Proof of Taylor's Expansion Theorem PhL 8.24.08

1. The Long Proof.

I didn't realize this was non-trivial until today. I am reading on page 124 of Ahlfors book.

Let's start with what is called Cauchy's Formula

$$F(z) = (1/2\pi i) \int_C dy\, F(y)/(y-z)$$

where C is a contour which goes clockwise around the pole in the y plane. This thing is only true as long as the function F(y) is itself not singular at the point y = z. In this case, we get that the contour integral picks up the normal pole residue without any derivatives required. Notice that the integrand F(y) /(y-z) has a pole at the point y = z, but the contour stays away from the pole, so nothing dramatic happens during the integration, it is all well defined.

Now consider for our function F the following function

$$F(z) = [\, f(z) - f(a) \,] / (z-a)$$

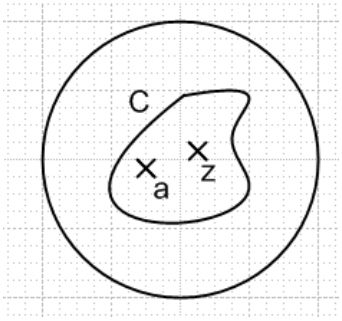
where f(z) is some function which is analytic in our entire region of interest. We are concerned that this function F(z) is not analytic right at the point z = a because we have a 0/0 situation there. But suppose we know that F(z) is at least *finite* at this point z = a,

$\lim_{z \to a} F(z) = f'(a)$ = some finite number since f(z) is analytic.

Then the above Cauchy Formula is valid for the function F. Again, it says

$$F(z) = (1/2\pi i) \int_C dy\, F(y)/(y-z) = (1/2\pi i) \int_C dy\, [\, f(y) - f(a) \,] / [\, (y-a)(y-z) \,]$$

and here is a picture showing the contour and the two poles



Because the contour stays away from both poles, nothing singular happens during the integration over C. If the two poles coincide, fine, nothing special happens, the integral is still well defined. The function of z

represented by the integral is analytic at z = a. Alfors wants to call this integral $f_1(z)$. It is identical with F(z) everywhere other than z = a, and it is well defined and analytic also at z=a. Therefore, the technical point he is making is this: $f_1(z)$ is an "extension" of the function F(z) which was undefined at z=0 and this extension is defined at z=a and is in fact analytic there.

So we write the above Cauchy formula as

$$[\, f(z) - f(a) \,] / (z-a) = f_1(z) = \text{analytic at } z=a.$$

We then have, given that f(z) is continuous and differentiable at point a, ie, f '(a) exists and is finite, we can write

$$f(z) = f(a) + (z-a)f_1(z)$$

This seems like a stupid trivial result but here is what it says: if f(z) is analytic in our disk, then it can be written as shown where $f_1(z)$ is also analytic in the disk.

Now do this again making the replacement $f \rightarrow f_1$. After all, f_1 is analytic just as f was. So:

$$f_1(z) = f_1(a) + (z-a)f_2(z)$$

where f_2 is the name we give to the integral in this case, and of course f_2 is analytic in our disk. Now combine these to get

$$f(z) = f(a) + (z-a)[\, f_1(a) + (z-a)f_2(z)] = f(a) + (z-a)\, f_1(a) + (z-a)^2\, f_2(z)$$

Obviously we can keep going, and we get this series

$$f(z) = f(a) + (z-a)\, f_1(a) + (z-a)^2\, f_2(a) + (z-a)^3\, f_3(a) + (z-a)^4\, f_3(a) + ...$$

$$= \Sigma_{n=0,\infty}\, (z-a)^n\, f_n(a)$$

where we now call $f = f_0$ just so we can make this sum.

