pi} dx \cos(nx)/\sqrt{a - b\cos(x)} = \sqrt{2/b}\ Q_{n-1/2}(a/b)$$

so we can set $a = r^2+r'^2$ and $b = 2rr'$ which tells us $\sqrt{2/b} = 1/\sqrt{rr'}$. We then get

$$I_n = 2\cos(n\psi')\ 1/\sqrt{rr'}\ * Q_{n-1/2}[(r^2+r'^2)/(2rr')]$$

Here then is what we have shown:

$$\int_{0}^{2\pi} d\psi \cos(n\psi) / \sqrt{r^2 + r'^2 - 2rr'\cos(\psi-\psi')} = 2\cos(n\psi')\ Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

New 3. Evaluation of the integral $\int_{0}^{2\pi} d\psi \sin(n\psi) / \sqrt{r^2 + r'^2 - 2rr'\cos(\psi-\psi')}$

Lets redo the above in the case of sin instead of cos.

$$I_n = \int_{0}^{2\pi} d\psi \sin(n\psi) / \sqrt{r^2 + r'^2 - 2rr'\cos(\psi-\psi')}$$

We can change the integration variable to $x = \psi-\psi'$ to get (and also shift the range)

$$I_n = \int_{-\pi}^{\pi} dx \sin(n[x+\psi']) / \sqrt{r^2 + r'^2 - 2rr'\cos x}$$

We then expand

$\sin(n[x+\psi']) = \sin(nx)\cos(n\psi') + \cos(nx)\sin(n\psi')$

The 1st term results in an even integral with an odd integrand and so makes no contribution. We then conclude that

$$I_n = \sin(n\psi') \int_{-\pi}^{\pi} dx \cos(nx) / \sqrt{r^2 + r'^2 - 2rr'\cos x}$$

$$= 2 \sin(n\psi') \int_{0}^{\pi} dx \cos(nx) / \sqrt{r^2 + r'^2 - 2rr'\cos x}$$

But above we claimed that

$$\int_{0}^{\pi} dx \cos(nx)/ \sqrt{a - b \cos(x)} = \sqrt{2/b}\ Q_{n-1/2}(a/b)$$

so we can set $a = r^2+r'^2$ and $b = 2rr'$ which tells us $\sqrt{2/b} = 1/\sqrt{rr'}$. We then get

$$I_n = 2 \sin(n\psi') 1/\sqrt{rr'} * Q_{n-1/2}[(r^2+r'^2)/(2rr')]$$

Here then is what we have shown:

$$\int_{0}^{2\pi} d\psi \sin(n\psi) / \sqrt{r^2 + r'^2 - 2rr' \cos(\psi-\psi')} = 2 \sin(n\psi') Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

New 4. Evaluation of $\int_{0}^{2\pi} d\psi \cos(n\psi - \psi_n) / \sqrt{r^2 + r'^2 - 2rr' \cos(\psi-\psi')}$

Here ψ_n are some constant phase shifts. We start off writing

$$\cos(n\psi-\psi_n) = \cos(n\psi)\cos(\psi_n) + \sin(n\psi)\sin(\psi_n)$$

Let's call our New 4 integral J_4, and those in the previous two sections J_2 and J_3. Then we can see that

$$J_4 = \cos(\psi_n)\, J_2 + \sin(\psi_n)\, J_3$$

$$= \cos(\psi_n)\, 2 \cos(n\psi') Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$
$$+ \sin(\psi_n)\ 2 \sin(n\psi') Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

$$= [\cos(n\psi')\cos(\psi_n) + \sin(n\psi')\sin(\psi_n)]\, 2\, Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

$$= 2 \cos(n\psi'-\psi_n) Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

We see that we basically get the New 2 result with the phase shifts added. The phase shifts make no difference in the integral, they just "pass through".

New 5. Alternate forms for New 2 and New 3 and New 4 results.

