**PhL on Jordan Form** PhL 1.4.05

Several authors have contributed bits and pieces to my understanding of this subject, but no author has given a clean linear presentation, so I will here attempt my own integration.

**1. The Jordan Block**.

Consider a Jordan Block of dimension 3 x 3,

$$J = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = C + \lambda_1 I$$

We have isolated a matrix called C here. As shown on page 33 of Matthews (apart from transpose), this is the companion matrix for the minimum polynomial $f(x) = x^3$ . That is, we have that $C^3 = 0$. As you keep raising the power of C, you keep shifting the row of 1's to the upper right until it finally shifts completely away and you have a completely zero matrix left. Here is a more detailed example with k = 4

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad C^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad C^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad C^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To see why this is happening, note that you can write $C_{jk} = \delta_{j+1,k}$ to represent the initial diagonal of ones in the first off diagonal. Then we have

$$(C^2)_{jk} = \Sigma_m \; C_{jm} C_{mk} = \Sigma_m \; \delta_{j+1,m} \delta_{m+1,k} = \delta_{j+2,k}$$

$$(C^3)_{jk} = \Sigma_m \; C^2_{jm} C_{mk} = \Sigma_m \; \delta_{j+2,m} \delta_{m+1,k} = \delta_{j+3,k} \qquad \text{and so on}$$

Recall the way you are supposed to find the min poly of a matrix: keep raising the power until you get a linear combination of lower powers. Normally when you do this with some arbitrary matrix C, you get something like this at some point,

$$C^k = -a_0 C^0 - a_1 C^1 + ... - a_{k-1} C^{k-1}$$

with some coefficients $a_i$ . We can rewrite this as (think $a_k = 1$)

$$a_0 C^0 + a_1 C^1 + ... + a_{k-1} C^{k-1} + C^k = 0$$

or

$$f(C) = 0 \qquad \text{where } f(x) = x^k + a_{k-1} x^{k-1} + .... + a_0$$

and then f(x) is the min poly for matrix C. In our special case, we get this:

$C^k = 0$ $\quad$ $k = 4$ $\quad$ $f(x) = x^k$

Notice that the lower powers C, $C^2$, $C^3$ cannot be expressed as lin coms of previous powers, because the ones are in the wrong place. That is WHY the min poly is $C^k$ .

So far, then, we have shown this fact:

**Theorem 1**: The min poly $m_C(\lambda)$ of the k x k matrix $(J - \lambda_1 I) = C$ is given by $m_C(\lambda) = \lambda^k$, and $m_C(C) = C^k = 0$.

**Theorem 2:** The min poly $m_J(\lambda)$ of the k x k matrix J is given by $(\lambda - \lambda_1)^k$.

Proof: We know that min poly of C is $m(C) = C^k$ which we can write as $(J - \lambda_1 I)^k$. We can then expand this thing to get $J^k - \lambda_1 J^{k-1} + ... = 0$, and this then must be the min poly for the matrix J. Any *lower* power of J (below $J^k$) cannot be expressed as a lincom of lower powers. So $m_J(J) = (J - \lambda_1 I)^k$. But now write this in terms of a scalar argument $\lambda$ to get $m_J(\lambda) = (\lambda - \lambda_1)^k$ .

**Theorem 3**: The char poly $ch_J(x)$ of the k x k matrix J is given by $ch_J(\lambda) = (\lambda - \lambda_1)^k$, so we can regard k as the algebraic multiplicity of the eigenvalue $\lambda_1$ of the matrix J.

Proof: $ch_J(\lambda) = \det( J - \lambda I) = (\lambda - \lambda_1)^k$ , just manually do the det, the off diagonal 1's have no effect.

**Comment**: The Jordan Block is constructed just to get this very result: the min poly = the char poly. Usually the min poly has a lower exponent, but not here!

**Theorem 4:** The matrix J has only one eigenvector (x,0,0,0...) and thus has geometric multiplicity = 1.

Proof: The eigenvector problem is this: $Jv = \lambda_1 v$, which is the same as $Cv = 0$ which tells us that $v_{i+1} = 0$ for i = 0,1,2... Thus, our eigenvector is $(v_0, 0, 0, 0....)$. We can regard this as the one eigenvector of the nullspace of matrix $C = (J - \lambda_1 I)$. The nullity of this space is then 1.

