Integral Theorems PhL 7.9.09

1. Gaussian: ($d\mathbf{S} = dS\,\hat{\mathbf{n}}$) // see later pages of this doc for proofs

$\int_V dV\ \nabla\bullet\mathbf{A} = \int_S d\mathbf{S}\bullet\mathbf{A}$ // divergence theorem = Gauss's theorem (1) Schaum 22.59

$\int_V dV\ \nabla^2\varphi = \int_S d\mathbf{S}\bullet\nabla\varphi = \int_S dS\ \partial_n\varphi$ // application of the above with $\mathbf{A} = \nabla\varphi$

$\int_V dV\ \nabla\varphi = \int_S d\mathbf{S}\ \varphi$ // "integral of a gradient theorem" (2) saved PDF's

$\int_V dV\ \psi(\nabla\varphi) = -\int_V dV\ (\nabla\psi)\varphi + \int_S d\mathbf{S}\ (\psi\varphi)$ // "parts integration" (2a) corollary to (2)

$\int_V dV\ \psi(\partial_i{}^k\varphi) = (-1)^k \int_V dV\ (\partial_i{}^k\psi)\ \varphi + \int_S dS\bullet\hat{\mathbf{i}}\ \{ \Sigma_{m=0}{}^{k-1}\ (-1)^m\ (\partial_i{}^m\psi)\ (\partial_i{}^{k-m-1}\varphi)\ \}$ (2b)

$\int_V dV\ \nabla\ \mathbf{x}\ \mathbf{A} = \int_S d\mathbf{S}\ \mathbf{x}\ \mathbf{A}$ // "integral of a curl theorem" (3) Schaum 22.64

Green's First Identity: Set $\mathbf{A} = \psi\nabla\varphi$ in (1) above and use $\nabla\bullet(\psi\nabla\varphi) = \nabla\psi\bullet\nabla\varphi + \psi\,\nabla^2\varphi$ to get:

$\int_V dV\ \nabla\bullet(\psi\nabla\varphi) = \int_S dS\ (\psi\nabla\varphi)\bullet\hat{\mathbf{n}} = \int_V dV\ \{\ \nabla\psi\bullet\nabla\varphi + \psi\,\nabla^2\varphi\ \}$ (4)

which we rearrange into its traditional form ($\partial\varphi/\partial n = \nabla\varphi\bullet\hat{\mathbf{n}}$ = change of φ in the $\hat{\mathbf{n}}$ direction)

$\int_V dV\ \psi\,\nabla^2\varphi = \int_S dS\ \psi(\partial\varphi/\partial n) - \int_V dV\ \{\nabla\psi\bullet\nabla\varphi\}$ (5a) Green #1

$\int_V dV\ \varphi\,\nabla^2\psi = \int_S dS\ \varphi(\partial\psi/\partial n) - \int_V dV\ \{\nabla\varphi\bullet\nabla\psi\}$ (5b)

where in the second line we have just swapped $\varphi \leftrightarrow \psi$. If we subtract these two lines, the $\nabla\ \nabla$ terms cancel and we get

$\int_V dV\ [\ \psi\,\nabla^2\varphi - \varphi\,\nabla^2\psi\ ] = \int_S dS\ [\ \psi(\partial\varphi/\partial n) - \varphi(\partial\psi/\partial n)\ ]$ (6) Green #2

2. Stokesian: (for n=2, $\mathbf{A} = f\hat{\mathbf{x}} + g\hat{\mathbf{y}}$, $\nabla\mathbf{x}\mathbf{A} = (\partial_x g - \partial_y f)\,\hat{\mathbf{z}}$, $d\boldsymbol{\ell}\bullet\mathbf{A} = fdx+gdy$, => "Green's Theorem")

$\int_S dS\bullet(\nabla\ \mathbf{x}\ \mathbf{A}) = \oint d\boldsymbol{\ell}\bullet\mathbf{A}$ **(7)** Schaum 22.60

$\int_S d\mathbf{S}\ \mathbf{x}\ \nabla\varphi = \oint d\boldsymbol{\ell}\ \varphi$ (8) Schaum 22.65

3. Other:

$\int_A{}^B \nabla\varphi\bullet d\boldsymbol{\ell} = \varphi(B) - \varphi(A)$ (9) MM 4-36 p 156

The previous page I will print and store in the TK binder. Here I will comment on some of these theorems.

