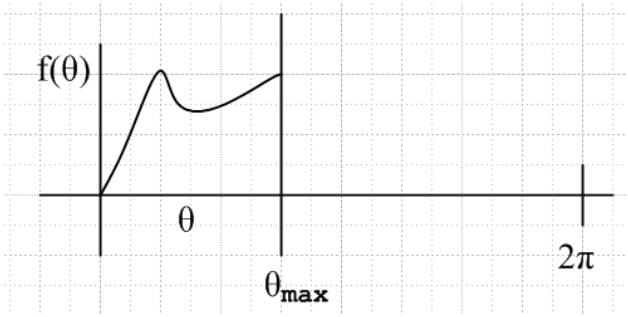
Fourier Series on non-periodic functions. PhL 10.20.09

If the function in question is defined over all argument and is non-periodic, we usually switch to the Fourier Transform. But suppose the function is only defined on part of a 2π interval, perhaps going out to some θ_{max}.



Let's say the function is not even defined beyond θ_{max} for some problem we are considering.

A. One approach is to say that f(θ) = 0 in the range (θ_{max}, 2π).

Then on a larger scale, we regard the function as periodic with the above shape in each 2π period. Since we then have a periodic function, we can apply our Fourier Series machinery.

Example 1: Just as an example, suppose f(θ) = sinθ and $\theta_{max} = \pi$ and we set it to 0 in the right half of the interval. Schaum p 131 then tells us that 2L = 2π and we have

$$a_n = (1/\pi) \int_0^{2\pi} d\theta\, f(\theta) \cos(n\theta) = (1/\pi) \int_0^{\pi} \sin(\theta) \cos(n\theta)$$

This integral from Schaum p 96 is this

$$\int_0^{\pi} \sin(\theta) \cos(n\theta) = [\ 0 \text{ if n = odd, else } 2/(1-n^2) \text{ if n = even }]$$

Therefore

$$a_n = \begin{cases} 0 & n \text{ odd} \\ (2/\pi)/(1-n^2) & n \text{ even} = 0,2,4.... \end{cases}$$

And then we also have

$$b_n = (1/\pi) \int_0^{2\pi} d\theta\, f(\theta) \sin(n\theta) = (1/\pi) \int_0^{\pi} \sin(\theta) \sin(n\theta)$$

This integral from Schaum p 96 is this

$$\int_0^{\pi} \sin(\theta)\sin(n\theta) = [\ \pi/2 \ \text{ if } n = 1, \text{ else } = 0\ ]$$

which tells us that $b_n = (1/\pi)(\pi/2)\,\delta_{n,1} = (1/2)\,\delta_{n,1}$

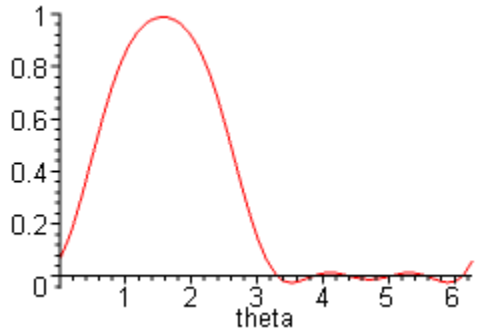
We can then examine the reconstructed function

$$f(\theta) = a_0/2 + \Sigma_{n=1}^{\infty}\,(\,a_n\cos(n\theta) + b_n\sin(n\theta)\,)$$

$$= (1/\pi) + (1/2)\sin(\theta) + (2/\pi)\,\Sigma_{n=2,4\ldots}\cos(n\theta)/(1-n^2)$$

One might duly wonder how this is going to reproduce "0" on the right half of the interval. The series certainly looks convergent, off hand. Before attempting a theoretical understanding, let Maple lead the way, and here we are only putting n = 2 and 4 in the series!

```
> g := 0:
> for n from 2 by 2 while n < 5 do
>    g := g + cos(n*theta)/(1-n*n)
> od:
> g:
> ff := (1/Pi) + 0.5*sin(theta) + (2/Pi)*g:
> plot(ff,theta=0..2*Pi);
```