Conclusions reached concerning the Jordan block J of dimension k with diagonal elements $\lambda_1$ :

(1) J has char poly = min poly = $(\lambda - \lambda_1)^k$

(2) J has one eigenvalue $\lambda_1$ and that eigenvalue has an algebraic multiplicity of k

(3) J has a geometric multiplicity of 1 and its only eigenvector is $(v_0, 0, 0, 0....)$

**2. Direct Sum of Jordan Blocks with same eigenvalue $\lambda_1$**

Suppose we now construct a larger matrix (dim k = $k_1$+ $k_2$) by doing a direct sum of 2 Jordan blocks, both of which have the SAME eigenvalue $\lambda_1$ :

$$A = J_{k1} \oplus J_{k2}$$

The first block has min poly $(\lambda - \lambda_1)^{k1}$, the second has $(\lambda - \lambda_1)^{k2}$. What is the min poly for the larger matrix? The result is derived on page 36 of Matthews. If we let C = A⊕B, then he shows that

$m_C = \text{lcm}(m_A, m_B)$ ' lcm = least common multiple'

We need to pause to prove this important result:

**Theorem 5:** If C = A⊕B, then $m_C = \text{lcm}(m_A, m_B)$.

Proof: Assume that $m_C$ is some polynomial f(x). Then we know that:

f(C) = 0 = f(A⊕B) = f(A) ⊕ f(B) which => f(A) = 0 and f(B) = 0.

This says that $m_A$ | f *and* $m_B$ | f . It says $m_A$ | f, for example, because $m_A$ is the lowest degree poly that satisfies $m_A(A) = 0$, so if f(A) is some *other* poly with f(A) = 0, then $m_A$ must divide into f. Now, we want to know the *lowest degree poly* f(x) that can be divided by *both* $m_A(x)$ and $m_B(x)$. This is exactly what $\text{lcm}(m_A(x), m_B(x))$ *is*. And if the f(x) so found is in fact the poly of lowest degree, then it must be the min poly of C. There can be no poly of lower degree that satisfies the requirement $m_A$ | f *and* $m_B$ | f.

Now we apply this theorem to our direct product situation. We have $m_A = (\lambda - \lambda_1)^{k1}$ and $m_B = (\lambda - \lambda_1)^{k2}$. The answer is that $\text{lcm}(m_A, m_B) = (\lambda - \lambda_1)^{MAX(k1,k2)}$. For example, if we have $m_A = (\lambda - \lambda_1)^2$ and we have $m_B = (\lambda - \lambda_1)^3$, then $(\lambda - \lambda_1)^3$ can be divided by both $m_A$ and $m_B$. So we can easily extend this idea to show the following

**Theorem 6:** Suppose we construct a matrix by combining multiple Jordan Blocks of various sizes, but which all have the same eigenvalue $\lambda_1$ ,

$$A = J_{k1} \oplus J_{k2} \oplus J_{k3} \oplus ....$$

Then the min poly for A is given by $m_A = (\lambda - \lambda_1)^{MAX(k1,k2,\ k3.....)}$ . This conclusion is reinforced on page 37 of Matthews.

**Comment**: If you are staring at the matrix A just described, you can easily see which submatrix has the largest dimension, call it $k_{max} = MAX(k_1, k_2, ....)$. Later, when we combine matrices $A_i$ for different eigenvalues $\lambda_i$ to make a still larger matrix B, and when we talk about the overall min poly for this larger matrix B, it will have a factor $(\lambda - \lambda_1)^{kmax}$ for our particular submatrix A. *So, the size of largest Jordan block in each λ manifold equals the exponent of the overall min poly for the matrix A.*

**Theorem 7:** For the same matrix A shown in Theorem 6, the char poly is given by $(\lambda - \lambda_1)^{k1 + k2 + ...}$ .

Proof: When we have *any* block diagonal form, we know that we can multiply the sub-determinants to get the total determinant, so in this case we can multiply the char polys of the sub-matrices, and that then leads to a sum of exponents as shown.

### 3. Primary Decomposition

Suppose we are given some matrix T and we have computed its char poly and min poly to be:

$$ch_T(\lambda) = (\lambda - \lambda_1)^{a1} (\lambda - \lambda_2)^{a2}$$

$$m_T(\lambda) = (\lambda - \lambda_1)^{b1} (\lambda - \lambda_2)^{b2}$$

where we know that $a_i \geq b_i$ because we know that $m_T(\lambda) \mid ch_T(\lambda)$. Then:

**Theorem 8:** According to the Primary Decomposition Theorem on page 61 of Matthews, we can decompose our vector space V, in which this matrix T acts, into a pair of subspaces,

$$V = V_1 \oplus V_2$$

where

$V_1 = N [ (T - \lambda_1 I)^{b1}]$ with $\upsilon = a_1$ N[X] means "Nullspace of matrix X "
$V_2 = N [ (T - \lambda_2 I)^{b2}]$ with $\upsilon = a_2$

where the exponent $b_i$ is the $k_{max}$ mentioned earlier for each subspace i = 1 and 2. So $b_i$ will be the size of the largest Jordan block in each of the two subspaces. Remember: $b_i$ is the min poly exponent. The $a_i$ are the algebraic multiplicities of each subspace.

