PlanetMath's Proof of Jordan PhL 1.7.05

Gumau's Proof of Jordan Decomposition

This guy is Guido Mauas, a Student of Mathematics, Buenos Aires, Argentina. He "owns" the little chunk of the PlanetMath "encyclopedia". I now regard this just as a summary of the detailed work that Matthews has done.

Guido uses α_i as the b_i, the exponents of the min poly. The primary decomp result is then clearly stated. He then talks about a "restriction" of operator T to ker p^b, just the way that Matthew's did (and that I do in my later document on this subject with Visio drawing).

He then looks at p: ker $p^b \rightarrow$ ker p^b, which is a little different from what I have done, but OK. We look for a basis for Ker p^b, the space which has algemult = a. The basis of K p^b is written as the union of the bases of the Jordan cyclic subspaces. Each of these is given as $B_{s,i}$ where i = overall eigenmanifold index, and s = the secondary cyclic subspace label. He writes $B_{s,i}$ as powers of p acting on some assumed generating vector which he calls $v_{s,i}$. The union of these is called just B_i.

Now finally we come to the last section of this 1 pager proof. Author wants to show directly that, given the cyclic basis above, we can derive that the Jordan Block looks like. I will do this in my own language:

(1) Define $p = (T-\lambda I)$ as usual. Write $p^{j+1} = p\ p^j = (T - \lambda)p^j$. Thus, $T\ p^j = p^{j+1} + \lambda\ p^j$.

(2) We now use this result to evaluate our matrix elements:

$$J_{kj} = \langle p^k v \mid T \mid p^j v \rangle = \langle p^k v \mid T\ p^j v \rangle = \langle p^k v \mid (p^{j+1} + \lambda\ p^j)\ v \rangle = \delta_{k,j+1} + \lambda\ \delta_{k,j}$$

$$= C_{jk} + \lambda\ I_{kj} = (C^T)_{kj} + \lambda\ (I)_{kj} \quad \text{which says} \quad J = C^T + \lambda\ I.$$

where I am referring back to my PhL notes on the companion C matrix. The only difference is that we have C instead of C^T. So basically our author has derived the shape of the J block.

Question : I have had to assume that $\langle p^k v \mid p^j v \rangle = \delta_{k,j}$. How can I verify this?

In a cyclic subspace as generated by vector v we have this basis: { v, pv, p^2v, p^3v ... }. This subject is discussed (thank goodness) on page 55 of Matthews.

Well, this is quite interesting. In the entire Matthews presentation, we deal only with vector spaces and the associated notions of spanning and linear independence. There is no norm, no metric, no scalar product! He has no need for such things. On page 56 when we want to know what the matrix looks like for our operator T, we find the columns of the matrix simply by applying the operator to the vectors of the basis. In this manner, he finds the companion matrix C that matches the min pol of vector v.

So let's go back to our derivation above to the point where we have:

$T\ p^j v = p^{j+1} v + \lambda\ p^j v.$ $\qquad p = (T-\lambda I)$

We now find the columns of our matrix T by applying T to the basis vectors one at a time. We make this ***association*** between basis vectors and column vectors:

$$p^0 v \rightarrow [1,0,0,0]^t$$
$$p^1 v \rightarrow [0,1,0,0]^t$$

Then $T(p^0 v) = p^1 v + \lambda\, p^0 v = [0,1,0,0]^t + [\lambda,0,0,0]^t = [\lambda, 1, 0, 0]^t$

Now once we make the above association, then yes, we can say that under the scalar product induced by the associated E^n, our basis vectors are orthonormal, and then we do have $< p^k v \mid p^j v>_{IND} = \delta_{k,j}$. But this is not the obvious scalar product that you would get directly from E^n

$$< p^k v \mid p^j v>_{IND} \neq [p^k v]^T [p^j v] = v^T (p^k)^T p^j v = \text{some huge mess and not just } \delta_{j,k}$$

An analogous situation would be to imagine two unit vectors in E^2 that are not orthogonal. The direct E^2 scalar product would show non-orthogonality, but my "induced" scalar product would show orthog. We are just associating each "position" in the column vector with a basis vector. I don't know what the official term for doing this is, I have used the word induced, maybe associated is better.