In my Sneddon raw notes for Section 6.3, I derive two *alternate* forms for the integrals shown in the New sections above.

$$\pi \int_0^\infty J_n(rx)J_n(r'x)\,dx = 2\,(rr')^{-n} \int_0^{\min(r,r')} ds\, s^{2n} / [\sqrt{r^2-s^2}\sqrt{r'^2-s^2}]$$

$$= Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

Thus, we can write

$$\int_0^{2\pi} d\psi \cos(n\psi) / \sqrt{r^2 + r'^2 - 2rr'\cos(\psi-\psi')} = 2\cos(n\psi') Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

$$= 2\cos(n\psi')\,\pi \int_0^\infty J_n(rx)J_n(r'x)\,dx = 4\cos(n\psi')\,(rr')^{-n} \int_0^{\min(r,r')} ds\, s^{2n} / [\sqrt{r^2-s^2}\sqrt{r'^2-s^2}]$$

and

$$\int_0^{2\pi} d\psi \sin(n\psi) / \sqrt{r^2 + r'^2 - 2rr'\cos(\psi-\psi')} = 2\sin(n\psi') Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

$$= 2\sin(n\psi')\,\pi \int_0^\infty J_n(rx)J_n(r'x)\,dx = 4\sin(n\psi')\,(rr')^{-n} \int_0^{\min(r,r')} ds\, s^{2n} / [\sqrt{r^2-s^2}\sqrt{r'^2-s^2}]$$

The New 4 result is just the New 2 one with phase shifts added and we get

$$\int_0^{2\pi} d\psi \cos(n\psi-\psi_n) / \sqrt{r^2 + r'^2 - 2rr'\cos(\psi-\psi')} = 2\cos(n\psi'-\psi_n) Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

$$= 2\cos(n\psi'-\psi_n)\,\pi \int_0^\infty J_n(rx)J_n(r'x)\,dx = 4\cos(n\psi'-\psi_n)\,(rr')^{-n} \int_0^{\min(r,r')} ds\, s^{2n} / [\sqrt{r^2-s^2}\sqrt{r'^2-s^2}]$$

If we take the extreme left and right elements of the above equality chain, we obtain the result quoted in Sneddon page 70 equation 3.4.6 !! This is the magic integral which allows the Copson solution of the Dirichlet disk to be solved!

0. Old Summary:

The integral of interest is this: (n = integer)

$$I = \int_0^{2\pi} dx \cos(nx)/\sqrt{a - b\cos(x)} = (4/\sqrt{a-b}) \int_0^{\pi/2} dx \cos(2nx)/\sqrt{1 - k^2\sin^2(x)}$$

$$= \int_{-\pi}^{\pi} dx \cos(nx)/\sqrt{a - b\cos(x)} = 2\int_0^{\pi} dx \cos(nx)/\sqrt{a - b\cos(x)}$$

where we have recast it into elliptic form (this is derived below).

Consider the Fourier Analysis of the function $1/\sqrt{a - b\cos(x)}$ on $(0,2\pi)$,

$$f(x) = 1/\sqrt{a - b\cos(x)} = a_0/2 + \Sigma_{n=1}^{\infty} a_n \cos(nx) \qquad \text{// expansion}$$

$$I = \pi a_n = \int_0^{2\pi} dx \cos(nx)/\sqrt{a - b\cos(x)} \qquad \text{// projection}$$

The integral I we study here is basically the Fourier coefficient for this expansion.

The result is, as shown below,

$$I = (2/\sqrt{a - b}\,) \Sigma_{s=0}^{n} (-1)^s \ (2n, 2n-2s)\ B[\ s+1/2, n-s+1/2\ ]\ F\ [s+1/2, 1/2; n+1; -2b/(a-b)\ ]$$

and here are two special cases also figured out below, where $k^2 = -2b/(a-b)$,

n=0 $\qquad I = 4/(a-b)\ K(k)$

n=1 $\qquad I = (4/\sqrt{a - b}\,)\ K(k) \qquad k^2 = -2b/(a-b)$

n=2 $\qquad I = (4/\sqrt{a - b}\,)\ \ [(a/b)\ K(k) + (1 - (a/b)\ E(k)\ ]$

I have confirmed the n=1 result with Maple, and the n=2 with Wolfram's integrator. the n=0 case comes from GR7 p409 where I take b→ -b. The ±b is the same here since changing the order in which you add things makes no difference.

