Hildebrand "Intro to Numer Analysis, Dover 1967, 1974 PhL 10.18.04

7.2 Least Squares (p 315) In this section, we approximate a function f(x) by a sum of basis function polynomials $\phi_r(x)$ and we have in mind a weight w(x). Suppose we insist that the least square error of the fit be minimal, as stated in 7.2.4. Note that the brackets <...> imply either integration or summation, a nice notation. Many results are derived here. The requirement of minimality tells us that $< wyR> = 0$, where R is just R = f-y. The actual minimized LS error is $<wR^2>_{min}$ as shown in 7.2.10. We would of course like to know the coefficients a_k that give this optimal fit, and these are the solutions to the system of equations shown in 7.2.5,6. If you arrange to have your basis polynomials be orthogonal, this system decouples and you get the simple result 7.2.13 for the solution coefficients a_r. As long as $w(x) > 0$, you can be sure that you never get a divide by zero in 7.2.13, so that is one reason I guess that we always require the weights to be positive definite. I have verified all equations in this section, but I don't yet know how they will be applied!

7.5 Orthogonal Polynomials (p 327). This is a rare section that discusses ortho polys in a general manner. Most books talk about the usual list of specific ones that are well known. We start with the requirement that our polys be orthogonal in 7.5.1 and see where that leads us! The auxiliary function U as defined in 7.5.2 is a useful tool all along the way. We quickly conclude that the U functions satisfy the ODE shown in 7.5.5. This is really a trivial equation in terms of the $\phi_r(x)$ because it just says the $r+1^{st}$ derivative of $\phi_r(x)$ must be zero. I am not sure where this ODE is later used, so we skip instead to the nice parts integration of the integral shown below 7.5.2 as done in 7.5.3. What you find is that you have nothing but parts left because you keep going until the integral vanishes because r^{th} derivative of $\phi_{r-1}(x) = 0$. As usual, you keep alternating signs with each parts. Since the q_r poly is arbitrary, you conclude that the U and all its derivatives vanish at the endpoints a and b (we are in continuous space in this section). So OK, now we have an ODE for U_r, and we have a set of BC's for U_r, so perhaps we have a completely defined ODE problem on our hands. We get a quote from Szego that in fact this ODE problem does in fact have a unique solution subject to some minor conditions like w(x) is pos def and powers are integrable over a,b, and it is OK if either endpoint is infinite! So this is a big theorem. It says that the interval points a,b and the weight function DEFINE a set of polys that are unique up to normalization.

Now we move on by trying to fit some f(x) with a y(x) expanded in our basis functions. If we insist that our fit do the least squares minimization, THEN we can write 7.5.10 for the a_r expansion coefficients, this came from section 7.2 above. In this expression, the denominator γ_r is something you can precompute if you know the polys.

Well fine. Unfortunately, this section does not derive the Rodrigues' Formula. This formula looks like 7.5.5, but it claims that U_r can be written in the form $U_r = w(x) [g(x)]^r$ where g(x) is a polynomial. This fact is certainly not obvious to me. Hilde does mention this formula in specific later sections. Maybe this is not generally true except for a certain class of weight functions. The generating function is also not mentioned. I think I would have to find a real book on this subject.