(1) The "divergence theorem" is the basic tool. It is known as "Gauss's Theorem" in the context of electrostatics, as for example in my Berkeley series E&M book of Purcell. Of course in the above, I intend this theorem to be stated in the space R_n , not just R_3. There are many proofs of this thing, I think the simplest is in that same book where you model a finite volume as a limit of little cubes. [I have written down my version of the proof of the divergence theorem in n = 1,2,3 and > 3 dimensions, see "Proof of the Divergence Theorem.doc".]

(2) Derivation of the "integral of a gradient theorem". Just apply the divergence theorem to **A(x)** = f(**x**) **a** where **a** is some arbitrary constant vector in R_n. We can write

$$\nabla\bullet\mathbf{A} = \nabla\bullet(f\mathbf{a}) = \nabla f \bullet \mathbf{a} + f\,\nabla\bullet\mathbf{a} = \nabla f \bullet \mathbf{a} \qquad \text{// since } \nabla\bullet\mathbf{a} = 0, \mathbf{a} \text{ being a constant.}$$

$$d\mathbf{S}\bullet\mathbf{A} = d\mathbf{S}\bullet(f\mathbf{a}) = (d\mathbf{S}\, f) \bullet \mathbf{a} = \mathbf{S}\, f \bullet \mathbf{a}$$

Thus, our divergence theorem (1) says this:

$$\int dV\ \nabla f \bullet \mathbf{a} = \int d\mathbf{S}\, f \bullet \mathbf{a}$$

which we can write as

$$\Big\{\int dV\ \nabla f\Big\} \bullet \mathbf{a} = \Big\{\int d\mathbf{S}\, f\Big\} \bullet \mathbf{a}$$

Since this must be true for arbitrary constant vector **a**, we must have $\int dV\ \nabla f = \int d\mathbf{S}\, f$, QED.

If this makes you uncomfortable, let **a** be the unit vector $\hat{\mathbf{i}}$. Then we have from above,

$$\Big\{\int dV\ \nabla f\Big\} \bullet \hat{\mathbf{i}} = \Big\{\int d\mathbf{S}\, f\Big\} \bullet \hat{\mathbf{i}}$$

$$\int dV\ \partial_i f = \int d\mathbf{S}\bullet\hat{\mathbf{i}}\,(f) = \int dS_i\,(f)$$

But this is just the i^{th} component of the equation $\int dV\ \nabla f = \int d\mathbf{S}\, f$. QED again. This theorem appears as equation (3.6) in web file called "chap3.pdf"

(3) Derivation of the "integral of a curl theorem".. Let **a** = constant vector again.

$$\mathbf{A} = \mathbf{a} \times \mathbf{F} \quad \text{so that } \nabla\bullet\mathbf{A} = \nabla \bullet (\mathbf{a} \times \mathbf{F}) = F \bullet (\nabla \times \mathbf{a}) - \mathbf{a} \bullet \nabla \times \mathbf{F} = -\,\mathbf{a} \bullet \nabla \times \mathbf{F}$$

$$d\mathbf{S} \bullet \mathbf{A} = d\mathbf{S} \bullet (\mathbf{a} \times \mathbf{F}) = \text{cyclic} = \mathbf{a} \bullet (\mathbf{F} \times d\mathbf{S})$$

The divergence theorem then says:

$$\int dV\,[-\mathbf{a}\bullet\nabla \times \mathbf{F}\,] = \int \mathbf{a}\bullet(\mathbf{F}\times d\mathbf{S})$$

$$\{\int dV\,\nabla\times\mathbf{F}\,\}\bullet\mathbf{a} = \{-\int \mathbf{F}\times d\mathbf{S}\,\}\bullet\mathbf{a}$$

$$\int dV\,\nabla\times\mathbf{F} = -\int \mathbf{F}\times d\mathbf{S} = +\int d\mathbf{S}\times\mathbf{F}$$

This is the "volume integral of a curl theorem". It is equation (3.7) in same "chap3.pdf" web file.