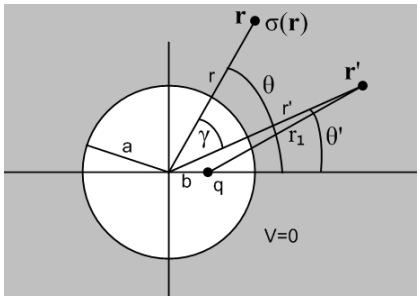
For general n, we get a sum of n+1 F functions with c = 1,2,3...n+1. You have to use the Gauss F shifter formulas to reduce the higher c functions to those having c = 1, and this is very painful to do by hand. Maple V does not seem able to do it at all, it does not even know how to express an F in terms of standard special functions (well, not true, see convert). In general, I think for any n you get some linear combination of K and E functions. For some reason, this Fourier Expansion is a very messy one. Expansion onto Legendre's is nicer. The two fundamental F functions at the bottom of the Gauss ladder are these:

$$F(1/2,\ 1/2,\ 1,\ z) = (2/\pi)K(k)$$
$$F(-1/2,\ 1/2,\ 1,\ z) = (2/\pi)E(k)$$

1. One place in which this integral arises.

In "ring and disk Green's problems", and in electrostatic in general, you often integrate over a surface to get a potential, and you want to enforce that potential to be 0. In the problem of a point charge inside the hole of a metal plate, we get this equation

$$V(\mathbf{r}') = q/r_1 + \int_a^{\infty} r\, dr \int_0^{2\pi} d\theta\ \ \sigma(r,\theta) / |\mathbf{r} - \mathbf{r}'| = 0$$



We want to solve this for some σ(r,θ) that makes the equation be true. One approach is to expand

$$\sigma(r,\theta) = \Sigma_{n=0}^{\infty} \sigma_n(r) \cos(n\theta)$$

and then you have this resulting equation

$$V(r',\theta') = q/r_1 + \Sigma_{n=0}^{\infty} \int_a^{\infty} r\, dr\, \sigma_n(r) \int_0^{2\pi} d\theta \cos(n\theta) / \sqrt{r^2 + r'^2 - 2rr' \cos(\theta-\theta')} = 0$$

This is where our integral arises. Define f = r'/r and this becomes

$$V(\mathbf{r}') = q/r_1 + \Sigma_{n=0}^{\infty} \int_a^{\infty} dr\, \sigma_n(r) \{ \int_0^{2\pi} d\theta \cos(n\theta) / \sqrt{1^2 + f^2 - 2f \cos(\theta-\theta')} \}$$

and there is our integral of interest in the curly brackets. By changing to θ" = θ-θ', expanding the cos, looking at symmetry, we end up with

$$I(f) = \cos(n\theta') F_n(f)$$

$$F_n(f) = \int_0^{2\pi} d\theta \cos n\theta / \sqrt{1^2 + f^2 - 2f \cos(\theta)}$$

$$V(r',\theta') = q/r_1 + \Sigma_{n=0}^{\infty} \cos(n\theta') \int_a^{\infty} dr\, \sigma_n(r) F_n(f).$$

Let $a = 1+f^2 > 1$ and $b = 2f > 0$

$$F_n(a,b) = \int_0^{2\pi} dx \cos(nx) / \sqrt{a - b \cos(x)} = I,$$ our integral of interest

We have then, for this application, that

$a+b = (1+f)^2 > 0$ $a-b = (1-f)^2 > 0$ just for the record

The resulting integral is so ugly that probably this is a bad way to expand the surface charge, although it seems the simplest expansion you could use for a polar coordinates problem.

2. Another place it arises.

Someone hands you the function $f(x) = 1/\sqrt{a - b\cos(x)}$ on the interval $(0,2\pi)$. They ask you: "what is the Fourier expansion of this function" ? You go to Schaum p 131 and set $L = \pi$ and the answer is this:

$$f(x) = 1/\sqrt{a - b\cos(x)} \quad = a_0/2 + \Sigma_{n=1}^{\infty} a_n \cos(nx)$$

where

$$\pi a_n = \int_0^{2\pi} dx \cos(nx)/\sqrt{a - b\cos(x)} \quad = I$$

So our integral of interest is merely the normal everyday Fourier projection of this particular function.

3. Casting into elliptic integral form

[This was my original approach] From above we have,

$$I = \int_0^{2\pi} dx \cos(nx)/\sqrt{a - b\cos(x)} \quad = 2 \int_0^{\pi} dx \cos(nx)/\sqrt{a - b\cos(x)}$$

Now consider the following algebra,

$\cos(x) = 1-2\sin^2(x/2)$ let $x' = x/2$ $\cos(x) = 1 - 2\sin^2(x')$

$a - b\cos(x) = a - b[1 - 2\sin^2(x')] = (a-b) + 2b\sin^2(x') = (a-b)[1 - k^2\sin^2(x')]$ $k^2 = -2b/(a-b)$

