Hilbert Spaces involved in the B&B paper PhL 1.19.05

There are a lot of Hilbert Spaces flying around all at once, and that causes immense confusion. So let's try to put this stuff now into some reasonable logic ordering, based on the discussion in document "Addendum... ".

Space #1. We start with our abstract position operator eigenvalue equation

$$X \mid x_i > = x_i \mid x_i >$$

but the x_i are our N *discrete points* which are the nodes of a polynomial $P_N(x_i)$. The states $\mid x_i >$ have these orthonormality and completeness properties,

$$1 = \Sigma_{i=0}^{N} A_i \, |x_i><x_i|$$

$$<x_i|x_j> = (1/A_i) \, \delta_{i,j}$$

The analogy with the continuous world is this:

$$1 = \int dx \, w(x) \mid x > < x \mid$$

$$< x \mid x' > \; = (1/w(x)) \, \delta(x - x')$$

The states $\mid x_i >$ form a basis in an N dimensional Hilbert space we might call "node space". The space is a Hilbert space because the $\mid x_i >$ are eigenvectors of a Hermitian operator X, more on this later.

Space #2. This is what we have called " |n> space". We could define a general "function space" where we could denote by | f > an abstract function f which maps some space into some other space. When we write | n>, we are referring to the particular function $\mid P_n>$ which is one of our basis functions. In the coordinate representation we write

$$<x_i \mid P_n> \; = \; <x_i \mid n > \; = \; P_n(x_i)$$

Now our n-space does not allow ANY function, it only allows functions which are polynomials of degree N-1 or less. Thus, this space is an N dimensional space.

Our completeness and orthogonality in n-space appears as follows:

$$1 = \Sigma_n \mid n> <n|$$

$$<n|m> = \delta_{n,m}$$

Consider now this expansion:

$$\mid n > \; = 1 \mid n > \; = \Sigma_{i=1}^{N} A_i \, |x_i><x_i|n > = \; \Sigma_{i=1}^{N} A_i \, P_n(x_i) \; |x_i>$$

This shows that | n > is just a linear combination of the $|x_i>$ states. So the vectors | n > are just an alternate basis in our "node space". They form an orthonormal basis.

Here is a result we get by playing with our two different bases:

$$<x_i|x_j> = (1/A_i)\,\delta_{i,j} \quad = <x_i \;\; \Sigma_n \,|\, n><n| \quad\; x_j> = \Sigma_n <x_i\,|n><n|\;\; x_j> = \Sigma_n\, P_n(x_i)\, P_n(x_j)$$

From this we conclude that

$$\Sigma_n\, P_n(x_i)\, P_n(x_j) = 0 \qquad\qquad i \neq j$$

$$\Sigma_n\, [\, P_n(x_i)\,]^2 = (1/A_i) = S_i$$

and we will see this summation used in various places.

Space #3. This is the "continuous position space" with | x >. In this space we have as noted earlier

$$1 = \int dx\; w(x)\,|\,x><x\,|$$

$$< x\,|\,x' > \;= (1/w(x))\;\delta(x - x')$$

$$X\,|\,x> \;= x\;\;|\,x>$$

This is an infinite dimensional vector space. We could associate this space with | f >, the space of functions, which is of course also of infinite dimension. We often list these functions as x^0, x^1, x^2 etc. We know that the space of polys of degree N-1 is an N dimensional *subspace* of the | f> space. The space and its subspace had the same identify operator "1". That is why we can do the following:

As an application of this we can write

$$< n\,|\,m> = < n\,|\,1\,|\,m> = < n\,|\int dx\; w(x)\,|\,x><x\,|\,m> = \int dx\; w(x) <n|x><x|m>$$

$$\Rightarrow\; \delta_{n,m} \;=\; \int dx\; w(x)\; P_n(x)\; P_m(x)$$

So "continuous position space" is the space which has w(x) and (a,b) and which we use to define our $P_n(x)$ ortho polys.

Space #4: This is what we call the J-space, another N dimensional space. This is an E^N space where we do things like our Jacobi matrix equation. The vectors in J-space are just N-tuples of numbers,

$$\mathbf{j} = [v_1, v_2, \ldots\ldots v_N\,] \qquad = \text{a vector in J space}$$

Here is a vector of particular interest in J-space,

$$\mathbf{V(x_i)} = [P_0(x_i), P_1(x_i), \ldots\ldots P_{N-1}(x_i)\,]$$

We think of (x_i) as a parameter on this vector, so $\mathbf{V(x_i)}$ is just an N-tuple of numbers.

