**This was installed 10.22.16, do not edit here!**

**Appendix A: Matrix rank equals the number of independent columns and rows ..... 1**

**Appendix B: Determinants ..... 6**

B.1 Definition of the determinant ..... 6

B.2 The permutation group, the permutation tensor ε, and expansions for det(M) ..... 8

B.3 Minors, Cofactors, and the Cofactor Expansions of det(M) ..... 14

B.4 Expressions for the inverse matrix $M^{-1}$ and Cramer's Rule ..... 16

**Appendix C: The Intersection of Constraint Surfaces ..... 19**

**Appendix D: Lagrange Multipliers in Classical Mechanics ..... 24**

**References ..... 40**

## Appendix A: Matrix rank equals the number of independent columns and rows

This Appendix proves our claim of (6.1.7) that, for a general m x n matrix M, the number of linearly independent columns and the number of linearly independent rows are the same, and both numbers are equal to the rank of the matrix. The reader will appreciate that this fact is non-obvious. Shilov proves it but the proof is a bit spread out over several sections. Here we provide a self-contained alternate proof.

We first quote a series of square-matrix theorems that are proved from scratch in our Appendix B. These are doubtless very familiar to the reader, but we just want to get them stated. The derivations of these theorems may be less familiar.

**Theorem 1**: $\det(M^{\mathbf{T}}) = \det(M)$. Switching rows with columns does not change a determinant. *(B.1.10)*

**Theorem 2:** det(M) can be represented in these two ways, where ε is the permutation tensor (B.2.8) :

$$\det(M) = \Sigma_{a_1 a_2 \cdots a_n} \varepsilon_{a_1 a_2 \cdots a_n} M_{1a_1} M_{2a_2} \cdots\cdots M_{na_n} \qquad (B.2.10a)$$

$$\det(M) = \Sigma_{a_1 a_2 \cdots a_n} \varepsilon_{a_1 a_2 \cdots a_n} M_{a_1 1} M_{a_2 2} \cdots\cdots M_{a_n n} \ . \qquad (B.2.10b)$$

**Theorem 3:** For a square matrix M, adding a multiple of one row to another does not change det(M). The same is true for adding a multiple of one column to another. *(B.2.12)*

**Theorem 4:** Swapping two rows (columns) of square matrix M causes $\det(M) \rightarrow -\det(M)$. *(B.2.13)*

**Corollary 4:** If two rows (columns) of square matrix M are the same, then det(M) = 0. *(B.2.14)*

**Theorem 5**: det(AB) = det(A)det(B) *(B.2.15)*

**Corollary 5**: det(ABC) = det(A)det(B)det(C) and so on. *(B.2.16)*

**Theorem 6** (Cofactor Expansions):

$\det(M) = \Sigma_n M_{sn}\, \text{cof}(M_{sn}) = \Sigma_n (\mathbf{r}_s)_n \text{cof}(M_{sn})$ work across row s s = 1,2....n *(B.3.16a)*

$\det(M) = \Sigma_n M_{ns}\, \text{cof}(M_{ns}) = \Sigma_n (\mathbf{c}_s)_n \text{cof}(M_{ns})$ work down column s s = 1,2....n *(B.3.16b)*

**Theorem 7:** $M^{-1} = \dfrac{C^T}{\det(M)} = \dfrac{[\text{cof}(M)]^T}{\det(M)} = \dfrac{\text{cof}(M^T)}{\det(M)}$ for square matrix M. *(B.4.8)*

**Theorem 8** (Cramer's Rule):

$$\mathbf{y} = M\mathbf{x} \quad \Rightarrow \quad x_s = \frac{\det(M[\mathbf{c}_s \rightarrow \mathbf{y}])}{\det(M)} \qquad (B.4.14)$$

To this list, we shall now add four new Theorems which will be proven right here. The definition of **linear independence** and **linear dependence** is given in the discussion surrounding (6.1.6) and won't be repeated here.

**Theorem 9:** Columns of square matrix M are linearly dependent $\Leftrightarrow$ det(M) = 0 (A.1)
Contrapositives: Columns of square matrix M are linearly independent $\Leftrightarrow$ det(M) $\neq$ 0

If we can prove Theorem 9 as stated, then the theorem is also true for rows. The reason is that swapping rows and columns corresponds to $M \leftrightarrow M^T$ and **Theorem 1** says $\det(M^T) = \det(M)$.

Proof of $\Rightarrow$: According to (6.1.6) and nearby discussion, our premise of linear dependence says that *either* one or more columns of M are zero, *or* at least one column can be written as a linear combination of the other columns, so perhaps $\mathbf{c}_j = \Sigma_{i\neq j} k_i \mathbf{c}_i$ . If a column is zero, det(M) = 0 from **Theorem 6** going down that column. If $\mathbf{c}_j = \Sigma_{i\neq j} k_i \mathbf{c}_i$, one can add $-\Sigma_{i\neq j} k_i \mathbf{c}_i$ to $\mathbf{c}_j$ without changing det(M) according to **Theorem 3**. But this makes the new column j vanish, so again det(M) = 0.

Proof of $\Leftarrow$: Here we shall prove the contrapositive instead of the claim:

claim: det(M) = 0 $\Rightarrow$ the columns of M are linearly dependent
contrapositive: the columns of M are linearly independent $\Rightarrow$ det(M) $\neq$ 0

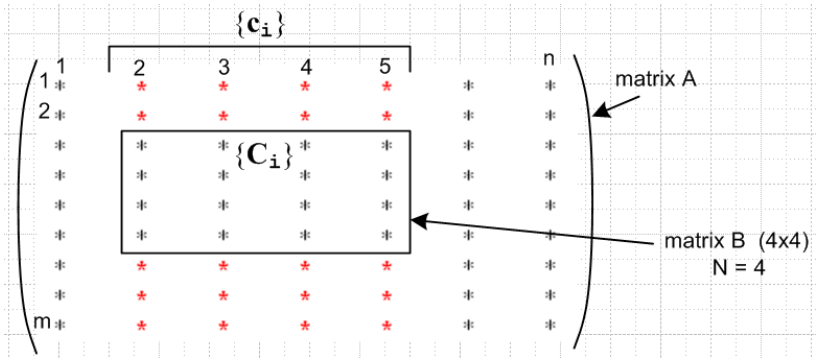
If the column vectors $\mathbf{c}_i$ of matrix M are linearly independent, then (6.1.6) says $\Sigma x_i \mathbf{c}_i = 0 \Rightarrow \mathbf{x} = 0$. Therefore we know that $\Sigma x_i M_{ji} = 0 \Rightarrow x_j = 0$ which is the same as $M\mathbf{x} = 0 \Rightarrow \mathbf{x} = 0$. Thinking of $M: E^n \rightarrow E^n$, we claim that mapping $M\mathbf{x} = \mathbf{y}$ is one-to-one. Certainly each $\mathbf{x}$ goes into a single $\mathbf{y}$, Could a $\mathbf{y}$ map back into two different values of $\mathbf{x}$, call them $\mathbf{x}$ and $\mathbf{x}'$ with $\mathbf{x} \neq \mathbf{x}'$ ? Then $M\mathbf{x} = \mathbf{y}$ and $M\mathbf{x}' = \mathbf{y}$ . Subtract to get $M(\mathbf{x}-\mathbf{x}') = 0$. But since $M\mathbf{z} = 0 \Rightarrow \mathbf{z} = 0$ (as just shown), we find $\mathbf{x} = \mathbf{x}'$ . Thus, the mapping $M\mathbf{x} = \mathbf{y}$ really is one-to-one, and that means it is invertible and the inverse is unique. From **Theorem 7** the inverse is in fact given by $M^{-1} = \text{cof}(M^T)/\det(M)$. For $M^{-1}$ to exist, one must have $\det(M) \neq 0$.

Another proof is very simple but perhaps less convincing. One can show (see our Tensor Analysis document in Refs) that if a set of column vectors $\mathbf{c}_i$ of matrix M spans an n-piped in $E^n$, the "volume" of that n-piped is given by $V = |\det(M)|$. If those vectors are linearly independent, then $V \neq 0$. This is obvious in 2D (volume = area) and 3D but less obvious for general $E^n$. For example, in 3D if a third vector lies in the plane of the other two (is dependent), there is no volume.

**Theorem 10** (mini-columns theorem). Let R be a subset of N rows of nxm matrix A, and let S be a subset of columns $\{\mathbf{c_i}\}$ in A. The matrix elements of A included in the intersection of these two sets form a set of "mini-columns" $\{\mathbf{C_i}\}$. This set of mini-columns forms a matrix B within A. Matrix B could be contiguous, or it could be non-contiguous in one or both directions. The claim and its contrapositive are:

(a) $\{\mathbf{C_i}\}$ linearly independent in B $\Rightarrow$ $\{\mathbf{c_i}\}$ linearly independent in A
(b) $\{\mathbf{c_i}\}$ linearly dependent in A $\Rightarrow$ $\{\mathbf{C_i}\}$ linearly dependent in B (A.2)

Below we shall prove (b) which then implies (a). A picture is worth a thousand words. Here matrix B happens to be contiguous in both directions :




(A.3)

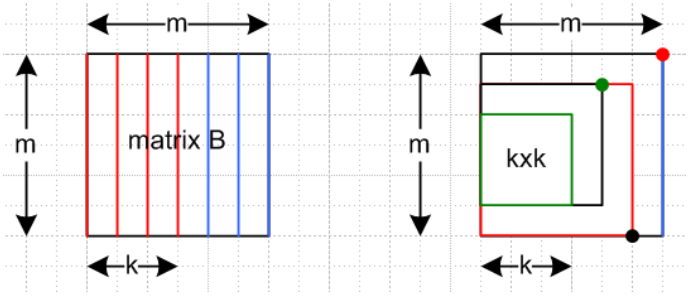
Proof of (b): If the set $\{\mathbf{c_i}\}$ is linearly dependent in A, then from (6.1.6) one can write $\Sigma_{i\in S}\, k_i\mathbf{c_i} = 0$ with $\mathbf{k} \neq 0$. This is a set of m equations which contains as a subset the set of N equations $\Sigma_{i\in S}\, k_i\mathbf{C_i} = 0$. This last equation then says that the $\{\mathbf{C_i}\}$ are linearly dependent in B. There is of course a similar "mini rows theorem".