(2a) The single parts integration formula. Derive it like this from (2):

$$\int dV\,\nabla(\psi\varphi) = \int d\mathbf{S}\,(\psi\varphi) \qquad \text{from (2)}$$

$$= \int dV\,[\,\psi\nabla\varphi + \varphi\nabla\psi] = \int dV\;\psi\nabla\varphi + \int dV\,\varphi\nabla\psi \qquad \text{QED}$$

(2b) Just apply the (2a) formula k times. I do this in my notes "attempts to prove Green's...", it is pretty trivial.

(4) (5a) (5b) (6) These are all derived "in line" on the first page of this doc.

(7) Stokes. Again, I would examine the Purcell derivation of this thing in 3D. [See separate document where I derive this theorem.]

(8) Let $\mathbf{A} = f\,\mathbf{a}$ where $\mathbf{a}$ is a constant vector. Then

$$\nabla\times\mathbf{A} = \nabla\times(f\mathbf{a}) = (\nabla f)\times\mathbf{a} + f\,\nabla\times\mathbf{a} = (\nabla f)\times\mathbf{a}$$

$$d\boldsymbol{\ell}\bullet\mathbf{A} = f\,d\boldsymbol{\ell}\bullet a$$

Then Stokes (7) says

$$\int_S d\mathbf{S}\bullet(\nabla f)\times\mathbf{a} = \oint f\,d\boldsymbol{\ell}\bullet a = \{\oint f\,d\boldsymbol{\ell}\}\bullet a$$

$$= \text{cyclic} = \int_S \mathbf{a}\bullet d\mathbf{S}\times\nabla f = \{\int_S d\mathbf{S}\times\nabla f\}\bullet\mathbf{a}$$

Since a is arbitrary , conclude that

$$\int_S d\mathbf{S}\times\nabla f = \oint d\boldsymbol{\ell}\, f$$

where $d\boldsymbol{\ell}$ is a vector (print does not show this well)

(9) $\int_{\mathbf{A}}^{\mathbf{B}} \nabla\varphi \bullet d\boldsymbol{\ell} = \int_{\mathbf{A}}^{\mathbf{B}} (d\varphi/d\boldsymbol{\ell}\,)\, d\boldsymbol{\ell} = \int_{\mathbf{A}}^{\mathbf{B}} (d\varphi/dx\,)\, dx = \varphi|^{\mathbf{B}}_{\mathbf{A}}$

Here you are integrating the projection of the gradient onto the integration path at each point.

What do these theorems look like in 1 dimension?

$\int_{\mathbf{V}} dV\, \nabla\bullet\mathbf{A} = \int_{\mathbf{S}} d\mathbf{S}\bullet\mathbf{A}$ // divergence theorem = Gauss's theorem (1) Schaum 22.59

$\int_{\mathbf{a}}^{\mathbf{b}} dx\, \partial_{\mathbf{x}}A = \int_{\mathbf{S}} d\mathbf{S}\bullet\mathbf{A} = \int_{\mathbf{S}} dS\, A(x) = A(b) - A(a)$

The "surface S" here is just the two points a and b. So the divergence theorem is then "calculus".

$\int_{\mathbf{V}} dV\, \nabla\varphi = \int_{\mathbf{S}} d\mathbf{S}\, \varphi$ // "integral of a gradient theorem" (2) saved PDF's

$\int_{\mathbf{a}}^{\mathbf{b}} dx\, \partial_{\mathbf{x}}\varphi = \int_{\mathbf{S}} dS\, \varphi = \varphi(b) - \varphi(a)$

So this theorem also reduces to "calculus".

$\int_{\mathbf{V}} dV\, \psi(\nabla\varphi) = -\int_{\mathbf{V}} dV\, (\nabla\psi)\varphi + \int_{\mathbf{S}} d\mathbf{S}\, (\psi\varphi)$ // "parts integration" (2a) corollary to (2)

$\int_{\mathbf{a}}^{\mathbf{b}} dx\, \psi(\partial_{\mathbf{x}}\varphi) = -\int_{\mathbf{a}}^{\mathbf{b}} dx\, (\partial_{\mathbf{x}}\psi)\varphi + \int_{\mathbf{S}} d\mathbf{S}\, (\psi\varphi) = -\int_{\mathbf{a}}^{\mathbf{b}} dx\, (\partial_{\mathbf{x}}\psi)\varphi + \psi(x)\varphi(x)|^{\mathbf{b}}_{\mathbf{a}}$

and yes, this is just 1D parts integration.