The constant is 4 because got 2 from range $(0,\pi)$, and another 2 from $dx = 2dx'$,

$$I = (4/\sqrt{a - b}) \int_0^{\pi/2} dx' \cos(2nx')/\sqrt{1 - k^2\sin^2(x')}$$

$$I = (4/\sqrt{a - b}) \int_0^{\pi/2} dx \cos(2nx)/\sqrt{1 - k^2\sin^2(x)} \qquad k^2 = (b-a)/2b = [1 - (a/b)]/2$$

$a = 1+f^2$ $f = r'/r$ $b = 2f$

$a-b = 1+f^2 -2f = (1-f)^2$ $k^2 = -2b/(a-b) = -4f/(1-f)^2$

So our integral is then

$$I = (4/\sqrt{a - b})\ J \qquad \text{where} \qquad J = \int_0^{\pi/2} dx \cos(2nx)/\sqrt{1 - k^2\sin^2(x)}$$

I would say this was in "elliptical integral form", and this might be helpful for searching. Notice that the integral J can only depend on a and b only through k^2. So we might write

$$I = \int_0^{2\pi} dx \cos(nx)/\sqrt{a - b\cos(x)} \quad = \quad (4/\sqrt{a - b}\,)\, J$$

4. Searching the literature for this integral

GR definite elementary trig: Section 3.67 and 3.84 are the only possible places in the index list on page xiii

Section 3.67 (p 386): Right off the bat we have this interesting integral

$$\int_0^{\pi/2} \sin^{\alpha}x \cos^{\beta}x /\sqrt{1 - k^2 \sin^2(x)} \quad = \text{ combination of beta function and HGF}$$

I think this is what we can use.

5. Doing the integral

Now consider this fact

$$e^{imx} = (\cos(x) + i \sin(x))^m = (C + iS)^m$$

$$= \Sigma_{k=0}^{m} (m,k)\, C^k (iS)^{m-k} = \Sigma_{k=0}^{m} (i)^{m-k} (m,k)\, C^k S^{m-k}$$

Let's define a new mirror summation index j = m-k which runs from 0 to m. Then

$$e^{imx} = \Sigma_{j=0}^{m} (i)^j \ (m,m-j)\, C^{m-j} S^j$$

The factor $(i)^j$ is real when j = even, so we can then say

$$\cos(mx) = \Sigma_{j=0,\,even}^{m} (i)^j \ (m,m-j)\, C^{m-j} S^j$$

If m is odd, the sum stops of course at m-1. We know $(i)^j = (i)^{2N} = (-1)^N = (-1)^{j/2}$ so

$$\cos(mx) = \Sigma_{j=0,\,even}^{m} (-1)^{j/2} \ (m,m-j)\, C^{m-j} S^j$$

Now make a new summation index s = j/2 so we then have

$$\cos(mx) = \Sigma_{s=0}^{m/2} (-1)^s \ (m, m-2s)\, C^{m-2s} S^{2s}$$

where again the sum ends one early if m/2 is fractional. Now let's replace m with 2n to get

$$\cos(2nx) = \Sigma_{s=0}^{n} (-1)^s \ (2n, 2n\text{-}2s)\ C^{2n-2s}S^{2s}$$

and now there is no longer confusion about where the sum ends. We can now install this into our integral:

$$J = J(n,k^2) = \int_0^{\pi/2} dx \cos(2nx)/\sqrt{1 - k^2 \sin^2(x)}$$

$$= \int_0^{\pi/2} dx \ \{ \Sigma_{s=0}^{n} (-1)^s \ (2n, 2n\text{-}2s)\ C^{2n-2s}S^{2s} \}/\sqrt{1 - k^2 \sin^2(x)}$$

$$= \Sigma_{s=0}^{n} (-1)^s \ (2n, 2n\text{-}2s) \int_0^{\pi/2} dx \ [\cos(x)]^{2n-2s} [\sin(x)]^{2s} \ / \sqrt{1 - k^2 \sin^2(x)}$$

and now we have our GR p 386 integral with $\alpha = 2s$ and $\beta = 2n\text{-}2s$.