**Theorem 11**. If all kxk minors in k columns of matrix A vanish, the k columns are linearly dependent. (A.4)

Contrapositive: If k columns are linearly independent, they must contain at least one non-zero kxk minor.

Once proven for columns, replacing $A \rightarrow A^{\mathbf{T}}$ gives the same theorem for rows.

Proof: Assume matrix A has m rows and n columns. In order for kxk minors to exist, one must have $k \leq \min(m,n)$. Gather up the k columns and make them be the leftmost k columns of a new matrix B. Since these columns have m elements, and since $k \leq m$, add m-k *arbitrary* new columns to the right of the k columns so that matrix B is then a square matrix. On the left below we show the k columns taken from matrix A in red, and then the arbitrary added columns are shown in blue.




(A.5)

Consider now the process of computing det(B) by going down the rightmost column using the standard cofactor sum formula of **Theorem 6**. This det(B) is a linear combination of (m-1)x(m-1) minors all in the left m-1 columns. Consider one of these minors. If we evaluate *it* using the same cofactor formula (of one less dimension), it will be a linear combination of (m-2)x(m-2) minors in the left m-2 columns. We keep going until we are evaluating a set of kxk cofactors in the left k columns. But these all vanish by the theorem premise. Thus, reversing this logic we conclude that det(B) = 0. The picture on the right above shows one minor at each level of the descent just described (red dot goes with red minor, etc).

Now suppose it were possible that the k red columns were independent. Since the added blue columns are arbitrary, we could certainly find a set of added blue columns so that all m columns were independent. But then we would have det(B) ≠ 0 by **Theorem 9**. But we just showed that det(B) = 0, so it must not be possible to have the k red columns be independent, so they must be dependent.

We are finally ready for the Main Act. Recall first that :

**Definition**: The **rank** r of an m x n matrix A is the dimension of the largest non-vanishing minor within A. [Shilov 1.92 ] Thus, $r \leq \min(m,n)$. *(6.1.5)* (A.6)

**Theorem 12**. For a general nxm matrix A, (A.7)

rank(A) = r ⇔ A has exactly r linearly independent columns

Once this theorem is proved for columns, it is also true for rows since $\text{rank}(A^T) = \text{rank}(A)$ by (A.6).

Proof of ⇒: If rank(A) = r, A must have at least one non-vanishing rxr minor. Think of this minor being a set of mini-columns as in Theorem 10. Since then det(minor) ≠ 0, this set of mini-columns is linearly independent according to **Theorem 9**. Thus, the set of corresponding full columns (those that pass down through the minor) are linearly independent from **Theorem 10** (a). So matrix A has at least these r independent columns.

Now let k = r+1. Since rank(A) = r we know that all kxk minors in A vanish. By **Theorem 11**, any set of k columns must be linearly dependent. Since k = r+1, *any* r+1 columns are linearly dependent, so the number of independent columns is just r, the number of columns passing down through the minor.

Proof of ⇐: If there are r linearly independent columns, then at least one rxr minor within those columns must be non-zero (contrapositive of **Theorem 11**). Suppose there were an (r+k)x(r+k) non-vanishing minor. Then the r+k mini-columns of that minor would be independent, and thus by **Theorem 10** there would be r+k independent full columns. But the theorem premise says there are only r independent columns, so all minors larger than rxr must vanish. Therefore rank(A) = r.

Sometimes the number of linearly independent rows of a matrix is called the **row rank**, and the number of independent columns is called the **column rank**. Theorem 12 and its row version show that

column rank = row rank = rank = dimension of largest non-vanishing minor (A.8)

Basis Minors and Basis Columns. If rank(A) = r, we know there must exist at least one non-vanishing rxr minor in A. Shilov refers to such a minor as a **basis minor** and the columns passing through this minor are called **basis columns**. We showed in **Theorem 10** that the set of such basis columns is linearly independent (which is why they are called basis columns). Obviously any of these columns can be written as a linear combination of the basis columns, such as $\mathbf{c}_2 = \Sigma_{i=1}^{k} k_i \mathbf{c}_i = 1\ \mathbf{c}_2$. If k = r+1, all kxk minors vanish and by **Theorem 11** all sets of k = r+1 columns are linearly dependent. Thus, every non-basis column is a linear combination of the basis columns. We have thus proved Shilov's Basis Minor Theorem which we quote from section 1.93 of his book (p 25)

THEOREM (***Basis minor theorem***). *Any column of the matrix A is a linear combination of its basis columns.* (A.9)

In his proof that column rank = rank, Shilov uses the above theorem as a starting point.

**Appendix B: Determinants**

An alterative title for this Appendix might be "More than you really want to know about determinants". We find it convenient to gather these Facts all in one place. Many of these Facts are used in Appendix A where we prove other Facts about matrix rank which in turn are used in Section 6 to support our alternate derivation of the Method of Lagrange Multipliers. It is hoped that the encapsulated presentation below might be useful to the reader for other applications of determinants. Of special interest to us is that permutations form a group which implies certain "rearrangement theorems" which in turn provide quick proofs of many well-known and some lesser-known Facts about determinants.

**B.1 Definition of the determinant**

First, we define $\mathbf{z_0}$ to be the n-component vector of increasing integers 1 to n,

$$\mathbf{z_0} \equiv \begin{pmatrix} 1 \\ 2 \\ \dots \\ n \end{pmatrix} . \tag{B.1.1}$$

Let **a** be some **permutation** (reordering) of these integers, so write

$$\mathbf{a} \equiv \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix} = A \begin{pmatrix} 1 \\ 2 \\ \dots \\ n \end{pmatrix} = A\,\mathbf{z_0} \tag{B.1.2}$$

where A is an nxn matrix which has 1's in the right places to create this permutation vector **a**. For example,

$$\mathbf{a} = A\mathbf{z_o} \qquad \leftrightarrow \qquad \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} .$$

The vector **a** can be obtained from the vector $\mathbf{z_0}$ by making $S_\mathbf{a}$ pairwise swaps of elements of $\mathbf{z_0}$. Although $S_\mathbf{a}$ is not unique, the number $(-1)^{S_\mathbf{a}}$ is unique. For example, to get from (1,2,3) to (1,3,2) one could swap the second pair so $S_\mathbf{a} = 1$, but one could then swap the first pair twice and then $S_\mathbf{a} = 3$. This number $(-1)^{S_\mathbf{a}}$ is of course ±1 and we shall call it the **parity** of the permutation **a**,

$$\text{Parity}(\mathbf{a}) \equiv (-1)^{S_\mathbf{a}} . \tag{B.1.3}$$

**Fact:** $$\text{Parity}(C\mathbf{a}) = \text{Parity}(CA\mathbf{z_0}) = (-1)^{S_\mathbf{c}+S_\mathbf{a}} = \text{Parity}(C^{-1}\mathbf{a}) \tag{B.1.4}$$

Proof: Doing permutation CA involves first doing A with its $S_\mathbf{a}$ swaps, and then doing C with its $S_\mathbf{c}$ swaps, for a total of $S_\mathbf{a}+S_\mathbf{c}$ swaps. The inverse permutation $C^{-1}$ obviously involves the same number of swaps as the permutation C.

Moving now toward the definition of the determinant of a matrix M, define (Π means "product")

$$\Pi(\mathbf{a},\mathbf{b};\,M) \equiv M_{\mathbf{ab}} \equiv M_{\mathbf{a}_1\mathbf{b}_1}M_{\mathbf{a}_2\mathbf{b}_2}\,.....\,M_{\mathbf{a}_n\mathbf{b}_n} \;=\; \text{product of n factors}\;. \qquad \text{(B.1.5)}$$

The notation $\Pi(\mathbf{a},\mathbf{b};\,M)$ is easier to deal with than $M_{\mathbf{ab}}$, but we shall use both these notations.

Suppose $\mathbf{c} = C\mathbf{z}_0$ is some arbitrary permutation of $\mathbf{z}_0$. Then:

**Fact:** $\Pi(C\mathbf{a},C\mathbf{b};\,M) = \Pi(\mathbf{a},\mathbf{b};\,M)$ (B.1.6)

Proof: Applying the same permutation to both the $a_k$ and $b_k$ indices of $M_{\mathbf{a}_1\mathbf{b}_1}M_{\mathbf{a}_2\mathbf{b}_2}\,.....\,M_{\mathbf{a}_n\mathbf{b}_n}$ merely reorders the terms in the product but the product stays the same.