Then

$$f'(z) = \Sigma_{n=1,\infty}\, n(z-a)^{n-1}\, f_n(a) = 1\, f_1(a) + \Sigma_{n=2,\infty}\, n(z-a)^{n-1}\, f_n(a)$$

$$f''(z) = \Sigma_{n=2,\infty}\, n(n-1)(z-a)^{n-2}\, f_n(a) = 2*1\, f_2(a) + \Sigma_{n=3,\infty}\, n(n-1)(z-a)^{n-2}\, f_n(a)$$

$$f^{(m)}(z) = \Sigma_{n=m,\infty}\, n(n-1)...(n-m+1)(z-a)^{n-m}\, f_n(a) = m!\, f_m(a) + \Sigma_{n=m+1,\infty}\, n(n-1)...(n-m+1)(z-a)^{n-m}\, f_n(a)$$

Therefore, when we go to z=a, the residual series vanishes in each case and we then get

$$f^{(m)}(a) = m!\, f_m(a) \qquad \Rightarrow f_m(a) = f^{(m)}(a)/\, m!$$

and then we can write our series as

$$f(z) = \Sigma_{n=0,\infty} (z-a)^n f^{(n)}(a)/ n!$$

and this is the Taylor series expansion around the point z = a.

2. The Short Proof.

Just expand f(z) in a power series around f(a)

$$f(z) = f(a) + k_1 (z-a) + k_2 (z-a)^2 + k_3(z-a)^3 + ...$$

which we know we can do because f(z) is analytic Now do m derivatives of f(z) to find this pattern:

$$f^{(m)}(a) = m!\, k_m$$

and then we get our Taylor expansion.

3. Do this in two variables

$$\begin{aligned} f(x,y) = {} & f(a,b) \\ & + k_{10} (x-a) + k_{01}(y-b) \\ & + k_{20}(x-a)^2 + k_{11}(x-a)(y-b) + k_{02}(y-b)^2 \\ & + k_{30}(x-a)^3 + k_{21}(x-a)^2 (y-b) + k_{12}(x-a)(y-b)^2 + k_{03}(y-b)^3 \end{aligned}$$

Comment: Motivation for assuming the above form:

First, treat y as a bystander variable and expand f(x,y) around x = a using 1D Taylor series, In this little comment, the k_{ij} functions are not necessarily the same as above, they are just nameless functions.

$$f(x,y) = f(a,y) + k_1(y)(x-a) + k_2(y)(x-a)^2 + k_3(y)(x-a)^3 + ...$$

or

$$\begin{aligned} f(x,y) = {} & f(a,y) \\ & + k_1(y)(x-a) \\ & + k_2(y)(x-a)^2 \\ & + k_3(y)(x-a)^3 + ... \end{aligned}$$

Now expand each function of y in a 1D Taylor series about y = b

$$\begin{aligned} f(x,y) = {} & [\, f(a,b) + s_1(b)(y-b) + s_2(b)(y-b)^2 + s_3(y)(y-b)^3 + ...] \\ & + [\, k_1(a,b) + k_{11}(b)(y-b) + k_{12}(y)(y-b)^2 + k_{13}(y)(y-b)^3 + ...](x-a) \\ & + [\, k_2(a,b) + k_{21}(b)(y-b) + k_{22}(y)(y-b)^2 + k_{23}(y)(y-b)^3 + ...](x-a)^2 \\ & + [\, k_3(a,b) + k_{31}(b)(y-b) + k_{32}(y)(y-b)^2 + k_{33}(y)(y-b)^3 + ...](x-a)^3 + ... \\ & + ... \end{aligned}$$

Now assume appropriate convergence so can reorder terms:

$$f(x,y) = f(a,b) + k_1(a,b)(x-a) + s_1(b)\,(y-b) \qquad \text{// lead term plus linear terms}$$
$$+ k_2(a,b)\,(x-a)^2 + k_{11}(b)\,(x-a)\,(y-b) + s_2(b)\,(y-b)^2$$
$$+ k_3(a,b)\,(x-a)^3 + k_{21}(b)\,(x-a)^2\,(y-b) + k_{12}(y)\,(x-a)\,(y-b)^2 + s_3(y)(y-b)^3$$
$$+ \ldots$$

This then shows why you might get all those different combinations at each power level. It just comes from doing this double expansion. So far we don't know how to evaluate the coefficients.