$$\int_0^{\pi/2} \sin^{\alpha}x \cos^{\beta}x /\sqrt{1 - k^2 \sin^2(x)} = (1/2)\ B[\ (\alpha+1)/2, (\beta+1)/2\ ]\ F\ [(\alpha+1)/2, 1/2; (\alpha+\beta+2)/2; k^2\ )$$

which is something we can at least comprehend. So

$$= (1/2)\ B[\ (2s+1)/2, (2n-2s+1)/2\ ]\ F\ [(2s+1)/2, 1/2; (2s+2n-2s+2)/2; k^2\ ]$$

$$= (1/2)\ B[\ s+1/2, n-s+1/2\ ]\ F\ [s+1/2, 1/2; n+1; k^2\ ]$$

This sure looks familiar to me! But I can't find it in the Legendre table. AS 556 shows other cases, but in all cases the a,b parameters are "correlated". So I guess we just leave it "as is". Then we have

$$J = \Sigma_{s=0}^{n} (-1)^s \ (2n, 2n\text{-}2s)\ (1/2)\ B[\ s+1/2, n-s+1/2\ ]\ F\ [s+1/2, 1/2; n+1; k^2\ ]$$

so there are no less than 6 gamma functions here, perhaps it could be simplified, but let's just leave it. We than have our result

$$I = \int_0^{2\pi} dx \cos(nx)/\sqrt{a - b \cos(x)}$$

$$= (4/\sqrt{a - b}\ )\ J(n,k^2)$$

$$= (2/\sqrt{a - b}\ ) \Sigma_{s=0}^{n} (-1)^s \ (2n, 2n\text{-}2s)\ B[\ s+1/2, n-s+1/2\ ]\ F\ [s+1/2, 1/2; n+1; -2b/(a-b)\ ]$$

so I think that is "not bad". This then is the General Formula for I.

6. Test for special case n = 0.

$$I = \int_0^{2\pi} dx\ 1/\sqrt{a - b\cos(x)}$$

$$= (2/\sqrt{a-b})\ \Sigma_{s=0}^{n}\ (-1)^s\ (2n, 2n-2s)\ B[\ s+1/2,\ n-s+1/2\ ]\ F\ [s+1/2,\ 1/2;\ n+1;\ -2b/(a-b)\ ]$$

$$= (2/\sqrt{a-b})\ (0, 0)\ B[1/2, +1/2\ ]\ F\ [1/2,\ 1/2;\ 1;\ -2b/(a-b)\ ]$$

$$= (2/\sqrt{a-b})\ \Gamma(1/2)^2/\Gamma(1)\ \ F\ [1/2,\ 1/2;\ 1;\ -2b/(a-b)\ ]$$

$$= (2/\sqrt{a-b})\ \pi\ \ F\ [1/2,\ 1/2;\ 1;\ -2b/(a-b)\ ]$$

Now look at AS page 591 top which says

$$K_{AS}(m) = (\pi/2)\ F(1/2,1/2;1;m) \qquad\qquad m = k^2 = -2b/(a-b)$$

so our answer was

$$I = (2/\sqrt{a-b})\ \pi\ *\ (2/\pi)\ K(k^2)\ \ = (4/\sqrt{a-b})\ K(k)$$

Here is Maple's evaluation of this integral

```
Int(1/sqrt(a - b*cos(x)),x=0..2*Pi);
```

$$\int_0^{2\pi} \frac{1}{\sqrt{a - b\cos(x)}}\,dx$$

```
int(1/sqrt(a - b*cos(x)),x=0..2*Pi);
```

$$4\,\frac{\text{EllipticK}\left(\sqrt{2}\sqrt{-\frac{b}{a-b}}\right)\sqrt{\frac{b+a}{a-b}}}{\sqrt{b+a}}$$

and we agree.

7. Test for special case n = 1.

$$I = \int_0^{2\pi} dx\ \cos(x)/\sqrt{a - b\cos(x)}$$

$$= (4/\sqrt{a-b})\ \ J$$

$$= (2/\sqrt{a-b})\ \Sigma_{s=0}^{n}\ (-1)^s\ (2n, 2n-2s)\ B[\ s+1/2,\ n-s+1/2\ ]\ F\ [s+1/2,\ 1/2;\ n+1;\ -2b/(a-b)\ ]$$

$$= (2/\sqrt{a-b})\ \Sigma_{s=0}^{1}\ (-1)^s\ (2, 2-2s)\ B[\ s+1/2,\ 1-s+1/2\ ]\ F\ [s+1/2,\ 1/2;\ 2;\ -2b/(a-b)\ ]$$

so there are going to be two terms:

$$= (2/\sqrt{a - b}) \{ (2, 2) B[1/2, 3/2] F [1/2, 1/2; 2; -2b/(a-b)]$$

$$- (2, 0) B[3/2, 1/2] F [3/2, 1/2; 2; -2b/(a-b)] \}$$

Now $B[3/2, 1/2] = \Gamma(3/2)\Gamma(1/2)/\Gamma(2) = (\pi/2)$, $(2,0) = 1$ so we get

$$= (2/\sqrt{a - b}) (\pi/2)\{ F [1/2, 1/2; 2; -2b/(a-b)] - F [3/2, 1/2; 2; -2b/(a-b)] \}$$