So it must be true. It must just be a fact that

$$(1/\pi) + (1/2)\sin(\theta) + (2/\pi)\,\Sigma_{n=2,4\ldots}\cos(n\theta)/(1-n^2) = \sin(\theta)\ H(\pi-\theta)$$

$$(2/\pi)\,\Sigma_{n=2,4\ldots}\cos(n\theta)/(1-n^2) = \sin(\theta)\ [H(\pi-\theta) - 1/2] - (1/\pi)$$

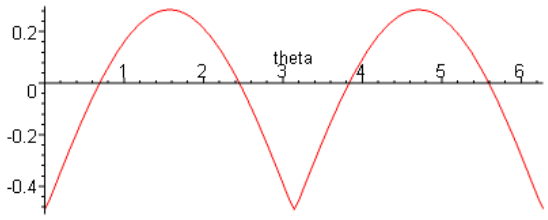
$$\Sigma_{n=2,4\ldots}\cos(n\theta)/(1-n^2) = (\pi/2)\sin(\theta)\ [H(\pi-\theta) - 1/2] - (1/2)$$

$$\Sigma_{m=1,2\ldots}\cos(2m\theta)/(1-(2m)^2) = (\pi/2)\sin(\theta)\ [H(\pi-\theta) - 1/2] - (1/2)$$

$$\Sigma_{m=1,2\ldots}\cos(2m\theta)/((2m)^2 - 1) = (\pi/2)\sin(\theta)\ [1/2 - H(\pi-\theta)] + (1/2)$$

The Fourier Series expansion generates an infinite number of weird expansion series like this one. It happens that this sum appears in GR p 39 (last on page) but says valid $0 \le \theta \le \pi/2$. But we know from a above that it is valid in fact for $0 \le \theta \le \pi$, and, as I have written it, for all of $(0,2\pi)$.

Bu the way, if you plot just the series (Maple g), things are perhaps less mysterious. If I use the first 50 terms, I find this plot;



$$g = \Sigma_{m=1,2\ldots}\cos(2m\theta)\,/((2m)^2 - 1)$$

The series just adds up to this function which is periodic with period π. The humps are sines and of course it we add this to a regular 2π sin the right side can cancel and that is where a zero right half can come from. So not too mysterious.

Let's do another example but "on the sphere"

Example 2: Suppose we have $f(\theta,\varphi)$ which is defined only on half of a sphere. We can decide to set it to zero on the other half. Then what happens?

$$f(\theta,\varphi) = \Sigma_{mn}\, Y_{nm}(\theta,\varphi) f_{nm} \qquad\qquad f_{nm} = (1/N_{nm}) \int d\Omega\, f(\theta,\varphi)\, Y_{nm}(\theta,\varphi)^*$$

Here I am using the Stakgold Appendix A normalization and my own notation for f_{nm} putting the n index first, which for some reason he does not do. Suppose in particular that $f(\theta,\varphi) = f(\theta)$. Then

$$f_{nm} = (1/N_{nm}) \int d\Omega\, f(\theta,\varphi)\, Y_{nm}(\theta,\varphi)^* = (1/N_{nm}) \int d\Omega\, f(\theta)\, Y_{nm}(\theta,\varphi)^*$$

$$= (1/N_{nm}) \int d(\cos\theta)\, f(\theta) P_n^m(\cos\theta) \int d\varphi\, e^{-im\varphi} = (2\pi\, \delta_{m,0}/N_{n0}) \int d(\cos\theta)\, P_n(\cos\theta)\, f(\theta)$$

Now suppose we take $f(\theta) = \cos\theta$ on the upper half sphere and $f = 0$ on the lower. Then we have

$$f_{nm} = (2\pi\, \delta_{m,0}/N_{n0}) \int_0^1 d(\cos\theta)\, P_n(\cos\theta) \cos\theta = (2\pi\, \delta_{m,0}/N_{n0}) \int_0^1 dx\, P_n(x)\, x$$