**Fact**: $\Pi(\mathbf{a},\mathbf{b};\,M^{\mathbf{T}}) = \Pi(\mathbf{b},\mathbf{a};\,M)$ (B.1.7)

Proof: $\Pi(\mathbf{a},\mathbf{b};\,M^{\mathbf{T}}) = M^{\mathbf{T}}_{\mathbf{a}_1\mathbf{b}_1}M^{\mathbf{T}}_{\mathbf{a}_2\mathbf{b}_2}\,....\,M^{\mathbf{T}}_{\mathbf{a}_n\mathbf{b}_n} \;=\; M_{\mathbf{b}_1\mathbf{a}_1}M_{\mathbf{b}_2\mathbf{a}_2}\,....\,M_{\mathbf{b}_n\mathbf{a}_n} = \Pi(\mathbf{b},\mathbf{a};\,M)\;.$

We shall now **define the determinant** of an n x n matrix M in the following admittedly obscure manner (later we will show that it reduces to more familiar forms),

$$\det(M) \equiv \frac{1}{n!}\,\Sigma_{\mathbf{a}}\,\Sigma_{\mathbf{b}}\;(-1)^{S_{\mathbf{a}}+S_{\mathbf{b}}}\;\Pi(\mathbf{a},\mathbf{b};\,M) = \frac{1}{n!}\,\Sigma_{\mathbf{a},\mathbf{b}}\;(-1)^{S_{\mathbf{a}}+S_{\mathbf{b}}}\;M_{\mathbf{ab}}$$

$$= \frac{1}{n!}\,\Sigma_{\mathbf{a}}\,\Sigma_{\mathbf{b}}\,(-1)^{S_{\mathbf{a}}+S_{\mathbf{b}}}\,M_{\mathbf{a}_1\mathbf{b}_1}M_{\mathbf{a}_2\mathbf{b}_2}\,.....\,M_{\mathbf{a}_n\mathbf{b}_n}\;. \qquad \text{(B.1.8)}$$

Here $\Sigma_{\mathbf{a}}$ means the sum over all permutations $\mathbf{a}$ of $\mathbf{z}_0$. In (B.1.8) the columns and rows of M are on a completely equal footing. Notice that $\mathbf{a}$ and $\mathbf{b}$ are in effect dummy summation indices. If we do $\mathbf{a}\leftrightarrow\mathbf{b}$, the expression for det(M) is unchanged. Thus one can rewrite (B.1.8) as

$$\det(M) = \frac{1}{n!}\,\Sigma_{\mathbf{b}}\,\Sigma_{\mathbf{a}}\;(-1)^{S_{\mathbf{b}}+S_{\mathbf{a}}}\;\;\Pi(\mathbf{b},\mathbf{a};\,M)\;. \qquad \text{(B.1.9)}$$

We are now ready for our first determinant theorem:

**Theorem 1**: $\det(M^{\mathbf{T}}) = \det(M)$. Switching rows with columns does not change a determinant. (B.1.10)

Proof:

$$\det(M^{\mathbf{T}}) = \frac{1}{n!}\,\Sigma_{\mathbf{b}}\,\Sigma_{\mathbf{a}}\,(-1)^{S_{\mathbf{b}}+S_{\mathbf{a}}}\;\;\Pi(\mathbf{b},\mathbf{a};\,M^{\mathbf{T}}) \qquad \text{// (B.1.9) applied to } M^{\mathbf{T}}$$

$$= \frac{1}{n!}\,\Sigma_{\mathbf{b}}\,\Sigma_{\mathbf{a}}\,(-1)^{S_{\mathbf{b}}+S_{\mathbf{a}}}\;\;\Pi(\mathbf{a},\mathbf{b};\,M) \qquad \text{// (B.1.7) applied to } M^{\mathbf{T}}$$

$$= \det(M) \qquad \text{// (B.1.8)}$$

**B.2 The permutation group, the permutation tensor ε, and expansions for det(M)**

**Definition.** A set of elements $\{g_i\}$ form a **group** G if :

- $g_i g_j$ is also in the group (closure)
- $g_i^{-1}$ exists for each $g_i$ in G, where $g_i^{-1}$ is also in G (inverse)
- $(g_i g_j)g_k = g_i(g_j g_k)$ (associative) (B.2.1)

Comment: It is implicit in the above definition that the group has some "operation" which gives meaning to $g_i g_j$, which we shall just think of as "multiplication". When group elements are represented by matrices, that operation is multiplication of those matrices, and that will apply to the permutation group below.

**Fact**: $g_i G = G$ (the rearrangement theorem) (B.2.2)

This says that multiplication of all the elements of a group by an element $g_i$ in the group creates a reordering of the group elements. Here G is the set of group elements.

Proof. Consider $g_i G = g_i\ [g_1, g_2....g_n] = [g_i g_1, g_i g_2....g_i g_n]$ = set of n elements. Unless two elements are the same, this must exhaust the entire group. How do we know that $g_i g_1$ and $g_i g_2$ might not be the same? Apply $g_i^{-1}$ from the left and that would say $g_1 = g_2$ which is not the case.

**Fact:** $\Sigma_g f(g) = \Sigma_g f(g_1 g)$ if $\Sigma_g$ runs over the entire group G (B.2.3)

Proof: In $\Sigma_g f(g_1 g)$, as g runs over G, the argument $g' \equiv g_1 g$ runs over G by the above rearrangement theorem. Thus, the sum $\Sigma_g f(g_1 g)$ is just a reordering of the terms in the sum $\Sigma_g f(g)$.

It is easy to show that the set of permutations of $\mathbf{z}_0$ forms a group and therefore the above facts can be used. For example, the product of two permutations is a permutation, and every permutation clearly has an inverse, and (AB)C = A(BC) for any matrices. Fact (B.2.3) can be written

$$\Sigma_B f(B\mathbf{z}_o) = \Sigma_B f(CB\mathbf{z}_0) \qquad // \ \mathbf{b} = B\mathbf{z}_0$$

Here the sum $\Sigma_B$ is over all permutation matrices which correspond to permutations **b** of $\mathbf{z}_0$, and C is some arbitrary permutation matrix associated with some permutation **c** of $\mathbf{z}_0$. An equivalent way of stating the above involves a direct summation $\Sigma_b$ over all permutation vectors **b** of $\mathbf{z}_0$ ,

$$\Sigma_b f(\mathbf{b}) = \Sigma_b f(C\mathbf{b}) . \qquad // \text{ permutation sum rearrangement theorem} \qquad (B.2.4)$$

Throwing in the arbitrary permutation C merely causes a reordering of the sum. This is an extremely powerful and useful fact. Consider then :

$$\det(M) \equiv \frac{1}{n!}\,\Sigma_{\mathbf{a}}\,[\Sigma_{\mathbf{b}}\,(-1)^{S_{\mathbf{a}}+S_{\mathbf{b}}}\;\Pi(\mathbf{a},\mathbf{b};M)\,] \qquad \text{// (B.1.8)}$$

$$= \frac{1}{n!}\,\Sigma_{\mathbf{a}}\;[\Sigma_{\mathbf{b}}\,(-1)^{S_{\mathbf{a}}+S_{\mathbf{b}}}\;\;\Pi(A^{-1}\mathbf{a},A^{-1}\mathbf{b};M)] \qquad \text{// (B.1.6) with } C = A^{-1};\; A^{-1}\mathbf{a} = A^{-1}A\mathbf{z}_0 = \mathbf{z}_0$$

$$= \frac{1}{n!}\,\Sigma_{\mathbf{a}}\,[\,\Sigma_{\mathbf{b}}\;\text{Parity}(A^{-1}\mathbf{b})\;\;\Pi(\mathbf{z}_0,A^{-1}\mathbf{b};M)\,] \qquad \text{// (B.1.4) with } C = A^{-1} \text{ and } \mathbf{a} = \mathbf{b}$$

$$= \frac{1}{n!}\,\Sigma_{\mathbf{a}}\,[\,\Sigma_{\mathbf{b}}\;\text{Parity}(\mathbf{b})\;\;\Pi(\mathbf{z}_0,\mathbf{b};M)\,] \qquad \text{// (B.2.4) with } C = A^{-1} \text{ (key step)}$$

$$= \Sigma_{\mathbf{b}}\;\text{Parity}(\mathbf{b})\;\;\Pi(\mathbf{z}_0,\mathbf{b};M) \qquad \text{// n! identical terms in } \Sigma_{\mathbf{a}}$$

$$= \Sigma_{\mathbf{a}}\;\text{Parity}(\mathbf{a})\;\;\Pi(\mathbf{z}_0,\mathbf{a};M) \qquad \text{// rename dummy sum variable}$$

$$= \Sigma_{\mathbf{a}}\;\text{Parity}(\mathbf{a})\;\;M_{1a_1}M_{2a_2}\,.....\,M_{na_n}\,. \qquad \text{// (B.1.5)} \qquad \text{(B.2.5)}$$

Since $\det(M^{\mathbf{T}}) = \det(M)$ from Theorem 1 (B.1.10), we can also write this result as

$$\det(M) = \Sigma_{\mathbf{a}}\;\text{Parity}(\mathbf{a})\;\;M_{a_1 1}M_{a_2 2}\,.....\,M_{a_n n} \qquad \text{(B.2.6)}$$

Proof: $\det(M) = \det(M^{\mathbf{T}}) = \Sigma_{\mathbf{a}}\;\text{Parity}(\mathbf{a})\;\;M^{\mathbf{T}}_{1a_1}M^{\mathbf{T}}_{2a_2}\,.....\,M^{\mathbf{T}}_{na_n} = \Sigma_{\mathbf{a}}\;\text{Parity}(\mathbf{a})\;\;M_{a_1 1}M_{a_2 2}\,.....\,M_{a_n n}$