So we then assume this general form (new k's),

$$f(x,y) = f(a,b)$$
$$+ k_{10}\,(x-a) + k_{01}(y-b)$$
$$+ k_{20}(x-a)^2 + k_{11}(x-a)\,(y-b) + k_{02}(y-b)^2$$
$$+ k_{30}(x-a)^3 + k_{21}(x-a)^2\,(y-b) + k_{12}(x-a)\,(y-b)^2 + k_{03}(y-b)^3$$

Notice that

$$\{\partial^2 x\, \partial^1 y\, f\}|^{a,b} = 2\, k_{21}$$

A general term in the series will have this form

$$T_{nm} = k_{nm}\,(x-a)^n\,(y-b)^m$$

and notice that

$$\{\partial^n x\, \partial^m y\, f\}|^{a,b} = n!\; m!\; k_{nm}$$

Any lesser derivative acting T_{nm} gives 0 when we evaluate at x=a and y=b.
Any greater derivative acting T_{nm} gives 0 because derivatives above $\partial^n x\, \partial^m y$ act on constant.
Thus, you can say

$$\{\partial^n x\, \partial^m y\, f\}|^{a,b}\, T_{n'm'} = \delta_{n,n'}\,\delta_{m,m'} \quad \{\partial^n x\, \partial^m y\, f\}|^{a,b} = \delta_{n,n'}\,\delta_{m,m'}(\, n!\; m!\; k_{nm}\,)$$

Now rewrite the original series as a sum of its T_{nm} terms

$$f(x,y) = \Sigma_{n=0}^{\infty}\; \Sigma_{m=0}^{\infty}\; T_{nm}(x,y)$$

Then we find that -- assuming we can do term-by-term differentiation --

$$\{\partial^{n'} x\, \partial^{m'} y\, f\}|^{a,b} = \Sigma_{n=0}^{\infty}\; \Sigma_{m=0}^{\infty}\; \{\partial^{n'} x\, \partial^{m'} y\, f\}|^{a,b}\; T_{nm}(x,y)$$

$$= \Sigma_{n=0}^{\infty}\; \Sigma_{m=0}^{\infty}\; \delta_{n,n'}\,\delta_{m,m'}(\, n!\; m!\; k_{nm}\,)$$

$= \ (n'! \ m'! \ k_{n'm'})$

Thus we have proven that (removing primes),

$\{\partial^n x \ \partial^m y \ f\}|^{a,b} = \ n! \ m! \ k_{nm} \ \ ,$

and our conclusion is this for the double Taylor series :

$$f(x,y) = \Sigma_{n=0}^{\infty} \ \Sigma_{m=0}^{\infty} \ T_{nm}(x,y) \ = \ \Sigma_{n=0}^{\infty} \ \Sigma_{m=0}^{\infty} \ \ k_{nm} \ (x-a)^n \ (y-b)^m$$

$$= \Sigma_{n=0}^{\infty} \ \Sigma_{m=0}^{\infty} \ \frac{\{\partial^n x \ \partial^m y \ f\}^{ab}}{n! \ m!} \ \ (x-a)^n \ (y-b)^m$$

Comment. Suppose in the above we have n=1 and m=1. That term is supposed to be

$$\{ \ (\partial^2 f/\partial x \partial y)|^{a,b} /(\ 1! \ 1!) \ \} \ (x-a)^1 \ (y-b)^1 \ = (\partial^2 f/\partial x \partial y)|^{a,b} \ (x-a)(y-b)$$

and this is correct because there is only one such term.

Now suppose we regard $x = x_1$ and $y = x_2$. Then the above expansion is written

$$f(x_1, x_2) = \Sigma_{n,m=0,\infty} \ k_{nm}(x_1-a)^n \ (x_2-b)^m$$

$$= \Sigma_{n,m=0,\infty} \ \{ \ \partial^{n+m} f/\partial^n x_1 \partial^m x_2|^{a,b} /(\ n! \ m!) \ \} \ (x_1-a)^n \ (x_2-b)^m$$

$$= \Sigma_{n,m=0,\infty} \ \{ \ (\partial_1^n \partial_2^m f) \ |^{a,b} /(\ n! \ m!) \ \} \ (x_1-a)^n \ (x_2-b)^m$$

The n=m=1 term still gives a single term. In the derivative denominator, we never vary the order of the variables. For example, we never have $\partial^{n+m} f/\partial^n x_2 \partial^m x_1$ and we never have $\partial_2^n \partial_1^m f$.