Notice that c = 2 now, so both these functions are "new". Here is one of the Gauss relations from Bateman p 103 [AS and GR agree that there are no z factors in this formula]

Reduction of the F's using Gauss shifting formulas.

Our targets are these functions

$F(1/2,1/2,1,z) = (2/\pi)K$
$F(-1/2,1/2,1,z) = (2/\pi)E$

which both have c = 1, but our two starting formulas have c = 2. So we want to select a Gauss formula that has three F's in it where one is at c+1 and two are at c, then we will set c = 1. Our candidate is Bateman (38) which says

$$c(1-z)F - cF(a-1) + (c-b) z F(c+1) = 0$$

and we go ahead and set c = 1

$$(1-z)F - F(a-1) + (1-b) z F(c=2) = 0$$

and we now need to write this out, where of course $z = k^2$

$$(1-z)F(a, b, 1, z) - F(a-1, b, 1, z) + (1-b) z F(a, b, 2, z) = 0$$

Our first F of interest is F [1/2, 1/2, 2, z]. To get it on the right, we select a = b = 1/2 so

$$(1-z)F(1/2, 1/2, 1, z) - F(-1/2, 1/2, 1, z) + (1/2) z F(1/2, 1/2, 2, z) = 0$$

Multiply through by (2/z) to get

$$(2/z)(1-z)F(1/2, 1/2, 1, z) - (2/z) F(-1/2, 1/2, 1, z) + F(1/2, 1/2, 2, z) = 0$$

and solve for the K of interest:

$$F(1/2, 1/2, 2, z) = - (2/z)(1-z)F(1/2, 1/2, 1, z) + (2/z) F(-1/2, 1/2, 1, z)$$

$= (2/z)[- (1-z) (2/\pi)K + (2/\pi)E]$

$= (4/\pi z)[-(1-z) K + E]$

Now go back to our general shifter formula

$(1-z)F(a, b, 1, z) - F(a-1, b, 1, z) + (1-b) z F(a, b, 2, z) = 0$

Our second F of interest is F [3/2, 1/2; 2; z] . This time we want a = 3/2 and b = 1/2. At least this will lower our c by one step. So we then have

$(1-z)F(3/2, 1/2, 1, z) - F(1/2, 1/2, 1, z) + (1/2) z F(3/2, 1/2, 2, z) = 0$

As before, multiply through by (2/z) to get

$(2/z) (1-z)F(3/2, 1/2, 1, z) - (2/z)F(1/2, 1/2, 1, z) + F(3/2, 1/2, 2, z) = 0$

and solve for the K of interest

$F(3/2, 1/2, 2, z) = - (2/z) (1-z)F(3/2, 1/2, 1, z) + (2/z)F(1/2, 1/2, 1, z)$ (*)

But now we still have an F that we don't know and which needs more work: F(3/2, 1/2, 1, z).
Let's try Bateman (36) on page 103

$(c-a-b)F - (c-a)F(a-1) + b(1-z) F(b+1) = 0$

Here c does not change, so set c = 1 everywhere

$(1-a-b)F(a, b, 1, z) - (1-a)F(a-1, b, 1, z) + b(1-z) F(a, b+1, 1, z) = 0$

Try b = 1/2 and a = 1/2 and we get

$(0)F(1/2, 1/2, 1, z) - (1/2)F(-1/2, 1/2, 1, z) + (1/2)(1-z) F(1/2, 3/2, 1, z) = 0$

