Convergence Questions PhL 9.26.09

Part A.

Consider this series:

$\Sigma_{n=1}^{\infty} \ln [(n-x)/(n-y)] = \Sigma_{n=1}^{\infty} a_n$

Does the series converge? We can see that $a_n \rightarrow \ln(1) = 0$, so at least it has a chance. Let's think of y as a fixed parameter and x = the variable. Maybe let's write this like so:

$f(z) = \Sigma_{n=1}^{\infty} \ln [(n-z)/(n-c)]$ c = constant (= y)

I know at least something about convergence of power series, but this is not a power series.

(1) Ratio Test. How about applying the "ratio test" :

http://en.wikipedia.org/wiki/Convergent_series

Ratio test. Assume that for all n, $a_n > 0$. Suppose that there exists r such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$$

If $r < 1$, then the series converges. If $r > 1$, then the series diverges. If $r = 1$, the ratio test is inconclusive, and the series may converge or diverge.

Here we go:

$a_{n+1} = \ln [(n+1-z)/(n+1-c)]$
$a_n = \ln [(n-z)/(n-c)]$

$r = a_{n+1}/ a_n = \ln [(n+1-z)/(n+1-c)]/ \ln [(n-z)/(n-c)]$

This is an ∞/∞ situation, so use l'Hopital to get ∂_n / ∂_n . I have Maple do this:

```
f := ln((n-z)/(n-c)):
g := ln((n+1-z)/(n+1-c)):
h := g/f:
j :=diff(f,n):
k :=diff(g,n):
m := k/j:
simplify(m);
```

$$\frac{(n-z)(n-c)}{(n+1-z)(n+1-c)}$$

So you can see that the limit is 1, so r = 1, inconclusive. Maple figures this out directly as well:

```
limit(h, n=infinity);
```

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(2) "Derivative Test"

$f(z) = \Sigma_{n=1}^{\infty} \ln [(n-z)/(n-c)]$

$f'(z) = - \Sigma_{n=1}^{\infty} 1/(n-z)$

The derivative series does not converge, but that proves nothing about the original series (well, maybe it shows it diverges, but let's move on).

(3) Comparison with integral test.

Consider the absolute version of the series

$fabs(z) = \Sigma_{n=1}^{\infty} | \ln [(n-z)/(n-c)] | = \Sigma_{n=1}^{\infty} a_n$

Looking way out in the tail, write the term as

$$(n-z)/(n-c) = (1-z/n)(1-c/n)^{-1} = (1-z/n)(1+c/n+ O(1/n^2)) = 1 + (c-z)/n + O(1/n^2)$$

$$\ln [(n-z)/(n-c)] = \ln[1 + (c-z)/n + O(1/n^2)] \approx (c-z)/n + O(1/n^2)$$

$$a_n \approx | \ln [(n-z)/(n-c)] | \approx |c-z| / n$$

The point here so far is just to show that a_n is a positive monotone decreasing sequence of numbers. Let's now assume that everything is real and that c > z. Then we have a monotone sequence of a_n. The means we are allowed to do this integral test in the first place.

Integral test. The series can be compared to an integral to establish convergence or divergence. Let $f(n) = a_n$ be a positive and monotone decreasing function. If

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^{t} f(x)\,dx < \infty,$$

then the series converges. But if the integral diverges, then the series does so as well.

So our integral of interest here is this:

$$\int dn \, | \ln [(n-z)/(n-c)] |$$

But for large n, we have just said that in the tail region we have $\ln [(n-z)/(n-c)] \approx (c-z)/n$ so the ln terms are the same sign as (c-z) . This is for real everything. For complex the condition is probably $Re(a_n)$ is monotone decreasing and so I would say

$$\int dn \,|\, \ln [(n-z)/(n-c)] \,|\, = \text{sign}[\text{Re}(c-z)] \int dn \ln [(n-z)/(n-c)]$$

so this sign is irrelevant, the point is that as we go out into the tail, sign does not change. Now we know that

$$\int dn \ln [(n-z)/(n-c)] =$$

```
int(ln((n-z)/(n-c)),n):
simplify(%);
```

$$\ln\left(\frac{n-z}{n-c}\right) n + z \ln\left(-\frac{z-c}{n-c}\right) - c \ln\left(-\frac{z-c}{n-c}\right) - z \ln\left(\frac{n-z}{n-c}\right)$$

We only care of course about the upper endpoint of the integral. Well this diverges, and we can have Maple just confirm that fact.

```
int(ln((n-z)/(n-c)),n):
f := simplify(%):
limit(f,n=infinity);
```

$$-\text{signum}(z-c)\,\infty$$

My conclusion: $\Sigma_{n=1}^{\infty} \ln [(n-z)/(n-c)]$ diverges!