It seems clear how you would extend this to three 3 variables:

$$f(x_1, x_2, x_3) = \Sigma_{n1,n2,n3=0,\infty} \ k_{n1,n2,n3} \ (x_1-x_{10})^{n1} \ (x_2 - x_{20})^{n2} \ \ (x_3 - x_{30})^{n3}$$

$$= \Sigma_{n1,n2,n3=0,\infty} \{ \ \partial^{n1+n2+n3} f/\partial^{n1} x_1 \partial^{n2} x_2 \partial^{n3} x_3|^{x10,x20,x30} /(\ n1! \ n2! \ n3!) \ \}$$
$$(x_1-x_{10})^{n1} \ (x_2 - x_{20})^{n2} \ \ (x_3 - x_{30})^{n3}$$

which I alternately write as

$$f(x,y,z) \ = \Sigma_{n=0}^{\infty} \ \Sigma_{m=0}^{\infty} \ \Sigma_{s=0}^{\infty} \ \frac{\{\partial^n x \ \partial^m y \ \partial^s z \ f\}^{ab}}{n! \ m! \ s!} \ \ (x-a)^n \ (y-b)^m \ (z-c)^s$$

4. Do it for a vector function of multiple variables

We just do the above for each component of the vector,

$$f_i(x,y) = \Sigma_{n,m=0,\infty}\ k^i{}_{nm}(x-a)^n\,(y-b)^m$$

$$= \Sigma_{n,m=0,\infty}\ \{\ \partial^{n+m}f_i/\partial^n x\partial^m y|^{a,b}\ /(\ n!\ m!)\ \}\ (x-a)^n\,(y-b)^m$$

5. Do it for a matrix function of multiple variables

Again, just do it for each component:

$$f_{ij}(x,y) = \Sigma_{n,m=0,\infty}\ k^{ij}{}_{nm}(x-a)^n\,(y-b)^m$$

$$= \Sigma_{n,m=0,\infty}\ \{\ \partial^{n+m}f_{ij}/\partial^n x\partial^m y|^{a,b}\ /(\ n!\ m!)\ \}\ (x-a)^n\,(y-b)^m$$

6. Do it for a matrix function of a matrix

The variables are now just the elements of the matrix which you list in some order. Here for a 2x2 we could say

$$f_{ij}\ (x_{11},x_{12};\ x_{21},\ x_{22}) =$$

$$\Sigma_{n11,n12,n21,n22\ =\ 0,\infty}$$
$$\{\ \partial^{n11+n12+n21+n22}\ f_{ij}/\partial^{n11}x_{11}\partial^{n12}x_{12}\partial^{n21}x_{21}\partial^{n21}x_{22}|^{x110,x210,x120,x220}\ /(\ n11!\ n12!\ n21!\ n22!)\ \}$$
$$(x_{11}-a_{11})^{n11}\ (x_{12}-a_{12})^{n12}\ \ (x_{21}-a_{21})^{n21}\ \ (x_{22}-a_{22})^{n22}$$

$$= \Sigma_{n11,n12,n21,n22\ =\ 0,\infty}\ k^{ij}{}_{n11,n12....}\ \ (\delta x_{11})^{n11}\ (\delta x_{12})^{n12}.....$$

Now suppose all the displacements are of the same general small magnitude perhaps λ, what would be the order λ term in the above expansion? We can list off the terms:

$$\{\ n11,\ n12,\ n21,n22\}\ \ = \{1000\} + \{0100\}+\ \{0010\}+\ \{0001\}$$

$$= \partial f_{ij}/\partial_{x11}\ (x_{11}-a_{11})\ \ + \partial f_{ij}/\partial_{x12}\ (x_{12}-a_{12}) + \partial f_{ij}/\partial_{x11}\ (x_{21}-a_{21}) + \partial f/\partial_{x22}\ (x_{22}-a_{22})$$

$$= \Sigma_{rs=1,2}\ \ (\partial f_{ij}/\partial_{xrs})\ \ (x_{rs}-a_{rs})\ \ = \Sigma_{rs=1,2}\ k^{ij}{}_{rs}\ \delta x_{rs}$$