Here then are the last two results: ( recall that $S_{\mathbf{a}}$ is the number of pairwise swaps to get from $\mathbf{z}_0$ to $\mathbf{a}$ )

$$\det(M) \equiv \Sigma_{\mathbf{a}}\;\text{Parity}(\mathbf{a})\;\;M_{1a_1}M_{2a_2}\,.....\,M_{na_n} \qquad \text{Parity}(\mathbf{a}) = (-1)^{S_{\mathbf{a}}} \qquad \text{(B.2.7a)}$$

$$\det(M) \equiv \Sigma_{\mathbf{a}}\;\text{Parity}(\mathbf{a})\;\;M_{a_1 1}M_{a_2 2}\,.....\,M_{a_n n} \qquad \text{Parity}(\mathbf{a}) = (-1)^{S_{\mathbf{a}}}\;. \qquad \text{(B.2.7b)}$$

Definition: The **permutation tensor** $\varepsilon_{ijk..}$ (n subscripts) :

- $\varepsilon_{12...n} = 1$
- for any index swap, ε changes sign: $\varepsilon_{..i..j..} = -\varepsilon_{..j..i..}$
- therefore, if any two indices are the same, $\varepsilon = 0$ $\varepsilon_{..i..i..} = -\varepsilon_{..i..i..} = 0$ (B.2.8)

**Fact**: If $\mathbf{a}$ is a permutation of $\mathbf{z}_0$, then $\varepsilon_{a_1a_2\cdots a_n} = \text{Parity}(\mathbf{a}) = (-1)^{S_{\mathbf{a}}} \equiv \varepsilon_{\mathbf{a}}$ . (B.2.9)

Proof: Since indices $a_i$ represent a permutation of $\mathbf{z}_0$ , it takes $S_{\mathbf{a}}$ swaps to get from $\varepsilon_{a_1a_2\cdots a_n}$ to $\varepsilon_{123...n}$ by the definition of ε. In dense notation, one could say $\varepsilon_{\mathbf{a}} = (-1)^{S_{\mathbf{a}}}\,\varepsilon_{\mathbf{z}_0}$ .

Using (B.2.9) in (B.2.7) then gives these two classic det(M) expressions :

**Theorem 2:** det(M) can be represented in these two ways:

$$\det(M) = \Sigma_{a_1a_2\cdots a_n}\,\varepsilon_{a_1a_2\cdots a_n}\,M_{1a_1}M_{2a_2}\,.....\,M_{na_n} \qquad \text{(B.2.10a)}$$

$$\det(M) = \Sigma_{a_1a_2\cdots a_n}\,\varepsilon_{a_1a_2\cdots a_n}\,M_{a_1 1}M_{a_2 2}\,.....\,M_{a_n n}\;. \qquad \text{(B.2.10b)}$$

In dense notation, one could write the above as (**a** is now a vector),

$$\det(M) = \Sigma_{\mathbf{a}} \varepsilon_{\mathbf{a}} M_{\mathbf{z_0 a}} \qquad \text{(B.2.10a)'}$$

$$\det(M) = \Sigma_{\mathbf{a}} \varepsilon_{\mathbf{a}} M_{\mathbf{a z_0}} \; . \qquad \text{(B.2.10b)'}$$

In most applications, the following simpler notation suffices:

$$\det(M) = \Sigma_{\mathbf{abc..q}} \varepsilon_{\mathbf{abc..q}} M_{\mathbf{1a}} M_{\mathbf{2b}} M_{\mathbf{3c}} .... M_{\mathbf{nq}} \qquad \text{(B.2.11a)}$$

$$\det(M) = \Sigma_{\mathbf{abc..q}} \varepsilon_{\mathbf{abc..q}} M_{\mathbf{a1}} M_{\mathbf{b2}} M_{\mathbf{c3}} .... M_{\mathbf{qn}} \; . \qquad \text{(B.2.11b)}$$

**Theorem 3:** For a square matrix M, adding a multiple of one row to another does not change det(M). The same is true for adding a multiple of one column to another. (B.2.12)

Proof: (rows) Suppose we replace $\mathbf{r_3} \rightarrow \mathbf{r_3} + \alpha \mathbf{r_2}$. This says $M_{\mathbf{3i}} \rightarrow M_{\mathbf{3i}} + \alpha M_{\mathbf{2i}}$ . Eq (B.2.11a) says :

$$\det(M') = \Sigma_{\mathbf{abc..q}} \varepsilon_{\mathbf{abc..q}} M_{\mathbf{1a}} M_{\mathbf{2b}} (M_{\mathbf{3c}} + \alpha M_{\mathbf{2c}}) .... M_{\mathbf{nq}}$$

$$= \det(M) + \alpha \Sigma_{\mathbf{abc..q}} \varepsilon_{\mathbf{abc..q}} M_{\mathbf{1a}} M_{\mathbf{2b}} M_{\mathbf{2c}} .... M_{\mathbf{nq}} \; .$$

Since $M_{\mathbf{1a}} M_{\mathbf{2b}} M_{\mathbf{2c}} .... M_{\mathbf{nq}}$ is symmetric under b↔c while $\varepsilon_{\mathbf{abc..q}}$ is anti-symmetric, the extra α term vanishes. That is to say, sum = $\Sigma_{\mathbf{bc}} A_{\mathbf{bc}} S_{\mathbf{bc}} = \Sigma_{\mathbf{cb}} A_{\mathbf{cb}} S_{\mathbf{cb}} = \Sigma_{\mathbf{cb}} (-A_{\mathbf{bc}})(S_{\mathbf{bc}})$ = - sum = 0. Using (B.2.11b) this argument shows that $\mathbf{c_3} \rightarrow \mathbf{c_3} + \alpha \mathbf{c_2}$ similarly does not alter det(M).

**Theorem 4:** Swapping two rows (columns) of square matrix M causes det(M) → - det(M). (B.2.13)

Proof: Let's swap *rows* 1 and 3 in M to get M'. Then from (B.2.11a),

$$\det(M') = \Sigma_{\mathbf{abc..q}} \varepsilon_{\mathbf{abc..q}} M_{\mathbf{3a}} M_{\mathbf{2b}} M_{\mathbf{1c}} .... M_{\mathbf{nq}}$$

$$= \Sigma_{\mathbf{abc..q}} [- \varepsilon_{\mathbf{cba..q}}] M_{\mathbf{3a}} M_{\mathbf{2b}} M_{\mathbf{1c}} .... M_{\mathbf{nq}} \qquad \text{// swap a↔c on } \varepsilon$$

$$= - \Sigma_{\mathbf{abc..q}} \varepsilon_{\mathbf{abc..q}} M_{\mathbf{3c}} M_{\mathbf{2b}} M_{\mathbf{1a}} .... M_{\mathbf{nq}} \qquad \text{// dummy rename a↔c}$$

$$= - \Sigma_{\mathbf{abc..q}} \varepsilon_{\mathbf{abc..q}} M_{\mathbf{1a}} M_{\mathbf{2b}} M_{\mathbf{3c}} .... M_{\mathbf{nq}} \qquad \text{// reorder product}$$

$$= - \det(M) \; .$$

Using Theorem 1 that det(A) = det($A^{\mathbf{T}}$), we then know that det($M'^{\mathbf{T}}$) = - det($M^{\mathbf{T}}$), so swapping *columns* 1 and 3 negates det(M). An obvious corollary follows if the two swapped columns have identical data :

**Corollary 4:** If two rows (columns) of square matrix A are the same, then det(M) = 0. (B.2.14)

Here is one more well-known determinant theorem which again demonstrates the power of the rearrangement theorem (B.2.4).

**Theorem 5**: $\det(AB) = \det(A)\det(B)$ (B.2.15)

Proof: Since we already use A in $\mathbf{a} = A\mathbf{z}_0$ , we shall prove $\det(XY) = \det(X)\det(Y)$ to avoid overloading symbols. Note that $\text{Parity}(\mathbf{a})\,\text{Parity}(\mathbf{b}) = (-1)^{s_a}(-1)^{s_b} = (-1)^{s_a+s_b} = \text{Parity}(A\mathbf{b})$ from (B.1.4). Then :

$\det(X)\det(Y) = [\,\Sigma_a\ \text{Parity}(\mathbf{a})\ \Pi(\mathbf{z}_0,\mathbf{a};\,X)]\,[\,\Sigma_b\ \text{Parity}(\mathbf{b})\ \Pi(\mathbf{z}_0,\mathbf{b};\,Y)]$ // (B.2.5) twice

$= \Sigma_a\ [\ \Sigma_b\ \text{Parity}(\mathbf{a})\ \text{Parity}(\mathbf{b})\ \ \Pi(\mathbf{z}_0,\mathbf{a};\,X)\ \ \Pi(\mathbf{z}_0,\mathbf{b};\,Y)\ ]$

$= \Sigma_a\ [\ \Sigma_b\ \ \ \text{Parity}(A\mathbf{b})\ \ \ \Pi(\mathbf{z}_0,\mathbf{a};\,X)\ \ \Pi(A\mathbf{z}_0,A\mathbf{b};\,Y)\ ]$ // (B.1.6) with C = A

$= \Sigma_a\ [\ \Sigma_b\ \ \ \text{Parity}(Ab)\ \ \ \Pi(\mathbf{z}_0,\mathbf{a};\,X)\ \ \Pi(\mathbf{a},Ab;\,Y)\ ]$ // $A\mathbf{z}_0 = \mathbf{a}$

$= \Sigma_a\ [\ \Sigma_b\ \ \ \text{Parity}(\mathbf{b})\ \ \ \Pi(\mathbf{z}_0,\mathbf{a};\,X)\ \ \Pi(\mathbf{a},\mathbf{b};\,Y)\ ]$ // (B.2.4) with C = A