Part B.

Consider this very similar series: (sum includes all integers n)

$$\Sigma_n \ln [(n-x)/(n-y)] = \ln(x/y) + \Sigma_{n=1}^{\infty} (a_n + a_{-n}) \qquad a_n = \ln [(n-x)/(n-y)]$$

Does the series converge? This would mean that we examine $\Sigma_{n=-N}^{N}$ and take $N \to \infty$. We then have

$$b_n = (a_n + a_{-n}) = \ln [(n-x)/(n-y)] + \ln [(n+x)/(n+y)] = \ln[(n^2-x^2)/ (n^2-y^2)]$$

As expected $b_n \to 0$ so we have a chance (ln1 =0).

Let's try the ratio test:

$b_n = \ln[(n^2-x^2)/ (n^2-y^2)]$
$b_{n+1} = \ln[((n+1)^2-x^2)/ ((n+1)^2-y^2)]$

$r = b_{n+1}/ b_n = \ln[((n+1)^2-x^2)/ ((n+1)^2-y^2)] \;/\; \ln[(n^2-x^2)/ (n^2-y^2)]$

For large n, we are getting ln(1)/ln(1) = ∞/∞, so try l'Hopital. I fiddle with Maple as before and find that r = 1, no conclusion.

Onto the integral test. Now we get something more interesting than last time.

$\int$ dn ln[$(n^2-x^2)/(n^2-y^2)$] =

$$fni := n \ln\left(\frac{n^2 - x^2}{n^2 - y^2}\right) - x \ln(x - n) + x \ln(x + n) + y \ln(y - n) - y \ln(y + n)$$

There are 5 terms here. Combine terms 2 and 3 to get

x ln [(x+n)/(x-n)] → x ln(-1) = x * (-iπ) = -iπx

Now so for some reason I could figure out we want -1 $= e^{-i\pi}$ so ln(-1) = -iπ. Terms 4 and 5 produce then iπy. So these four terms produce iπ(y-x). The big question is the first term!

n ln[$(n^2-x^2)/(n^2-y^2)$] = ∞ * 0

so write this as

= ln[$(n^2-x^2)/(n^2-y^2)$] / (1/n)

= ln[$(m^{-2}-x^2)/((m^{-2}-y^2)$] / m as m→ 0

= ln[$(1-x^2m^2)/(1-y^2m^2)$] / m

Now Maple computes

∂_m ln[$(1-x^2m^2)/(1-y^2m^2)$] =

$$-2\frac{m(x^2 - y^2)}{(-1 + m^2 x^2)(-1 + m^2 y^2)}$$

$\partial_m m = 1$

Ratio is then as shown, and clearly → 0 as $1/m^3$. So our conclusion is this:

$\int^{\infty}$ dn ln[$(n^2-x^2)/(n^2-y^2)$] = iπ(y-x) = convergent!

Maple can do this all at once like so:

```
fn := ln((n^2 - x^2)/(n^2 - y^2));
```

$$fn := \ln\left(\frac{n^2 - x^2}{n^2 - y^2}\right)$$

```
fni := int(fn,n);
```

$$fni := n \ln\left(\frac{n^2 - x^2}{n^2 - y^2}\right) - x \ln(x - n) + x \ln(x + n) + y \ln(y - n) - y \ln(y + n)$$

```
limit(fni,n=infinity);
```

$$I y \pi - I x \pi$$

By the integral test, our series thus converges, hurray!

$\Sigma_{n\neq 0}$ ln [(n-x)/(n-y)] converges for all x and y

From other work, I know what it converges to, namely

Σ_n ln [(n-x)/(n-y)] = Π_n [(n-x)/(n-y)] = sin(πx)/sin(πy)