Here is our Jacobi matrix equation in J-space

$$J\ \mathbf{V(x_i)} = x_i\ \mathbf{V(x_i)}$$

which we can write as

$$\Sigma_m\ J_{nm}\ [\mathbf{V(x_i)}]_m = x_i\ [\mathbf{V(x_i)}]_n$$

which we can rewrite as

$$\Sigma_m\ J_{nm}\ P_m(x_i) = x_i\ P_n(x_i)$$

or

$$\Sigma_m\ J_{nm}\ < x_i \mid m > = x_i < x_i \mid n >$$

We can now replace $x_i < x_i \mid n > = < x_i \mid X \mid n >$ using an earlier fact, and then we can simply remove the projection from both sides, since must be true for any x_i , and we get (changing name of dummy sum index)

$$\Sigma_{m'}\ J_{nm'}\ \mid m' > = X \mid n >$$

Now close both sides with <m | to get

$$J_{nm} = <m \mid X \mid n >$$

This shows that the "Jacobi matrix" matrix elements are the matrix elements of the position operator X taken in the | n> basis of our N dimensional Hilbert Space.

Now one other fact about our vector $\mathbf{V(x_i)}$**.** We can compute it's magnitude squared in J-space,

$$\| \mathbf{V(x_i)} \|_{J\text{-space}} = \mathbf{V(x_i)} \bullet \mathbf{V(x_i)} = \Sigma_n\ [\ P_n(x_i)\ ]^2 = S_i$$

Sample Calculation

Consider, where we now identify matrix J with matrix T_N,

$$\Sigma_m\ J_{nm}\ P_m(x_i) = x_i\ P_n(x_i) \qquad \text{or} \qquad T_N\ \mathbf{V(x_i)} = x_i\ \mathbf{V(x_i)}$$

The "vector sense" of this eigenvalue equation for a given fixed index "i" is that of our J-space. After all, T_N is a matrix in J-space. The "classic result" for first order perturbation theory is this

$$\delta x_i = \frac{\mathbf{(V(x_i),[\delta T_N]V(x_i))}}{\mathbf{(V(x_i),V(x_i))}}$$

where recall that the denominator is S_i . This is how we obtain (2.13) of Jim's ".. 2 classes of log..." paper, or the corresponding equation on page 5 of the first B&B paper.

Now that we have obtained the correct "official result" we can rewrite the above as follows

$$\text{numerator} = (\ \mathbf{V(x_i)},[\delta T_N]\mathbf{V(x_i)}) = \Sigma_{nm}\ [\mathbf{V(x_i)}]_{\mathbf{n}}[\delta T_N]_{nm}[\mathbf{V(x_i)}]_{\mathbf{m}} = \ \Sigma_{nm}\ P_n(x_i)\ [\delta T_N]_{nm}\ P_m(x_i)$$

$$= \Sigma_{nm} \langle x_i \mid n\rangle \langle n \mid [\delta T_N] \mid m\rangle \langle m \mid x_i\rangle \ = \langle x_i \ \mid [\delta T_N] \mid \mid x_i \rangle$$

$$\text{denominator} \ = \langle x_i \ \mid x_i\rangle \ = S_i$$

and we would say as our perturbation theory result

$$\delta x_i \ = \frac{\langle x_i \ \mid [\delta T_N] \mid \mid x_i \rangle}{\langle x_i \ \mid x_i\rangle}$$

Now, recall that the abstract operator T_N is really X, so we are talking about δX. This X operator really does change if we change our weight function. Think of it this way

$$X(\alpha) \mid x_i, \alpha \rangle = x_i \mid x_i, \alpha \ \rangle$$

$$1 = \Sigma_{i=0}^{N}\ A_i \mid x_i, \alpha \rangle\langle x_i, \alpha \mid$$

$$\langle x_i, \alpha \mid x_j, \alpha \rangle = S_i(\alpha)\ \delta_{i,j}$$

where α is a parameter we can vary. We would then quote directly from 1st order perturbation theory to make this claim

$$\delta x_i \ = \frac{\langle x_i;\ \alpha \ \mid [\delta X(\alpha)] \ \mid x_i;\ \alpha \ \rangle}{\langle x_i;\ \alpha \ \mid \ x_i;\ \alpha \rangle}$$

I am certainly tempted to identify the J-space with the n-space, continue manana.