$= \Sigma_b\ \ \text{Parity}(\mathbf{b})\ \Sigma_a\ \Pi(\mathbf{z}_0,\mathbf{a};\,X)\ \Pi(\mathbf{a},\mathbf{b};\,Y)$ // move $\Sigma_a$

$= \Sigma_b\ \ \text{Parity}(\mathbf{b})\ \Sigma_a\ \Pi(\mathbf{z}_0,\mathbf{b};\,XY)$ // see below

$= \det(XY)$ . // (B.2.5)

The idea here is to get **a** in the right place on both Π's so that

$\Sigma_a\ \Pi(\mathbf{z}_0,\mathbf{a};\,X)\ \Pi(\mathbf{a},\mathbf{b};\,Y) = \Sigma_{a_1a_2\cdots a_n}\ X_{1a_1}X_{2a_2}\ \ldots\ldots\ X_{na_n}\ \ Y_{a_1b_1}Y_{a_2b_2}\ \ldots\ldots\ Y_{a_nb_n}$

$= \Sigma_{a_1a_2\cdots a_n}\,(X_{1a_1}Y_{a_1b_1})(X_{2a_21}Y_{a_2b_2})\ldots..(X_{na_n}Y_{a_nb_n})$

$= (XY)_{1b_1}\,(XY)_{2b_2}\ldots(XY)_{nb_n}\ = \Pi(\mathbf{z}_0,\mathbf{b};\,XY)$

which in dense notation one would write as

$\Sigma_a\ X_{z_0a}Y_{ab} = (XY)_{z_0b}$ .

**Corollary 5**: $\det(ABC) = \det(A)\det(B)\det(C)$ and so on. (B.2.16)

Proof: $\det(ABC) = \det(A[BC]) = \det(A)\det(BC) = \det(A)\det(B)\det(C)$ .

It is not hard to slightly generalize the results of Theorem 2 to obtain :

**Theorem 2A:** If **b** is a permutation of $\mathbf{z}_0$, then det(M) can be represented in these two ways:

$$\det(M) = \varepsilon_{b_1 b_2 \cdots b_n} \Sigma_{a_1 a_2 \cdots a_n} \varepsilon_{a_1 a_2 \cdots a_n} M_{b_1 a_1} M_{b_2 a_2} \cdots\cdots M_{b_n a_n} \quad \text{(B.2.17a)}$$

$$\det(M) = \varepsilon_{b_1 b_2 \cdots b_n} \Sigma_{a_1 a_2 \cdots a_n} \varepsilon_{a_1 a_2 \cdots a_n} M_{a_1 b_1} M_{a_2 b_2} \cdots\cdots M_{a_n b_n} \ . \quad \text{(B.2.17b)}$$

In dense notation, one could write the above as (**a** and **b** are now vectors),

$$\det(M) = \varepsilon_b \Sigma_a \varepsilon_a M_{ba} \quad \text{(B.2.18a)}$$

$$\det(M) = \varepsilon_b \Sigma_a \varepsilon_a M_{ab} \ . \quad \text{(B.2.18b)}$$

Since $\varepsilon_b^2 = 1$, one can move $\varepsilon_b$ to the left side of these equations, so in the dense notation,

$$\varepsilon_b \det(M) = \Sigma_a \varepsilon_a M_{ba} \quad \text{(B.2.19a)}$$

$$\varepsilon_b \det(M) = \Sigma_a \varepsilon_a M_{ab} \ . \quad \text{(B.2.19b)}$$

Equations (B.2.19) are valid even if **b** is not a permutation of $\mathbf{z}_0$. In this case one has 0 = 0.
For the special case $\mathbf{b} = \mathbf{z}_0$ Theorem 2A reduces to Theorem 2 since $\varepsilon_{\mathbf{z}_0} = \varepsilon_{12\ldots n} = 1$.

Proof of Theorem 2A: We shall prove (B.2.17a) and then (B.2.17b) follows by taking $M \to M^T$ and using $\det(M^T) = \det(M)$.

$\det(M) = \Sigma_a$ Parity(**a**) $\Pi(\mathbf{z}_0, \mathbf{a}; M)$ // line 6 of (B.2.5)

$= \Sigma_a$ Parity(**a**) $\Pi(B\mathbf{z}_0, B\mathbf{a}; M)$ // (B.1.6) with C = B

$= \Sigma_a$ Parity(B**a**) Parity(**b**) $\Pi(B\mathbf{z}_0, B\mathbf{a}; M)$ // Parity(B**a**) Parity(**b**) = Parity(**a**)

$= \Sigma_a$ Parity(**a**) Parity(**b**) $\Pi(B\mathbf{z}_0, \mathbf{a}; M)$ // (B.2.4) that $\Sigma_a f(B\mathbf{a}) = \Sigma_a f(\mathbf{a})$

$=$ Parity(**b**) $\Sigma_a$ Parity(**a**) $\Pi(\mathbf{b}, \mathbf{a}; M)$ // $\mathbf{b} = B\mathbf{z}_0$

$= \varepsilon_{b_1 b_2 \cdots b_n} \Sigma_{a_1 a_2 \cdots a_n} \varepsilon_{a_1 a_2 \cdots a_n} M_{b_1 a_1} M_{b_2 a_2} \cdots M_{b_n a_n}$ // (B.1.5) and (B.2.9)

$= \varepsilon_b \Sigma_a \varepsilon_a M_{ba}$ . // previous line in dense notation (B.2.20)

Here it is assumed that B and hence **b** is associated with a permutation of $\mathbf{z}_0$, In this case, $(\varepsilon_b)^2 = 1$ so one can move $\varepsilon_b$ to the left side to get

$$\varepsilon_b \det(M) = \Sigma_a \varepsilon_a M_{ba} \ . \quad \text{(B.2.21)}$$

In this form, the equation is valid whether or not **b** is a permutation of $\mathbf{z}_0$ . If in $\mathbf{b} = (b_1,b_2...b_n)$ the $b_i$ are arbitrary elements of the set {1,2,...n} where two or more $b_i$ are the same (that is, **b** is not a permutation of $\mathbf{z}_0$), then $\Sigma_{\mathbf{a}}\ \varepsilon_{a_1a_2\cdots a_n}\ M_{b_1a_1}M_{b_2a_2}\ ....\ M_{b_na_n} = 0$ by symmetry. For example, if $b_1 = b_2$ then

$$
\begin{aligned}
S \ &\equiv\ \Sigma_{\mathbf{a}}\ \varepsilon_{a_1a_2\cdots a_n}\ M_{b_1a_1}M_{b_1a_2}\ ....\ M_{b_na_n} \\
&=\ \Sigma_{\mathbf{a}}\ \varepsilon_{a_2a_1\cdots a_n}\ M_{b_1a_2}M_{b_1a_1}\ ....\ M_{b_na_n} && \text{// rename dummy indices } a_1 \leftrightarrow a_2 \\
&=\ \Sigma_{\mathbf{a}}\ [-\varepsilon_{a_1a_2\cdots a_n}]\ M_{b_1a_1}M_{b_1a_2}\ ....\ M_{b_na} && \text{// (B.2.8) for } \varepsilon \text{ and slide } M_{b_1a_2} \text{ to the right} \\
&=\ -\Sigma_{\mathbf{a}}\varepsilon_{a_1a_2\cdots a_n}M_{b_1a_1}M_{b_1a_2}\ ....\ M_{b_na} = -S \qquad \Rightarrow\ S = 0\ . && \text{(B.2.22)}
\end{aligned}
$$

So in this case the right side of (B.2.21) is zero. But from (B.2.8) $\varepsilon_{\mathbf{b}} = 0$ so the left side is also zero.

Reader Exercise: Prove that

$$\varepsilon_{i_1i_2\cdots i_n}\det(M) = \Sigma_{\mathbf{A}}\ \text{parity}(A)\ M_{1,A(i_1)}M_{2,A(i_2)}\ .....M_{n,A(i_n)}\ . \qquad \text{(B.2.23)}$$

Proof in dense notation:

$$
\begin{aligned}
\varepsilon_{\mathbf{i}}\det(M) \ &= \varepsilon_{\mathbf{i}}\Sigma_{\mathbf{a}}\ \varepsilon_{\mathbf{a}}\ M_{\mathbf{z}_0,\mathbf{a}} && \text{// this is } \varepsilon_{\mathbf{i}} \text{ times usual expansion (B.2.7a)} \\
&= \text{parity(I)}\ \Sigma_{\mathbf{A}}\ \text{parity(A)}\ M_{\mathbf{z}_0,\mathbf{A}\mathbf{z}_0} && \text{// } \mathbf{a} = A\mathbf{z}_0 \\
&= \text{parity(I)}\ \Sigma_{\mathbf{A}}\ \text{parity(AI)}\ M_{\mathbf{z}_0,\mathbf{A}\mathbf{I}\mathbf{z}_0} && \text{// rearrangement theorem } \Sigma_{\mathbf{A}}f(A) = \Sigma_{\mathbf{A}}\ f(AI) \\
&= \Sigma_{\mathbf{A}}\ \text{parity(A)}\ M_{\mathbf{z}_0,\mathbf{A}\mathbf{i}} && \text{// } \mathbf{i} = I\mathbf{z}_0 \qquad \text{QED}
\end{aligned}
$$

**B.3 Minors, Cofactors, and the Cofactor Expansions of det(M)**

Definition: The **minor** of a matrix element $M_{rs}$ is the determinant of the submatrix of M obtained by crossing out the $r^{th}$ row and the $s^{th}$ column. It is a challenge, however, to write this out in symbols. Let's start with a more detailed version of (B.2.11a), where the $a_i$ are *column* summation indices,

$$\det(M) = \Sigma_{a_1 a_2 a_3 a_4 a_5 \ldots a_n} \varepsilon_{a_1 a_2 a_3 a_4 a_5 \ldots a_n} M_{1a_1} M_{2a_2} M_{3a_3} M_{4a_4} M_{5a_5} \ldots\ldots M_{na_n} . \qquad \text{(B.3.1)}$$

We shall make the following conjecture for the form of minor($M_{23}$)

$$\text{minor}(M_{32}) = (-1)^{3-2} \Sigma_{a_1 a_2 a_4 a_5 \ldots a_n} \varepsilon_{a_1 a_2 2 a_4 a_5 \ldots a_n} M_{1a_1} M_{2a_2} M_{4a_4} M_{5a_5} \ldots\ldots M_{na_n} \qquad \text{(B.3.2)}$$

Compared to det(M) shown in (B.3.1), we have made these changes :

- Removed the factor $M_{3a_3}$ ( since row-3 matrix elements cannot appear in minor($M_{32}$) )
- Removed the sum over $a_3$.
- Replaced $a_3$ by the number 2 on the ε tensor.
- added a sign factor $(-1)^{3-2}$.