Now for fun, let's go for the λ^2 term:

$$\{\ n11,\ n12,\ n21,n22\} = \{1100\} + \{1010\}+\ \{1001\}+\ \{0110\} +\ \ \{0101\} + \{0011\}$$
$$+\ \{2000\} + \{0200\}+\ \{0020\}+\ \{0002\}$$

$$= (\ \partial^2 f_{ij}/\partial_{x11}\partial_{x12})\ \ \delta x_{11}\delta x_{12}\ +5\ \text{similar}\ \ \ +\ \ (\partial^2 f_{ij}/\partial^2{}_{x11})/2!(\delta x_{11})^2\ +\ \ 3\ \text{similar}$$

Try to state in more general terms, for an NxN matrix:

$$f_{ij}(x_{11},x_{12}\ldots.) = \Sigma_{n11,n12\ldots=0,\infty}\,(k^{ij}{}_{n11,n12\ldots.})\;(\delta x_{11})^{n11}\,(\delta x_{12})^{n12}\ldots..$$

Now let's say that $(\delta x_{11}) = \lambda(\delta y_{11})$ just to expose the smallness. Then we get

$$f_{ij}(x_{11},x_{12}\ldots.) = \Sigma_{n11,n12\ldots=0,\infty}\,(k^{ij}{}_{n11,n12\ldots.})\;(\delta y_{11})^{n11}\,(\delta y_{12})^{n12}\ldots..\;\;\lambda^{n11+n12\ldots}$$

So the point is that if we want the order M contribution in this sum, we need to do this

$$f_{ij}(x_{11},x_{12}\ldots.) = \Sigma_{n11,n12..}\;\;(k^{ij}{}_{n11,n12\ldots.})\;(\delta x_{11})^{n11}\,(\delta x_{12})^{n12}\ldots..$$

where we restrict the sum to values such that $n_{11}+n_{12}+\ldots = M$.

Conclusion: it is not easy to write the terms higher than order $\lambda=1$ in some kind of closed compact form. However, it is easy to do it for the $\lambda=1$ terms. For example, with a 3x3 matrix there will be 9 terms which we could indicated by {100 000 000} and so on, sliding the 1. There are nine possible δx objects and we can write

$$f_{ij}(x_{11},x_{12}\ldots) = \Sigma_{rs=1,3}\;(\partial f_{ij}/\partial_{xrs})\;(x_{rs}-a_{rs})\;\;=\Sigma_{rs=1,3}\,k^{ij}{}_{rs}\,\delta x_{rs}$$

and we can even put this into a sort of matrix notation

$$F(X)_{\text{first order}}\;=\Sigma_{rs=1,3}\;(\partial F/\partial X_{rs})\;(X_{rs}-A_{rs})\;\;=\Sigma_{rs=1,3}\,K_{rs}\,\delta X_{rs}$$

where F, X and each of the K_{rs} objects is a 3x3 matrix, and δX_{rs} are just numbers. Don't make the mistake of thinking K_{rs} is an element of a matrix, it is a whole matrix. The form on the right is the sum of nine matrices K_{rs} each weighted by the scalar δX_{rs} .

We can generalize to an NxN matrix

$$F(X)|^{\text{first order}}\;=\Sigma_{rs=1,N}\;(\partial F/\partial X_{rs})\;(X_{rs}-A_{rs})\;\;=\Sigma_{rs=1,N}\,K_{rs}\,\delta X_{rs}$$

where we are expanding X near that matrix A. Suppose A = 1, the identity matrix, then

$$F(X)|^{\text{first order}}\;=\Sigma_{rs=1,N}\;(\partial F/\partial X_{rs})|^{X=1}\;(X_{rs}-\delta_{rs})\;\;=\Sigma_{rs=1,N}\,K_{rs}\,(X_{rs}-\delta_{rs})$$

or we write through first order:

$$F(X) = F(1) + \Sigma_{rs=1,N}\,K_{rs}\,(X_{rs}-\delta_{rs})\qquad\qquad\text{F, X and K are all NxN matrices}$$

as appears, for example, in **equation 2.14 page 20 of B&D** where F(1) = 1 and $K_{rs} = (-i/4)\,\sigma_{rs}$ and we have the case N=4. We even know something about the matrices

