On Certain Slowly Convergent Series Occurring in Plate Contact Problem PhL 10.17.04
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This guy is an emeritus math prof at Purdue and has worked a lot in computational math and ortho polys and all that good stuff. This paper references 6 earlier papers of the author, and these contain some key ingredients that one would need to have for a complete understanding, which I don't need to have.

I think the series studied here arise in structural engineering, so called "plate contact problems", which I think means an area of contact between a plate and some other object. I could find little on the web on this subject, so we forget it.

The main concern of the paper is the series $R_p(z)$ shown in (1.1), where p is some integer like 2 or 3. You want to add up this infinite series, but it has no known closed form answer, so you do it numerically. When z is near the unit circle, however, this series converges slowly, so perhaps you would have to add up millions of terms to get an answer to the accuracy of 25 digits, no comment is made on this however other than the series converges slowly. There is certainly no exponential convergence. The other series shown in (1.2) is functionally related to the first. [Well, if the error is related to the last term, if you want the last term to be smaller than 10^{-20} for 20 places accuracy, that term needs to have $k = 10^{10} / 2$, so you need to add 5 billion terms for the p = 2 case !!!]

The author is interested in two different regions of z. One is the real axis [0,1], and the other is the unit circle. In the case of the unit circle, the sums are known for p=2 and 3, as shown in (1.6). Notice that this author does not waste brackets to clarify the fact that α is part of the argument of cos and sin. The sum for any p is known when z=1, but z = 1 is a trouble point for his method.

2. Summation of R_p for p=2,3. The author wants to show a method here. Our sum of interest is converted to the Stieltjes integral as shown in 2.7. We don't care about the z=1 case since we already have a closed formula, BUT, this formula uses the Riemann zeta, so in fact the method can be used to compute this function to very high accuracy.

So what is the method? The S-integral is just an integral of the form $\int dt\ w(t)/(t-x)$, a form I have seen in dispersion relations and many places (principle part stuff, optics, etc). Here the x is z^{-2} . The method is really derived in some other paper, but here is what you do. You regard w(t) as a weight function which is related to some (unknown) set of ortho polys. You somehow know that the form of the poly recursion formula is as shown in 2.10 -- at least it looks typical. He has a method of computing the α and β coefficients. You then show that the integral you need to solve your problem is the limit $\upsilon \to \infty$ of a sequence which is a messy continued fraction thing if you wrote it out, which you don't need to do. You pick some reasonably large υ like $\upsilon = 99$ and you compute all 100 terms in the sequence as shown, going backwards until you step down to index n = -1. That final value which you reach IS the integral in this approximation of $\upsilon = 99$. The exact answer is shown in 2.14 where $\upsilon = \infty$. Computation of the sequence is cheap. So this is his method of evaluating the sums. The result seems to apply to any value of p.

3. Generating the α and β coefficients. He starts off defining some weird "moments" $m(\alpha; p)$ which are integrals of your w(t) against shifted Legendre polys. At least for p = 2 and 3 he can get closed forms for these moments, as shown on page 330. It seem that his differentiation method could get the moments for larger p as well. The actual way you get the α and β from these moments is not shown, but it is claimed to be a well-known method [a Chebyshev algorithm, his ref 4] So OK, let's assume we have our α and β formulas so we can compute them easily when we need to. They are just numbers.

4. Examples. First, he talks about his method of finding a reasonable υ (like $\upsilon = 50$) to work with for a desired ε. It seems that any ε is achievable by this method. He does a bracketing trick and some tests to determine the range [0, A] where $A < 1$, for example. But A depends on υ and ε. Table 4.1 shows some results for $p = 2$ and 3. For example, if you use $\upsilon = 30$ and require $\varepsilon = 10^{-20}$ which I guess means 20 places of accuracy, you can get out to $A = .9395$ in the unit interval. If you need to get closer than this to 1, you need to user larger υ. He does the same idea for the unit circle location of z. The problem here is now getting to a small angle away from 0 degrees, again $z = 1$ is the problem point. The relative angle is called $\omega(\upsilon,\varepsilon)$ and Table 4.2 gives the results. In Table 4.3 we have more of the same idea for the unit interval case. As you increase your A requirement from .8 to .999, you see how υ increases. He uses 20 places of accuracy for this Example 1. Then Example 2 is similar for on the unit circle.

On page 334 he applies the same method to the fancier sinh and cosh sums in (1.8) -- these also arise in those "plate contact problems". We get formula 5.1 which has expo convergence and we need th ϕn functions. Each is written as a Stieltjes as in 5.2, and the method is used as before. This is more than I want to know.

Appendix. Here he lists the α and β coefficients for $p = 2$ and 3 out to $n = 99$ which I think apply to the first problem considered. I wonder why he shows these? They would allow someone to use the method to sum the series for $p = 2$ and $p = 3$, so I guess this data is provided as a service to the reader interested in doing those sums!

Comments: This method replaces adding billions of terms with doing a few quick sequence computations, in which you use the pre-computed numbers in the appendix. Pretty amazing.