$- F(-1/2, 1/2, 1, z) + (1-z) F(1/2, 3/2, 1, z) = 0$

Solve for the one we want

$(1-z) F(1/2, 3/2, 1, z) = F(-1/2, 1/2, 1, z)$

Now insert this into (*) above to get

$F(3/2, 1/2, 2, z) = - (2/z) (1-z)F(3/2, 1/2, 1, z) + (2/z)F(1/2, 1/2, 1, z)$

$= - (2/z) F(-1/2, 1/2, 1, z) + (2/z)F(1/2, 1/2, 1, z)$

$$= (2/z)\ [\ -(2/\pi)E\ + (2/\pi)K\ ]$$

$$= (4/\pi z)\ [\ K - E]$$

So far then we have determined that

$$F\ [1/2,\ 1/2,\ 2,\ z] = \ (4/\pi z)[\ -(1-z)\ K\ + E\ ]$$

$$F(3/2,\ 1/2,\ 2,\ z)\ \ = (4/\pi z)\ [\ K - E]$$

Our integral is then

$$I = \ \int_0^{2\pi} dx\ \cos(x)/\sqrt{a - b\ \cos(x)}$$

$$= (2/\sqrt{a - b}\ )\ \ (\pi/2)\{\ F\ [1/2,\ 1/2;\ 2;\ -2b/(a-b)\ ]\ - F\ [3/2,\ 1/2;\ 2;\ -2b/(a-b)\ ]\ \ \}$$

$$= (2/\sqrt{a - b}\ )\ \ (\pi/2)\ (4/\pi z)\ \ \{\ [\ -(1-z)\ K\ + E\ ]\ - [\ K - E]\ \ \}$$

$$= (2/\sqrt{a - b}\ )\ (2/z)\ \ \{\ [\ -(2-z)\ K\ + 2E\ ]\ \}$$

$$= (2/\sqrt{a - b}\ )\ (2/z)\ 2\ \{\ [\ -(1-z/2)\ K\ + E\ ]\ \}$$

$$= (8/\sqrt{a - b}\ )\ \ \{\ [\ -(1-z/2)\ K\ + E\ ]\ \}/z \qquad\qquad z = k^2\ = k^2 = -2b/(a-b)$$

$$= (4/\sqrt{a - b}\ )\ \ \{\ [\ -(1-z/2)\ K\ + E\ ]\ \}\ (2/z)$$

$$= (4/\sqrt{a - b}\ )\ \ [\ -(2/z\ -\ 1)\ K\ + (2/z)E\ ]$$

So we know that

$$2/z = 1 - (a/b) \qquad\qquad \text{// scratch}$$

so the result of MY n=1 case is this

$$I\ =\ (4/\sqrt{a - b}\ )\ \ [(a/b)\ K(k)\ + (1 - (a/b)\ E(k)\ ]$$

Here is the Wolfram *indefinite* integral for this n = 1 case:

$$\int \frac{\cos(x)}{\sqrt{a - b\cos(x)}}\, dx =$$

$$\frac{2\sqrt{\frac{a-b\cos(x)}{a-b}}\left(a\,F\left(\frac{x}{2}\middle|-\frac{2b}{a-b}\right)+(b-a)\,E\left(\frac{x}{2}\middle|-\frac{2b}{a-b}\right)\right)}{b\sqrt{a-b\cos(x)}}$$

If I evaluate this at π and 0 I get

$= 2\sqrt{(a+b)/(a-b)}\ \{ a\,F(\pi/2 \mid k^2) + (b-a)\,E(\pi/2 \mid k^2)\ \}/\ b\sqrt{a+b}$

$= (2/\sqrt{a-b})\ \{ a/b\,K(k^2) + (b-a)/b\,E(k^2)\ \}$

$= (2/\sqrt{a-b})\ \{ (a/b)\,K(k^2) + [1-(a/b)]\,E(k^2)\ \}$

Then my integral is this (goes to 2π, so add factor 2)

$I = \int_0^{2\pi} dx\ \cos(nx)/\sqrt{a - b\cos(x)} \quad = (4/\sqrt{a-b})\ \{ (a/b)\,K(k^2) + [1-(a/b)]\,E(k^2)\ \}$

and this agrees with my calculation.

8 What does Bateman Vol 4 have to say about this integral?

$I = \int_0^{2\pi} dx\ \cos(nx)/\sqrt{a - b\cos(x)}$

But this is a Fourier *Series* transform, not a Fourier Transform, so this won't appear in Bateman. Does anyone publish a table of Fourier series transforms? I cannot find tables with "uncommon" functions. I thought in passing about doing the Laplace transform and then evaluating, but that was a red herring, you still have to do the integral!