$K_{rs} = (\partial F/\partial X_{rs})|^{X=1}$ = the matrix obtained by differentiating matrix F with respect to

the single variable X_{rs} which is one element of the X matrix

8. Taylor expansion in N variables (added 3.3.12)

The expansion around the point (x,y) = (a,b) given above is the following

$$f_i(x,y) = \Sigma_{n,m=0,\infty}\ k^i_{nm}(x-a)^n\,(y-b)^m$$

$$= \Sigma_{n,m=0,\infty}\ \{\ \partial^{n+m}f_i/\partial^n x\partial^m y|^{a,b}\ /(\ n!\ m!)\ \}\ (x-a)^n\,(y-b)^m$$

Now write $a = x_0$ and then $x-x_0 = dx$ etc to get

$$f_i(x,y) = \Sigma_{n,m=0,\infty}\ k^i_{nm}(dx)^n\,(dy)^m$$

$$= \Sigma_{n,m=0,\infty}\ \{\ (\partial^{n+m}f_i/\partial^n x\partial^m y)|^{x0,y0}\ /(\ n!\ m!)\ \}\ (dx)^n\,(dy)^m$$

Is there some way to put this into vector notation where $\mathbf{x} = (x,y)$? First rewrite as

$$f_i(x_1,x_2) = \Sigma_{n,m=0,\infty}\ k^i_{nm}(dx_1)^n\,(dx_2)^m$$

$$= \Sigma_{n,m=0,\infty}\ \{\ (\partial^{n+m}f_i/\partial^n x_1\partial^m x_2)|^{x10,x20}\ /(\ n!\ m!)\ \}\ (dx_1)^n\,(dx_2)^m$$

Probably the adding a third variable gives

$$f_i(x_1,x_2,x_3) = \Sigma_{n,m,k=0,\infty}\ k^i_{nmk}\ (dx_1)^n\,(dx_2)^m\,(dx_3)^k$$

$$= \Sigma_{n,m,k=0,\infty}\ \{\ (\partial^{n+m+k}f_i/\partial^n x_1\partial^m x_2\partial^k x_3)|^{x10,x20,x30}\ /(\ n!\ m!\ k!)\ \}\ (dx_1)^n\,(dx_2)^m\,(dx_3)^k$$

Does this derivative object have a name?

$$(\partial^{n+m+k}f_i/\partial^n x_1\partial^m x_2\partial^k x_3)\ =\ [\partial_1{}^n\partial_2{}^m\partial_3{}^k\ f_i(\mathbf{x})]$$

Not so bad. So then we have

$$f_i(x_1,x_2,x_3) = \Sigma_{n,m,k=0,\infty}\ [\partial_1{}^n\partial_2{}^m\partial_3{}^k\ f_i(\mathbf{x})]^{\mathbf{x}=x0}\ /(\ n!\ m!\ k!)\ \}\ (dx_1)^n\,(dx_2)^m\,(dx_3)^k$$

$$f_i(x_1,x_2,x_3) = \Sigma_{n1,n2,n3=0,\infty}\ [\partial_1{}^{n1}\partial_2{}^{n2}\partial_3{}^{n3}\ f_i(\mathbf{x})]^{\mathbf{x}=x0}\ /(\ n1!\ n2!\ n3!)\ \}\ (dx_1)^{n1}\,(dx_2)^{n2}\,(dx_3)^{n3}$$

Then our general formula might be written

$$f_i(\mathbf{x})\ =\ \Sigma_n\ [\partial_1{}^{n1}\partial_2{}^{n2}\partial_3{}^{n3}.....\ f_i(\mathbf{x})]^{\mathbf{x}=x0}\ /\ (n1!\ n2!\ n3!\ ...)\ \}\ (dx_1)^{n1}\,(dx_2)^{n2}\,(dx_3)^{n3}\ .....$$

$$f_i(\mathbf{x})\ =\ \Sigma_n\ [\partial_1{}^{n1}\partial_2{}^{n2}\partial_3{}^{n3}.....\ f_i(\mathbf{x})]^{\mathbf{x}=x0}\ /\ \Pi_{s=1}{}^N\,(n_s)!\ \}\ \Pi_{s=1}{}^N\,(dx_s)^{ns}$$

$$f_i(\mathbf{x})\ =\ \Sigma_n\ [\partial_1{}^{n1}\partial_2{}^{n2}\partial_3{}^{n3}.....\ f_i(\mathbf{x})]^{\mathbf{x}=x0}\ \ \Pi_{s=1}{}^N\,[(dx_s)^{ns}/n_s!]$$

$$f_i(\mathbf{x})\ =\ \Sigma_k\ [\partial_1{}^{k1}\partial_2{}^{k2}\partial_3{}^{k3}.....\ f_i(\mathbf{x})]^{\mathbf{x}=x0}\ \ \Pi_{s=1}{}^N\,[(dx_s)^{ks}/k_s!]$$

NB: The order of variables in the ∂'s are always monotonic increasing!!

