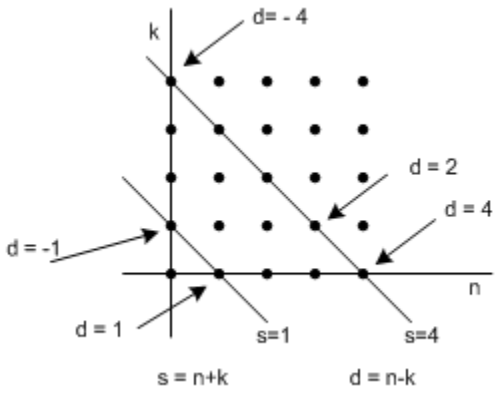
Double Summation Reordering Theorems PhL 3.29.09

I am tired of reinventing this every time I run into it!

1.Here are some notes from Schiff Chap 8 Approx Bound State.doc

$= \Sigma_{k=0} \Sigma_{n=0} \lambda^{k+n} W^{(m)}{}_k \psi^{(m)}{}_n$ //drawing in phys/QM/Schiff/Schiff support



s = n+k the "sum" index s+d = 2n s-d = 2k
d = n-k the "difference" index

RHS = $\Sigma_{s=0,1,2} \lambda^s [\Sigma_{d=s,s-2...-s} W^{(m)}{}_{(s-d)/2} \psi^{(m)}{}_{(s+d)/2}]$

Here is what this is really saying

Theorem 1:

$\Sigma_{k=0,1..\infty} \Sigma_{n=0,1..\infty} F_{k,n} = \Sigma_{s=0,1..\infty} \Sigma_{d=s,s-2...-s} F_{(s-d)/2,\,(s+d)/2}$

Suppose both summations on the left start instead at 1. Then let p = k+1 q = n+1

$\Sigma_{p=1,2..\infty} \Sigma_{q=1,2..\infty} F_{p-1,q-1} = \Sigma_{s=0,1..\infty} \Sigma_{d=s,s-2...-s} F_{(s-d)/2,\,(s+d)/2}$

Now define $F_{p-1,q-1} = G_{p,q}$ or $F_{a,b} = G_{a+1,b+1}$

Theorem 2:

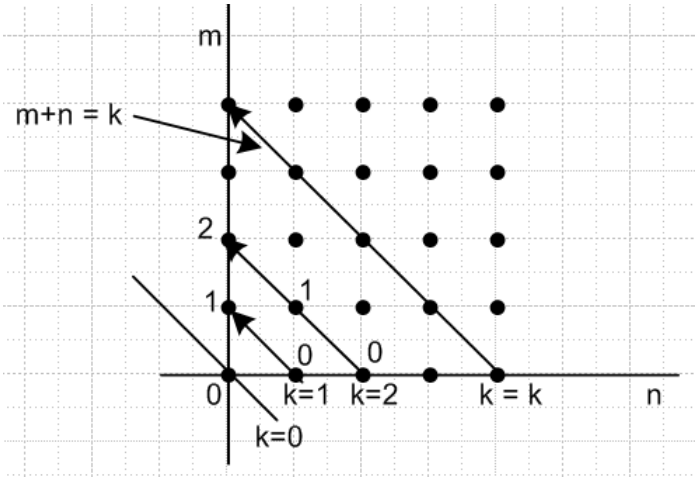
$\Sigma_{p=1,2..\infty} \Sigma_{q=1,2..\infty} G_{p,q} = \Sigma_{s=0,1..\infty} \Sigma_{d=s,s-2...-s} G_{(s-d)/2+1,\,(s+d)/2+1}$

The following is a similar theorem that is needed when showing $e^x e^y = e^{x+y}$:

Theorem 3: **[** added this here on 11.20.16]

$$\Sigma_{m=0}^{\infty} \Sigma_{n=0}^{\infty} \; f_{n,m} = \Sigma_{k=0}^{\infty} \; \Sigma_{m=0}^{k} f_{k-m,m} \qquad\qquad m+n = k$$

The proof is just this picture (same vsd as above)



On the left of theorem 3 you step through the points one column at a time (or one row at a time), whereas on the right you step through the same points in a different order which is along the arrows shown, starting with the leftmost arrow at the origin which you cannot see.