Nothing is said here about geometric multiplicity, but we know that geo mult is the number of Jordan blocks in each subspace. We also know these facts:

$N [ (T - \lambda_1 I)^1]$ has $\upsilon$ = geo mult 1
$N [ (T - \lambda_2 I)^1]$ has $\upsilon$ = geo mult 2

These nullspaces are the "eigenmanifolds" of Stakgold. Here we have unity exponents.

### 4. The Cyclic Subspace Business

Following Matthews page 54, we imagine some **"vector specific" min poly $m_{T,v}(T)v = 0$** rather than the more general min poly $m_T(T) = 0$. This specific thing has to divide the normal min poly.

**Theorem 9:** If f is any polynomial and v is some vector in V (the space in which T acts), then the set of all vectors of the form f(T)v forms a *subspace* of V. This is called the **T-cyclic subspace generated by the vector v**, and this subspace is a **T-invariant subspace**, and it is represented as $\mathbf{C}_{T,v}$.

Proof: I think the subspace part follows from usual linearity stuff. That is, $f_1(T)$ and $f_2(T)$ make a sum that is in the same space, so it is closed under addition, etc. Now suppose w is in $C_{T,v}$. Then $w = f(T)v$ for some polynomial f. Then $Tw = Tf(T)v = g(T)v$ which is also in the space since g is just some other polynomial.

**Theorem 10:** In the situation described in Theorem 9, assume that for vector v we have some vector-specific min poly $m_{T,v}(T)$ of degree k. Recall that v is the generator vector of the subspace $C_{T,v}$. Then the following vectors form a basis in $C_{T,v}$: $\{ v, Tv, T^2v, .... T^{k-1}v \}$ This is **the T-cyclic basis**.

This theorem has a proof on page 55,56 of Matthews which I have not read.

**Theorem 11:** Again in the situation of the previous two theorems, if we work in the T-cyclic basis, then the matrix T is none other than the companion matrix of the min poly $m_{T,v}$.

This is shown on page 56 Matthews.

**Application of the Previous Theorems.** Suppose $m_{T,v}(\lambda) = \lambda^k$. This is a poly of degree k, and we can form the T-cyclic basis of theorem 10: $C_{T,v}$: $\{ v, T'v, T'^2v, .... T'^{k-1}v \}$ where we use T' everywhere. Moreover, the matrix T' in this cyclic basis is given (according to theorem 11) by the companion matrix which corresponds to $m_{T',v}(\lambda) = \lambda^k$, and we know from earlier work that this matrix is exactly the matrix C we discussed in section 1 above, the matrix with all zeros except for the first off-diagonal of ones.

Now if we define T as the shift $T' = (T - \lambda_1 I)$, we can do what we did in section 1 above and claim that $m_{T,v}(\lambda) = (\lambda - \lambda_1)^k$ is the v-specific min poly for operator T. Furthermore, the T-cyclic basis is now

$$C_{T,v}: \{ v, (T - \lambda_1 I)v, (T - \lambda_1 I)^2 v, .... (T - \lambda_1 I)^{k-1} v \}$$

The matrix $T = T' + \lambda_1 I$ is then none other than J, the k x k Jordon Block matrix with eigenvalue $\lambda_1$.

**Comments**: So, we have now learned that the Jordan Block arises when you take as your basis vectors the strange set $\{ v, (T - \lambda_1 I)v, (T - \lambda_1 I)^2 v, .... (T - \lambda_1 I)^{k-1} v \}$.

**Status**: At this stopping point, we have "come down from above" with the primary decomposition to get a set of subspaces $V_1$ and $V_2$ , and we have "come up from below" with some strange cyclic subspace gizmo to create one Jordan block. It remains to connect these two worlds.

**Status 1/7/05**. Things are now pretty well connected, I think I am done with this subject for now, after spending quite a few days on it. Results are all in a binder and in a directory.