9. Doing the integral by using a certain integral representation of Q [5.2.10]

Consider the integral representation (10) shown on page 156 of Bateman for Q as a (0,π) integral, but consider it in the case that ν = n-1/2 *where n is an integer*. In this situation we have n = ν+1/2 and

$\cos(\nu\pi) = \cos(n\pi - \pi/2) = -\sin(n\pi) = 0$

so *there is no second term*! Then we have this:

$Q_{n-1/2}^{\mu}(z) = e^{i\pi\mu}\,(2\pi)^{-1/2}(z^2-1)^{\mu/2}\ \Gamma(\mu+1/2)\ \int_0^{\pi} dt\ (z-\cos t)^{-\mu-1/2}\cos(nt)\,dt$

There is a little smudge in Bateman, but this same integral representation appears in GR7 p 961

8.713

1. $$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}\,\Gamma\left(\mu+\frac{1}{2}\right)}{\sqrt{2\pi}}\,(z^2-1)^{\frac{\mu}{2}}\left\{\int_0^\pi \frac{\cos\left(\nu+\frac{1}{2}\right)t\,dt}{(z-\cos t)^{\mu+\frac{1}{2}}} - \cos\nu\pi\int_0^\infty \frac{\left\lfloor_0^{-(\nu+\frac{1}{2})t}\,dt}{(z+\cosh t)^{\mu+\frac{1}{2}}}\right\}$$
$$\left[\mathrm{Re}\,\mu > -\tfrac{1}{2},\quad \mathrm{Re}(\nu+\mu) > -1,\quad |\arg(z\pm 1)| < \pi\right]$$ MO 89

Now set μ = 0 to get

$Q_{n-1/2}(z) = (2\pi)^{-1/2}\ \Gamma(1/2)\ \int_0^\pi dx\ \cos(nx)\ (z-\cos(x))^{-1/2}$

$= (2\pi)^{-1/2}\ \pi^{1/2}\ \int_0^\pi dx\ \cos(nx)\ (z-\cos(x))^{-1/2}$

$= (1/\sqrt{2})\ \int_0^\pi dx\ \cos(nx)\ (z-\cos(x))^{-1/2}$

which is beginning to look a lot like our desired integral! Let's go back to our desired integral

$I = \int_0^{2\pi} dx\ \cos(nx)/\sqrt{a - b\cos(x)}$

$= 2\ \int_0^\pi dx\ \cos(nx)/\sqrt{a - b\cos(x)}$

$= 2\ (1/\sqrt{b})\ \int_0^\pi dx\ \cos(nx)/\sqrt{(a/b) - \cos(x)}$

$= 2\ (1/\sqrt{b})\ \sqrt{2}\ Q_{n-1/2}(a/b) = 2\sqrt{2/b}\ Q_{n-1/2}(a/b)$

Wow! This means that

$a_n(a,b) = I/\pi = (2/\pi)\sqrt{2/b}\ Q_{n-1/2}(a/b)$

As a check on this, rewrite this way (bear with me for a moment)

$a_n(\alpha,\beta) = I/\pi = (2/\pi)\sqrt{2/\beta}\ Q_{n-1/2}(\alpha/\beta)$

then set

$\alpha = b^2 + \rho^2 \qquad \beta = 2c\rho \qquad \alpha/\beta = (b^2+\rho^2)/(2c\rho)$

$a_n(\alpha,\beta) = (2/\pi)\ (1/\sqrt{c\rho}\ )\ Q_{n-1/2}[(b^2+\rho^2)/(2c\rho)]$

This is the same as the result I obtained in this document:

Iris Green's by Dirichlet method Attempt #3.doc page 21 $b^2 = c^2 + d^2$

So that clinches the deal! This is a MUCH better form for the result! So here is our final result:

$$I = \int_0^{2\pi} dx\ \cos(nx)/\sqrt{a - b\cos(x)} = 2\sqrt{2/b}\ Q_{n-1/2}(a/b)$$

Comments: This result says, apart from defining the constant, that $Q_{n-1/2}(a/b)/\sqrt{b}$ is the Fourier Series Coefficient of the function $1/\sqrt{a - b\cos(x)}$. I have looked very hard at both my GR editions and I cannot find this integral anywhere in there. If is of course only valid for integral n. Bateman does not give tables of Fourier *Series* transforms, so not in the ET volumes.

Another interesting way to write this is as follows:

$$Q_{n-1/2}(\cosh\xi) = (2\pi)^{-1/2}\ \Gamma(1/2)\ \int_0^{\pi} dx\ \cos(nx)\ (\cosh(\xi)-\cos(x))^{-1/2}$$

and this has the definite "look" of something going on in toroidal coordinates.