In Stakgold volume 2 page 2 we had the concept of a vector "multi-index"

k such that $k = k_1 + k_2 + \ .. \ + k_N$

The derivative is just called $D^{\mathbf{k}}$ so the above becomes

$$f_i(\mathbf{x}) = \Sigma_{\mathbf{k}} \ [D^{\mathbf{k}} f_i(\mathbf{x})]^{\mathbf{x}=x^0} \ \Pi_{s=1}^{N} [(dx_s)^{k_s}/k_s!] \qquad\qquad D^{\mathbf{k}} = \partial_1^{k1}\partial_2^{k2}\partial_3^{k3}.....$$

Good, that is a fairly compact result I think. Let's now write the terms of various smallness orders.

zeroth term = $f_i(\mathbf{x}_0)$

first order term $= (\partial_1 f_i)dx_1 + \text{etc} = (\partial_j f_i)\, dx_j = \nabla(f_i) \bullet d\mathbf{x} = [(\nabla \mathbf{f})\, \mathbf{x}]_i = (\nabla \mathbf{f})_{ij} x_j$

And then so far we have, in vector notation,

$$\mathbf{f}(x) = \mathbf{f}(x_0) + (\nabla \mathbf{f})|^{\mathbf{x}^0}\, \mathbf{x}$$

and we see the utility of the $\nabla \mathbf{f}$ notation.

second order term $= (\partial_1\partial_1 f_i)dx_1 dx_1/2! + (\partial_2\partial_2 f_i)dx_2 dx_2/2! + ...$
$+ (\partial_1\partial_2 f_i\ ) dx_1 dx_2$ // there IS no term $(\partial_2\partial_1 f_i\ ) dx_2 dx_1$!!!

Example: $f_i(\mathbf{x}) = xy$

$f_i(\mathbf{x}+\mathbf{dx}) = (x+dx)(y+dy) = xy + ydx + xdy + dxdy$

So how can we put this into some reasonable notation?

second order term (N=2) $= (\partial_1\partial_1 f_i)dx_1 dx_1/2! + (\partial_2\partial_2 f_i)dx_2 dx_2/2! + (\partial_1\partial_2 f_i\ ) dx_1 dx_2$

$= (1/2!) \Sigma_{n+m=2} (\partial_n\partial_m f_i)\, dx_n dx_m$

In *this* double sum, we have two terms appearing $(\partial_1\partial_2 f_i)\, dx_1 dx_2$ and $(\partial_2\partial_1 f_i)\, dx_2 dx_1$ which is different from the mindset of the general equation. Only one of these terms should count, and that is corrected for by the outside factor (which provides the 1/2! for the other two terms).

Is there some vector or tensor way to write this second order term? I guess we could define

$$(\nabla\nabla \mathbf{f})_{inm} \equiv (\partial_n\partial_m f_i)$$

but that doesn't help much with vector notation.

Notice that n,m are coordinate index labels, whereas the k_i integers are powers.

$$f_i(\mathbf{x})|^{1st} = \Sigma_{k=1} \; [\partial_1^{k1}\partial_2^{k2}\partial_3^{k3}..... f_i(\mathbf{x})]^{x=x^0} \; \Pi_{s=1}^{N} [(dx_s)^{ks}/k_s!] \; = N \text{ terms}$$

$$f_i(\mathbf{x})|^{2nd} = \Sigma_{k=2} \; [\partial_1^{k1}\partial_2^{k2}\partial_3^{k3}..... f_i(\mathbf{x})]^{x=x^0} \; \Pi_{s=1}^{N} [(dx_s)^{ks}/k_s!] \; = N + (N,2) \text{ terms}$$

In writing this, I was trying to find a way to associate Γ^{c}_{ab} with these second order terms.