The Jordan Form N spaces picture PhL 1.7.05

See Figure 1 below. Just explaining what this is a picture *of* is going to take a lot of words! We are talking about composite mappings of the form $p^{h-1}(T)$: U→V then p(T):V→W, for a total mapping that has the form $p^h(T)$: U → W. The left egg is U, the middle egg is V, and we only show the zero point of the right egg W. We are interested in h = 1,2,3... b (b = min poly factor exponent). Our picture is drawn for b = 3, and we show the three h values h = 1,2 and 3. We are always interested on the left in Ker(p^h). We know that dim Ker(p^b) = a, the algemult. And we know that dim Ker (p) = γ, the geomult. We know from Matthews page 58 that the nullspaces in the left egg are nested as shown, without equality. In other words, we know that Ker(p^3) ⊃ Ker(p^2) ⊃ Ker(p). So this sequence of kernel spaces is getting smaller, while the corresponding sequence of image spaces in the V egg is getting larger. Think of nullity+rank = constant as the explanation for this fact. As nullity goes down (smaller spaces on the left), the rank goes up (size of domain on the right).

As an example of what this picture shows, consider a point in Ker(p^3) like the one shown as • . We operate on this point with $p^2(T)$ and we end up with a point in the space we have labeled Im(p^2). All of this region is then going to map under the final p operator into 0 in the W egg. Now it may be that some points in K(p^3) map under p^2 directly into the 0 in the V egg, but OK, the final p mapping will still take us to 0 in the W egg. All points in K(p^3) under the combined mapping p^2 then p must end up at 0 in W. But they end up in an intermediate zone in V which we have called Im(p^2).

Now we have a bit of confusion. When we say Im(p^2) here and in the picture, we really mean the image or range of the operator Matthews calls Q_0 , not the operator Q. Q_0 is domain-restricted to one of the kernels on the left. For example, we might have Q = p^2 with domain being all of U, but Q_0 = p^2 with domain being only Ker(p^3). If we did not do this restriction, the image areas in the V egg would be larger, and in fact they would probably have regions which lie outside the largest circle in the V egg which is in fact Ker(p). But I only care about these "restricted" image regions, because I only care about starting off in one of the kernel regions on the far left.

Because my images in the V egg are for the "restricted" operators, I think we can identify each of these regions with one of Matthew's $N_{h,p}$ spaces. Normally we are supposed to think of an N-space as the intersection (what I called a cross-hatched region elsewhere) of an Image area with Ker(p) in the V egg, but with this restricted meaning of Image, the nested set of images all lie inside Ker(p) as shown, so we don't have to think about intersection. The outermost region in the V egg is $N_{1,p}$ = Ker(p). [At least I think this is all correct, I could be wrong.]

So to summarize: on the left we have a set of nullspaces which are *decreasing* in size, and on the right we have a set of N-spaces that are *increasing* in size and which are maxing out at $N_{1,p}$ = ker(p). When we say "increasing in size", we mean that we are adding one or more new basis vectors. For example, suppose $N_{3,p}$ has 1 basis vector and $N_{2,p}$ has 2 and $N_{1,p}$ has 3. In Matthews notation, this means $\upsilon_{3,p}$ = 1 and $\upsilon_{2,p}$ = 2 and $\upsilon_{1,p}$ = 3. This is in fact how we get the Matthew's dot diagram shown in our figure. As you move outward to each larger N space, Matthews suggests that you "re use" the basis vectorsyou already have and add new ones as needed to this set. This is his notion of "extending" the basis.