Notice that since there is a 2 on the ε tensor, any time a summation index $a_i = 2$ there is no contribution since then the ε tensor has two indices the same, so in effect the value 2 has been removed from all the residual summations $a_i$. That is good, since column 2 is supposedly "crossed out" in minor($M_{32}$). The factor $(-1)^{3-2}$ is added so that the "diagonal term" in the minor will be positive. This sign $(-1)^{3-2}$ gets used up if we slide the "2" to its natural position (position 2) on the ε tensor,

$$\text{minor}(M_{32}) = \Sigma_{a_1 a_2 a_4 a_5 \ldots a_n} \varepsilon_{a_1 2 a_2 a_4 a_5 \ldots a_n} M_{1a_1} M_{2a_2} M_{4a_4} M_{5a_5} \ldots\ldots M_{na_n} . \qquad \text{(B.3.3)}$$

The diagonal term in minor($M_{32}$) is then positive, matching the mechanical crossing-out method,

$$\varepsilon_{12345..n} M_{11} M_{22} M_{44} M_{55} \ldots\ldots M_{nn} = + M_{11} M_{22} M_{44} M_{55} \ldots\ldots M_{nn} . \qquad \text{(B.3.4)}$$

A more compact notation for (B.3.2) would be

$$\text{minor}(M_{32}) = (-1)^{3-2} \Sigma_{a_i, i \neq 3} \varepsilon_{a_i, a_3 = 2} \Pi_{i \neq 3} (M_{i,a_i}) . \qquad \text{(B.3.5)}$$

Starting over, we could show similarly that

$$\text{minor}(M_{42}) = (-1)^{4-2} \Sigma_{a_i, i \neq 4} \varepsilon_{a_i, a_4 = 2} \Pi_{i \neq 4} (M_{i,a_i}) . \qquad \text{(B.3.6)}$$

We now have a sign factor $(-1)^{4-2}$ because the "2" on ε has to be slid 2 positions to get to its natural location (position 4 on ε). More generally we can say

$$\text{minor}(M_{s2}) = (-1)^{s-2} \Sigma_{a_i, i \neq s} \varepsilon_{a_i, a_s = 2} \Pi_{i \neq s} (M_{i,a_i}) \qquad \text{(B.3.7)}$$

where the slide is now s-2 places. Still more generally we find, replacing 2 by r,

$$\text{minor}(M_{sr}) = (-1)^{s\pm r}\ \Sigma_{a_i,\,i\neq s}\ \varepsilon_{a_i,a_s=r}\ \Pi_{i\neq s}\,(M_{i,a_i})\ . \qquad \text{(B.3.8)}$$

This then is our "best form" expression for a minor of matrix element $M_{sr}$. Either sign will do, since

$$(-1)^{s-r} = (-1)^{s-r}(-1)^{2r} = (-1)^{s+r} \qquad \text{since } (-1)^{2r} = 1\ .$$

Choosing the + sign in (B.3.8) and putting $(-1)^{s+r}$ on the left side we get,

$$(-1)^{s+r}\ \text{minor}(M_{sr}) = \Sigma_{a_i,\,i\neq s}\ \varepsilon_{a_i,a_s=r}\ \Pi_{i\neq s}\,(M_{i,a_i})\ . \qquad \text{(B.3.9)}$$

The left side here is called the **cofactor** of $M_{sr}$, written $\text{cof}(M_{sr})$. Thus we have shown that

$$\text{cof}(M_{sr}) \equiv (-1)^{s+r}\ \text{minor}(M_{sr}) = \Sigma_{a_i,\,i\neq s}\ \varepsilon_{a_i,a_s=r}\ \Pi_{i\neq s}\,(M_{i,a_i})\ . \qquad \text{(B.3.10)}$$

**Fact**: Neither $\text{minor}(M_{sr})$ nor $\text{cof}(M_{sr})$ are functions of the $M_{sr}$ matrix element of M! In (B.3.10) this is so because: (1) row s is excluded in $\Pi_{i\neq s}(M_{i,a_i})$; (2) $a_i = r$ is excluded by the factor $\varepsilon_{a_i,a_s=r}$ . More intuitively, this is so because to get $\text{minor}(M_{sr})$ we "cross out" row s and column r. Thus,

$$\frac{\partial \text{cof}(M_{sr})}{\partial M_{sr}} = \frac{\partial \text{minor}(M_{sr})}{\partial M_{sr}} = 0 \qquad \text{for any pair r,s in 1,2...n}\ . \qquad \text{(B.3.11)}$$

Finally, suppose we replace r with an integer which we call $a_s$. Doing this causes

$$\varepsilon_{a_i,a_s=r} \rightarrow \varepsilon_{a_i,a_s=a_s} = \varepsilon_{a_i} = \varepsilon_{a_1a_2a_4a_5\ldots a_n} = \text{the normal } \varepsilon \text{ tensor form}\ . \qquad \text{(B.3.12)}$$

Then (B.3.10) becomes the following,

$$\text{cof}(M_{sa_s}) \equiv (-1)^{s+a_s}\ \text{minor}(M_{sa_s}) = \Sigma_{a_i,\,i\neq s}\ \varepsilon_{a_i}\ \Pi_{i\neq s}\,(M_{i,a_i})\ . \qquad \text{(B.3.13)}$$

Now start again with det(M) of (B.3.1) and rewrite it in our compact notation:

$$\begin{aligned}
\det(M) &= \Sigma_{a_1a_2a_3a_4a_5\ldots a_n}\,\varepsilon_{a_1a_2a_3a_4a_5\ldots a_n}\,M_{1a_1}M_{2a_2}M_{3a_3}M_{4a_4}M_{5a_5}\ldots\ldots M_{na_n} \\
&= \Sigma_{a_i}\,\varepsilon_{a_i}\,\Pi_i(M_{i,a_i}) \qquad \text{// next, extract the } a_s \text{ sum and its } M_{sa_s} \text{ factor :} \\
&= \Sigma_{a_s}\,M_{sa_s}\,[\Sigma_{a_i,\,i\neq s}\ \varepsilon_{a_i}\,\Pi_{i\neq s}\ (M_{i,a_i})] \qquad \text{// next use (B.3.12) to get} \\
&= \Sigma_{a_s}\,M_{sa_s}\,[\text{cof}(M_{sa_s})] \qquad \text{// next, change summation index from } a_s \text{ to n} \\
&= \Sigma_n\,M_{sn}\,\text{cof}(M_{sn})\ . \qquad \text{// valid for any s in 1,2...n}
\end{aligned} \qquad \text{(B.3.14)}$$

This is the **cofactor expansion** of det(M) where one "works across row s" and n is a column index. Another form of this expansion is

$$\det(M) = \det(M^{\mathbf{T}}) = \Sigma_{\mathbf{n}}\, M^{\mathbf{T}}{}_{\mathbf{sn}}\, \mathrm{cof}(M^{\mathbf{T}}{}_{\mathbf{sn}})$$

$$= \Sigma_{\mathbf{n}}\, M_{\mathbf{ns}}\, \mathrm{cof}(M_{\mathbf{ns}}) \qquad \text{(B.3.15)}$$

and in this form one "works down column s" where n is now a row index. We have just proven :

**Theorem 6** (Cofactor Expansions):

$$\det(M) = \Sigma_{\mathbf{n}}\, M_{\mathbf{sn}}\, \mathrm{cof}(M_{\mathbf{sn}}) = \Sigma_{\mathbf{n}}\, (\mathbf{r}_{\mathbf{s}})_{\mathbf{n}}\, \mathrm{cof}(M_{\mathbf{sn}}) \qquad \text{work across row s} \qquad s = 1,2....n \qquad \text{(B.3.16a)}$$

$$\det(M) = \Sigma_{\mathbf{n}}\, M_{\mathbf{ns}}\, \mathrm{cof}(M_{\mathbf{ns}}) = \Sigma_{\mathbf{n}}\, (\mathbf{c}_{\mathbf{s}})_{\mathbf{n}}\, \mathrm{cof}(M_{\mathbf{ns}}) \qquad \text{work down column s} \qquad s = 1,2....n \qquad \text{(B.3.16b)}$$

The term "cofactor" presumably arose because each sum term consists of a "factor" $M_{\mathbf{ij}}$ and a "co-factor" which cooperates with the factor to make the sum, as coauthors cooperate to write a book. A common notation is to define $M^{\mathbf{ij}} \equiv \mathrm{cof}(M_{\mathbf{ij}})$ and then the cofactor expansions become

$$\det(M) = \Sigma_{\mathbf{n}}\, M_{\mathbf{sn}} M^{\mathbf{sn}}$$
$$\det(M) = \Sigma_{\mathbf{n}}\, M_{\mathbf{ns}} M^{\mathbf{ns}} .$$

Although compact, this is not so handy for tensor notation with *its* up and down indices (see Lucht Ref).