2. Double Integration reordering theorems.

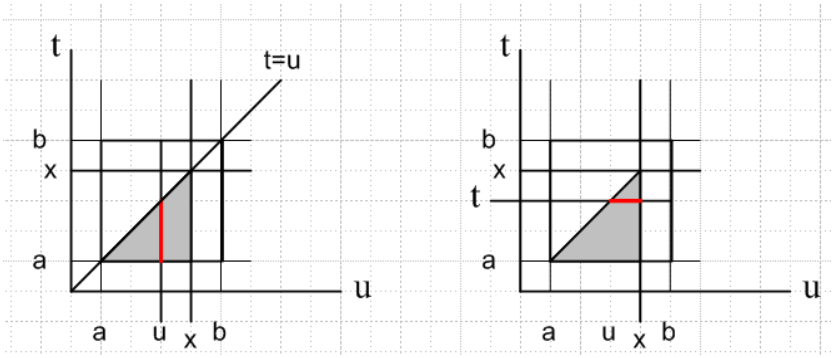
[added 7.28.10 while reading Sneddon on dual integral equations] This is another thing I am getting tired of doing -- drawing the picture every time this stuff occurs. It goes without saying that the reordering is allowed (interchange of order of integration) under suitable mathematical conditions, such as all integrals must converge in some sense.

2A. The Lower Triangle Reordering Theorem : In this theorem, the integration region is the lower right triangle of a square which in each dimension is bounded by (a,x). The parameter b merely suggests an upper limit for x. The diagonal of the square is t = u. In Sneddon "x" was a variable and a was a parameter, but for our theorem here, we can think of x and a as being on an equal footing.

The theorem states:

$$\int_{\text{lower triangle}} du\, dt \; F(u,t) =$$

$$\int_a^x du \int_a^u dt \; F(u,t) \qquad = \qquad \int_a^x dt \int_t^x du \; F(u,t)$$



The red slice shows the innermost sum for the respective integral. Here are some special cases:

2A.1 $\int_0^x du \int_0^u dt\ F(u,t) = \int_0^x dt \int_t^x du\ F(u,t)$ a = 0

2A.2 $\int_a^\infty du \int_a^u dt\ F(u,t) = \int_a^\infty dt \int_t^\infty du\ F(u,t)$ x = ∞

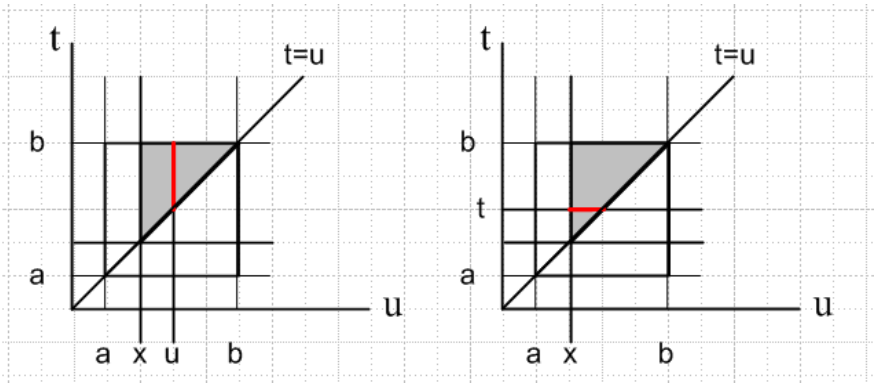
2A.3 $\int_0^\infty du \int_0^u dt\ F(u,t) = \int_0^\infty dt \int_t^\infty du\ F(u,t)$ a=0 and x=∞

2B. The Upper Triangle Reordering Theorem : In this theorem, the integration region is the upper left triangle of a square which in each dimension is bounded by (x,b). The parameter a merely suggests a lower limit for x. The diagonal of the square is t = u. In Sneddon "x" was a variable and b was a parameter, but for our theorem here, we can think of x and b as being on an equal footing.

The theorem states:

$$\int_{\text{upper triangle}} du\ dt\ F(u,t) =$$

$$\int_x^b du \int_u^b dt\ F(u,t) = \int_x^b dt \int_x^t du\ F(u,t)$$



The red slice shows the innermost sum for the respective integral. Here are some special cases:

2B.1 $\int_0^b du \int_u^b dt\ F(u,t) = \int_0^b dt \int_0^t du\ F(u,t)$ x=0

2B.2 $\int_x^\infty du \int_u^\infty dt\ F(u,t) = \int_x^\infty dt \int_x^t du\ F(u,t)$ $b = \infty$

2B.3 $\int_0^\infty du \int_u^\infty dt\ F(u,t) = \int_0^\infty dt \int_0^t du\ F(u,t)$ x=0 and $b = \infty$