$\int_{\mathbf{V}} dV\, \nabla \mathbf{x} \mathbf{A} = \int_{\mathbf{S}} d\mathbf{S} \mathbf{x} \mathbf{A}$ // "integral of a curl theorem" (3) Schaum 22.64

This has no 1D analog! There is no 1D curl.

Green's First Identity: Set $A = \psi\, \partial_{\mathbf{x}}\, \varphi$ in (1) above and use $\nabla\bullet(\psi\nabla\varphi) = \nabla\psi \bullet \nabla\varphi + \psi\, \nabla^2\varphi$ to get:

$\partial_{\mathbf{x}}\, (\psi\, \partial_{\mathbf{x}}\varphi) = \partial_{\mathbf{x}}\psi\, \partial\, \varphi + \psi\, \partial_{\mathbf{x}}{}^2\varphi$ to get:

$\int_{\mathbf{a}}^{\mathbf{b}} dx\, \partial_{\mathbf{x}}\, (\psi\, \partial_{\mathbf{x}}\, \varphi) = \int_{\mathbf{S}} dS\, (\psi\, \partial_{\mathbf{x}}\, \varphi)\bullet\, \hat{\mathbf{n}} = \int_{\mathbf{a}}^{\mathbf{b}} dx\, \{\, \partial_{\mathbf{x}}\, \psi\, \partial_{\mathbf{x}}\, \varphi + \psi\, \partial_{\mathbf{x}}{}^2\varphi\, \}$ (4)

which we rearrange into its traditional form ($\partial\varphi/\partial n = \nabla\varphi\bullet\, \hat{\mathbf{n}}$ = change of φ in the $\hat{\mathbf{n}}$ direction)

$$\int_a^b dx\, \psi\, \partial_x^2 \varphi = \int_S dS\, \psi(\partial\varphi/\partial x) - \int_V dV\, \{\partial_x \psi\, \partial_x \varphi\} \qquad (5a) \qquad \text{Green \#1}$$

$$\int_a^b dx\, \varphi\, \partial_x^2 \psi = \int_S dS\, \varphi(\partial\psi/\partial x) - \int_V dV\, \{\partial_x \varphi\, \partial_x \psi\} \qquad (5b)$$

(This is all just parts integration on a certain functions.) where in the second line we have just swapped $\varphi \leftrightarrow \psi$. If we subtract these two lines, the $\nabla\,\nabla$ terms cancel and we get

$$\int_a^b dx\, [\, \psi\ \partial_x^2 \varphi - \varphi\ \partial_x^2 \psi\,] = \int_S dS\, [\, \psi(\partial\varphi/\partial x) - \varphi(\partial\psi/\partial x)\,] \qquad (6) \qquad \text{Green \#2}$$

$$= [\, \psi(\partial\varphi/\partial x) - \varphi(\partial\psi/\partial x)\,]\Big|_a^b$$

This then is just the sum of two statements of parts integration. Completely valid.

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2. Stokesian:

$$\int_S d\mathbf{S} \bullet (\nabla \times \mathbf{A}) = \oint d\boldsymbol{\ell} \bullet \mathbf{A} \qquad \mathbf{(7)} \qquad \text{Schaum 22.60}$$

$$\int_S d\mathbf{S} \times \nabla\varphi = \oint d\boldsymbol{\ell}\, \varphi \qquad (8) \qquad \text{Schaum 22.65}$$

Since there is no 1D cross product or curl, these cannot have any 1D meaning.

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3. Other:

$$\int_A^B \nabla\varphi \bullet d\boldsymbol{\ell} = \varphi(B) - \varphi(A) \qquad (9) \qquad \text{MM 4-36 p 156}$$

$$\int_A^B \partial_x \varphi\, dx = \varphi(B) - \varphi(A)$$

and again this is just "calculus".

Conclusion on 1D: every theorem is valid in 1D as shown here except those involving curls or cross products. They all reduce to either "calculus" or parts integration.