**B.4 Expressions for the inverse matrix $M^{-1}$ and Cramer's Rule**

Consider the expansion (B.3.16a) working across row s,

$$\det(M) = \Sigma_{\mathbf{n}}\, M_{\mathbf{sn}}\, \mathrm{cof}(M_{\mathbf{sn}}) . \qquad \text{(B.4.1)}$$

Suppose we replace the elements of row s ($M_{\mathbf{sn}}$) with the elements of some other row r in M ($M_{\mathbf{rn}}$). In doing so we have created a new matrix, call it M'. Notice that $\mathrm{cof}(M'_{\mathbf{sn}}) = \mathrm{cof}(M_{\mathbf{sn}})$ *because*, although going from M to M' we have altered row s, we have not altered $\mathrm{cof}(M_{\mathbf{sn}})$ since this depends only on the entries in all the rows *other than* row s (think "crossing out" row s for minor($M_{\mathbf{sn}}$) ). See Fact (B.3.11) and text above. Therefore we find,

$$\det(M') = \Sigma_{\mathbf{n}}\, M'_{\mathbf{sn}}\, \mathrm{cof}(M'_{\mathbf{sn}}) = \Sigma_{\mathbf{n}}\, M_{\mathbf{rn}}\, \mathrm{cof}(M_{\mathbf{sn}}) . \qquad \text{(B.4.2)}$$

But by Corollary 4 (B.2.14) det(M') = 0 since two rows of M' are the same. Thus

$$0 = \Sigma_{\mathbf{n}}\, M_{\mathbf{rn}}\, \mathrm{cof}(M_{\mathbf{sn}}) . \qquad r \neq s \qquad \text{(B.4.3)}$$

Combining this with (B.4.1) gives,

$$\Sigma_{\mathbf{n}}\, M_{\mathbf{rn}}\, \mathrm{cof}(M_{\mathbf{sn}}) = \det(M)\delta_{\mathbf{r,s}} . \qquad \text{(B.4.4)}$$

Now for clarity define a cofactor matrix C in this manner

$$C_{sn} \equiv \text{cof}(M_{sn}) \; . \tag{B.4.5}$$

In matrix notation, we could define the matrix cof(M) to be matrix C and then

$$C = \text{cof}(M) \qquad \Rightarrow \qquad C_{sn} = [\text{cof}(M)]_{sn} = \text{cof}(M_{sn}) \; . \tag{B.4.6}$$

Now (B.4.4) says

$$\Sigma_n M_{rn} \, C_{sn} = \det(M) \delta_{r,s}$$

or

$$\Sigma_n M_{rn} \, C^T_{ns} = \det(M) \delta_{r,s}$$

and finally in matrix notation,

$$MC^T = \det(M) \, 1$$

or

$$M \left[ \frac{C^T}{\det(M)} \right] = 1 \; . \tag{B.4.7}$$

Therefore we find this classic square matrix inversion formula,

**Theorem 7:** $$M^{-1} = \frac{C^T}{\det(M)} = \frac{[\text{cof}(M)]^T}{\det(M)} = \frac{\text{cof}(M^T)}{\det(M)} \; . \tag{B.4.8}$$

To verify the last equality in (B.4.8) consider :

$= [\text{cof}(M)]^T_{ns} = [\text{cof}(M)]_{sn}$ // meaning of transpose

$= \text{cof}(M_{sn})$ // (B.4.6)

$= \text{cof}(M^T_{ns})$

$= [\text{cof}(M^T)]_{ns}$ // (B.4.6) applied to $M^T$

and therefore we have this matrix identity,

$$[\text{cof}(M)]^T = [\text{cof}(M^T)] \; . \tag{B.4.9}$$

Using (B.4.8) it is trivial to solve a non-singular (detM≠0) system of linear equations $\mathbf{y} = M\mathbf{x}$ :

$$\mathbf{y} = M\mathbf{x} \qquad \Rightarrow \qquad \mathbf{x} = M^{-1} \mathbf{y} = \frac{1}{\det(M)} \, \text{cof}(M^T) \, \mathbf{y} = \frac{1}{\det(M)} \, [\text{cof}(M)]^T \mathbf{y} \; . \tag{B.4.10}$$

In components,

$$x_s = \frac{1}{\det(M)} \Sigma_n \, ([\mathrm{cof}(M)]^T)_{sn} \, y_n = \frac{1}{\det(M)} \Sigma_n \, y_n \, [\mathrm{cof}(M)]_{ns} = \frac{1}{\det(M)} \Sigma_n \, y_n \, \mathrm{cof}(M_{ns}) \, . \qquad \text{(B.4.11)}$$

Now recall the cofactor expansion (B.3.16b),

$$\det(M) = \Sigma_n \, M_{ns} \, \mathrm{cof}(M_{ns}) = \Sigma_n \, (\mathbf{c}_s)_n \, \mathrm{cof}(M_{ns}) \, . \qquad \text{(B.4.12)}$$

If one replaces column $\mathbf{c}_s$ in M by $\mathbf{y}$ one gets

$$\det(M[\mathbf{c}_s \rightarrow \mathbf{y}]) = \Sigma_n \, y_n \, \mathrm{cof}(M_{ns}) \qquad \text{// (B.4.12)}$$

$$= \det(M) \, x_s \, . \qquad \text{// (B.4.11)} \qquad \text{(B.4.13)}$$

Solving this for $x_s$ we obtain another classic result,

**Theorem 8** (Cramer's Rule, 1750): // Gabriel Cramer (1704-1752), Swiss, made it to age 47

$$\mathbf{y} = M\mathbf{x} \quad \Rightarrow \quad x_s = \frac{\det(M[\mathbf{c}_s \rightarrow \mathbf{y}])}{\det(M)} \, . \qquad \text{(B.4.14)}$$

**Comment: Names of Matrices**

Historically the matrix $C^T = [\mathrm{cof}(M)]^T = [\mathrm{cof}(M^T)]$ was called the **adjoint** of matrix M and was often denoted by $\hat{M}$ and then (B.4.8) reads $M^{-1} = \hat{M} / |M|$. But this then conflicted with another usage, namely the matrix $M^\dagger \equiv (M^T)^* = (M^*)^T$ being the **adjoint** of M (the conjugate transpose of M). When $M^\dagger = M$, M is said to be **self-adjoint** and has significance in quantum (matrix) mechanics: the quantum operators of all physical observables are self-adjoint and thus have real eigenvalues. Nowadays the matrix $[\mathrm{cof}(M)]^T$ is called the **classical adjoint** of M, while $M^\dagger$ is then the **Hermitian adjoint** (or **Hermitian conjugate**) of M and is sometimes written $M^H$. If $M^\dagger = M$ then M is said to be **Hermitian**. Historically when $[\mathrm{cof}(M)]^T$ was the adjoint of M, $M^\dagger$ was called the associate of M. The classical adjoint sometimes appears as the adjugate or adj(M). Transpose matrices $M^T$ are often denoted $\tilde{M}$ .

On a related matter, some older authors used the word **minor** to refer to what we now call a cofactor. See for example Morse & Feshbach page 509. These authors never use the word cofactor.

**Appendix C: The Intersection of Constraint Surfaces**

In the proof of Theorem 3 in Section 6.2 (a) we rely on our ability, at least in theory, to eliminate $x_1$, $x_2$, $x_3$ from the three constraint equations shown in (6.2.3).

$$\begin{aligned} a(x_1, x_2, x_3, x_4, x_5, x_6) &= 0 \\ b(x_1, x_2, x_3, x_4, x_5, x_6) &= 0 \\ c(x_1, x_2, x_3, x_4, x_5, x_6) &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= X^1(x_4, x_5, x_6) \\ x_2 &= X^2(x_4, x_5, x_6) \\ x_3 &= X^3(x_4, x_5, x_6) \ . \end{aligned} \qquad (6.2.3)$$

How does one know this is possible? The equations might be very complicated, some of the coordinates might not appear in some of the equations, and so on.

We look first at three examples in $E^3$ then in Example 4 we return to our question.

Example 1: Consider these two constraint equations in $E^3$ :

$$\begin{aligned} a(x_1,x_2,x_3) &= x_1^2 + x_2^2 + x_3^2 - 2^2 = 0 \\ b(x_1,x_2,x_3) &= (x_1-2)^2 + x_2^2 + x_3^2 - 2^2 = 0 \ . \end{aligned} \qquad (C.1)$$

Each constraint represents a sphere of radius 2 and so is a 2-dimensional smooth surface in $E^3$. The first sphere is centered at the origin, while the second has its center at (2,0,0). Subtracting the first equation from the second gives $-4x_1+4 = 0$ so $x_1 = 1$, then inserting this into both equations gives

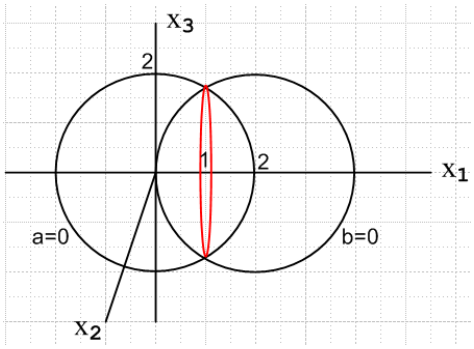
$$\begin{aligned} 1 + x_2^2 + x_3^2 - 2^2 &= 0 \\ 1 + x_2^2 + x_3^2 - 2^2 &= 0 \end{aligned} \quad \Rightarrow \ x_2^2 + x_3^2 = 3 \ . \qquad (C.2)$$

The intersection of the two constraint surfaces is a circle, a 1-dimensional smooth surface in $E^3$.