The integral here, not being full range, does not equal $k\ \delta_{n,1}$. Rather, it is just some set of numbers I will here call p_n that are easy to compute in maple as ratios of integers. If n is odd and ≥ 3, we know that the full interval integral is 0, and can be written as twice our integral, so our integral = 0 for such cases. It n is even, we get the numbers p_n. I don't think there is any closed form. Well, we could try GR p 795 7.121 with $\varphi = \pi/2$ so $\cos\varphi = 0$ and $\sin\varphi = 1$. It says then that

$$\int_0^1 dx\, P_n(x)\, x = 1/[(n-1)(n+2)] * 1\, P_n(0) = P_n(0) / [(n-1)(n+2)]$$

But Maple disagrees with the sign of this result. So let's instead use AS p 786 top which says

$$\int_0^1 dx\, P_{2m}(x)\, x = (-1)^m\, \Gamma(m-1/2)\, \Gamma(1) / [\, 2\, \Gamma(-1/2)\, \Gamma(m+2)\,]$$

$$= (-1)^{m+1}\, \Gamma(m-1/2) / [\, 4\sqrt{\pi}\ \Gamma(m+2)\,] \qquad m = 0,1,2,3$$

Maple agrees with this formula. So its a bit of a mess

$$\int_0^1 dx\, P_n(x)\, x = (-1)^{n/2+1}\, \Gamma(n/2-1/2) / [\, 4\sqrt{\pi}\ \Gamma(n/2+2)\,] = p_n \qquad n = \text{even only}$$

so fine, there is a closed form if you want one. Remember that if n is odd and ≥ 3, $p_n = 0$. We need then one extra value by hand: $p_1 = 1/3$.

So we have found for our example with $f(\theta) = \cos\theta$ on the upper hemisphere and 0 on the lower that

$$f_{nm} = \delta_{m,0}\, (\, 2\pi\, p_n / N_{n0}\,)$$

Now let's reconstruct the function.

$$f(\theta,\varphi) = \Sigma_{mn}\, Y_{nm}(\theta,\varphi) f_{nm} = 2\pi\, \Sigma_{n=1}\, (p_n / N_{n0})\, P_n(\cos\theta)$$

Again, you might wonder how this is going to come out being 0 on the lower hemisphere, so let's plot it in Maple. We need this extra fact

$$N_{n0} = 4\pi/(2n+1)$$

So we want to compute and plot

$$f(\theta) = \Sigma_{n=1}\, (n+1/2)\, p_n\, P_n(\cos\theta)$$

$$p_n = (-1)^{n/2+1}\, \Gamma(n/2-1/2) / [\, 4\sqrt{\pi}\ \Gamma(n/2+2)\,] \qquad n = 0,2,4,6....$$
$$p_n = 0 \qquad n = 3,5,7...$$
$$p_n = 1/3 \qquad n = 1$$

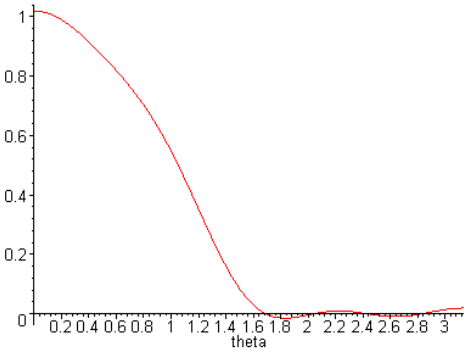
which we can rewrite as

$$f(\theta) = (3/2)\, (1/3) \cos\theta + \Sigma_{n=0,2,4..}\, (n+1/2)\, p_n\, P_n(\cos\theta)$$

$$= (1/2) \cos\theta + \Sigma_{n=0,2,4..}\, (n+1/2)\, p_n\, P_n(\cos\theta)$$

Now we have Maple plot this for the sum having just a few terms:

```
> restart;
> with(orthopoly):
> g := 0:
> for n from 0 by 2 while n < 7 do
>    g := g + P(n,cos(theta))* (n+1/2)* (-1)^(n/2+1) * GAMMA(n/2-1/2)/ ( 4 * sqrt(Pi)*GAMMA(n/2+2))
> od:
>
> g:
> plot(g + (1/2)*cos(theta),theta = 0..Pi);
```



and there is the expected result, just as in the simpler Fourier series example.