Now let's re-examine Matthew's example on page 73 which in fact has the dot diagram shown, and we will use his other numbers in this discussion as well. We are suppose to "start off" at the top of the dot diagram, which is to say, we start thinking about the smallest nested N space which is $N_{3,p}$. We want to find basis vectors for this space. His suggestion for doing this is as follows:

(1) *first*, find basis vectors over on the left for space Ker(p^3). In his example, dim Ker p^3 = 6 = a, so there are in fact 6 basis vectors. To mechanically do this of course requires significant work, but we

would have Maple do that work for us. We would compute that matrix $p^3(A) = (A - \lambda I)^3$ [it happens that $\lambda = 0$ in his example] , and we would then ask Maple to find the eigenvectors of this matrix. In his example, this is a single eigenmanifold with $\lambda = 0$, and there really will be 6 distinct eigenvectors since we know that dim Ker $p^3 = 6$. Matthews calls these eigenvectors X_1 through X_6.

(2) *second*, Matthews suggests that you apply p^2 to all the basis vectors that you found for Ker p^3 and in this way you will generate a set of vectors which "span" $N_{3,p}$. He uses a certain bracket $\langle$ ... $\rangle$ notation to indicate a set of spanning vectors, so here he would say $N_{3,p} = \langle p^2X_1, p^2X_2, p^2X_3, ... p^2X_6\rangle$ and this notation means "the space $N_{3,p}$ is spanned by the vectors .... ". Of course we know that $\upsilon_{3,p} = 1$, so that tells us that this set of spanning vectors contains only one LI vector! Perhaps several of them are 0 (under the action of p^2), and then any remaining ones must be "multiples" of each other. Could we get $p^2X_2 = 0$, for example? Sure! This would just mean that the basis vector X_2 of K p^3 is also a basis vector of K p^2. You could imagine creating a whole world of basis vectors over in the left egg by a similar extension idea. In any event, we know that the set shown above will have a viable basis vector for $N_{3,p}$. Let's assume it is $p^2 u_1$. Then we can simplify to say that $N_{3,p} = \langle p^2 u_1\rangle$.

Matthews then moves on to think about basis vectors for the next larger space $N_{2,p}$. He re-uses the one we just found, which we have called $p^2 u_1$. He then computes a set of basis vectors (BV's) for K p^2 (and we know there are 5 of these), and then says $N_{2,p} = \langle p^2 u_1 , pY_1, pY_2, pY_3, ... pY_5\rangle$ where he prepends our already-found BV. we know that $\upsilon_{2,p} = 2$, so one of the pY_i guys will be our pu_2 second BV, and we will then say $N_{2,p} = \langle p^2 u_1, pu_2\rangle$. Notice that when we worked with K p^3, the mapping operators were p^2. And now when we are dealing with K p^2, the mapping operators are p.

We then go the last step and we end up with $N_{1,p} = \langle p^2 u_1, p^1u_2 , p^0 u_3\rangle$, and this set of BV's works for our Ker(p) space which has $\upsilon_{1,p} = 3$ = geomult.

Now we ask: WHY does Matthews suggest we keep re-using previously computed BV's and adding new ones to extend as needed? After all, we could do it some other way. The reason is this: Look at our result $N_{1,p} = \langle p^2 u_1, p^1u_2 , p^0 u_3\rangle$. The first BV is not only a BV for Ker(p), but it is also the "maximal" BV of a cyclic subspace C that is associated with the Jordan block. The BV's for this C space are $\langle p^0 u_1, p^1u_1 , p^2 u_1\rangle$. This C space is generated by vector u_1 and has the min poly $p^2 = p^{3 - 1} = p^{e - 1}$. It has dimension e = 3, and it is associated with the first column of the dot diagram. So to restate: this BV has two roles to play. It is a BV for Ker(p), and it is a BV for J_3. Similarly, p^1u_2 is a BV for Ker(p), and is also a BV for J_2.

So the plan in the above example is to find viable vectors u_1, u_2, and u_3 in U and from these we can compute both the BV's for Ker(p) as well as the matrix P which gets us to Jordan form. The "other" BV's for the C spaces that are not in our Ker(p) BV list are called "generalized eigenvectors" and we can see that there is no question as to their existence. In Matthew's scheme, you can trivially compute them by applying powers of p to the starting u_i . It is all pretty automated!