Given $a(x_1,x_2,x_3) = 0$ and $b(x_1,x_2,x_3) = 0$, is it possible to eliminate $x_1$ and $x_2$ from the two constraint equations? The answer is yes, as follows,

$$\begin{aligned} x_1 &= X^1(x_3) = 1 \\ x_2 &= X^2(x_3) = \pm\sqrt{3 - x_3^2} \ . \end{aligned} \qquad (C.3)$$

The following schematic drawing shows the intersection surface in red,




(C.4)

Notice in this example that there are two possible solutions for $x_2$. One can regard the intersection surface (the circle $x_2^2 + x_3^2 = 3$) as having two pieces (front half, back half) and the ± sign selects one of these pieces.

It is easy to imagine more complicated examples for two constraints in $E^3$ where perhaps one or both of the constraint surfaces contain disjoint pieces (e.g. a hyperboloid), and where the intersection surface also has several disjoint pieces.

Example 2: Consider these two constraint equations in $E^3$ :

$$\begin{aligned} a(x_1,x_2,x_3) &= x_1^2 + x_2^2 + x_3^2 - 2^2 = 0 \\ b(x_1,x_2,x_3) &= \quad x_2^2 + x_3^2 - 1^2 = 0 \,. \end{aligned} \tag{C.5}$$

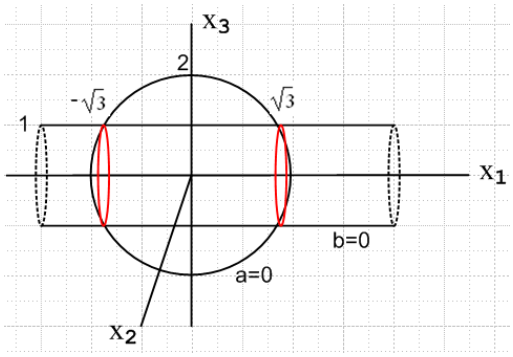
The second equation is missing the $x_1$ coordinate. The first equation describes the same origin-centered radius 2 sphere of the previous example. The second equation is that of a circle of radius 1, but as a function of three variables it is in fact a cylinder whose axis is the $x_1$ axis. If $x_2$ and $x_3$ lie on the circle, any value of $x_1$ satisfies the second equation. Missing coordinates result in surfaces which are "extruded" in the dimensions of the missing coordinates. Subtracting the second equation from the first gives $x_1^2 - 3 = 0$ so $x_1 = \pm\sqrt{3}$ , then inserting this into both equations gives

$$\begin{aligned} 3 \; + x_2^2 + x_3^2 - 2^2 &= 0 \\ x_2^2 + x_3^2 - 1^2 &= 0 \qquad \Rightarrow \; x_2^2 + x_3^2 = 1 \,. \end{aligned} \tag{C.6}$$

Given $a(x_1,x_2,x_3) = 0$ and $b(x_1,x_2,x_3) = 0$, is it possible to eliminate $x_1$ and $x_2$ from the two constraint equations? The answer is yes, as follows,

$$\begin{aligned} x_1 &= X^1(x_3) = \pm\sqrt{3} \\ x_2 &= X^2(x_3) = \pm\sqrt{1 - x_3^2} \,. \end{aligned} \tag{C.7}$$

The following schematic drawing shows the intersection surface in red,




(C.8)

In this example there are two possible solutions for $x_1$ and two for $x_2$ so overall there are four solutions. The intersection surface consists of the two circles each having a front half and a back half.

Example 3: Consider these two constraint equations in $E^3$ :

$$\begin{aligned} a(x_1,x_2,x_3) &= x_1^2 + x_2^2 + x_3^2 - 2^2 = 0 \\ b(x_1,x_2,x_3) &= x_1^2 + x_2^2 - 1^2 = 0 . \end{aligned} \qquad \text{(C.9)}$$

Now $x_3$ is missing from the second equation instead of $x_1$. Subtracting the second equation from the first gives $x_3^2 - 3 = 0$ so $x_3 = \pm\sqrt{3}$ , then inserting this into both equations gives

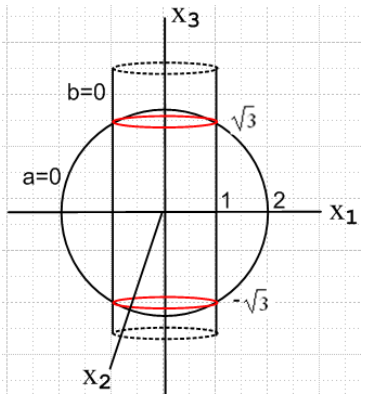
$$\begin{aligned} & x_1^2 + x_2^2 + 3 - 2^2 = 0 \\ & x_1^2 + x_2^2 - 1^2 = 0 \quad \Rightarrow \; x_1^2 + x_2^2 = 1 . \end{aligned} \qquad \text{(C.10)}$$

Given $a(x_1,x_2,x_3) = 0$ and $b(x_1,x_2,x_3) = 0$, is it possible to eliminate $x_1$ and $x_2$ from the two constraint equations? The answer is yes, as follows, where $\alpha$ is an arbitrary real parameter in [-1,1],

$$\begin{aligned} & x_1 = X^1(x_3 = \pm\sqrt{3}) = \alpha \qquad -1 \le \alpha \le 1 \\ & x_2 = X^2(x_3 = \pm\sqrt{3}) = \pm\sqrt{1 - \alpha^2} . \quad \text{// the two } \pm \text{ signs are independent} \end{aligned} \qquad \text{(C.11)}$$

In the previous two examples the argument $x_3$ was a free parameter whose variation mapped out the smooth constraint intersection surface piece(s). In Example 3 $x_3$ is fixed at $\pm\sqrt{3}$ and a new parameter $\alpha$ must be introduced to map out the intersection surfaces. There are again four solution half-circles.

The following schematic drawing shows the intersection surface in red,




(C.12)

Example 4: Now consider these three constraint equations in $E^6$ :

$$\begin{aligned} & a(x_1, x_2, x_3, x_4, x_5, x_6) = 0 \\ & b(x_1, x_2, x_3, x_4, x_5, x_6) = 0 \\ & c(x_1, x_2, x_3, x_4, x_5, x_6) = 0 . \end{aligned} \qquad \textit{(6.2.2)} \qquad \text{(C.13)}$$

We assume that each equation describes a smooth surface of dimension 5 in $E^6$ (each is a hypersurface) and each surface may have several pieces. In a problem with constraints, one *assumes* that there is some

non-null surface of intersection of all the constraint surfaces. This intersection surface may have multiple pieces and each piece (barring pathological cases) is itself a smooth surface of dimension 3 in $E^6$. A candidate solution point **r** must lie on one of these intersection surface pieces. In general, each constraint knocks down the dimension of the intersection surface by one degree of freedom.

So let us assume that $x_4$, $x_5$, $x_6$ are the last three components of some point(s) on the overall intersection surface. Each such point of course has some $x_1$, $x_2$, $x_3$ components. If $x_4$, $x_5$, $x_6$ are varied slightly, $x_1$, $x_2$, $x_3$ will also vary slightly, and this is what we mean by writing the functions

$$\begin{aligned} x_1 &= X^1(x_4, x_5, x_6) \\ x_2 &= X^2(x_4, x_5, x_6) \\ x_3 &= X^3(x_4, x_5, x_6)\ . \end{aligned} \qquad \textit{(6.2.3)} \qquad \text{(C.14)}$$

If the intersection surface has multiple pieces which have the same $x_4$, $x_5$, $x_6$ value, then any of the three functions $X^n$ above may be multi-valued, as occurred in our earlier examples. Conversely, if we select a triplet $x_4$, $x_5$, $x_6$ for which there are no points on the intersection surface, the solution $x_1$, $x_2$, $x_3$ does not exist. We don't care about such points in $E^6$.

So this then is what we mean in Section 6.2 (a) when we say above (6.2.3) that one can use the three constraint equations to eliminate the three variables $x_1$, $x_2$, $x_3$. In the concluding equations (6.2.11) one can regard the functions $X^1$, $X^2$, $X^3$ as being any of the multi-valued functions just discussed if the intersection surface has multiple pieces. For example, we had in (6.2.11) that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} \begin{pmatrix} X^1{}_4 \\ X^2{}_4 \\ X^3{}_4 \\ 1 \end{pmatrix} \qquad \textit{part of (6.2.11)} \qquad \text{(C.15)}$$

and here $X^1{}_4$ refers to $(\partial/\partial x_4)X^1(x_4, x_5, x_6)$ evaluated at coordinates $x_4$, $x_5$, $x_6$ for some candidate solution point **r** on the intersection surface. The fact that the matrix in (C.15) must have zero determinant is not affected by the possible existence of multiple solution $X^1$, $X^2$, $X^3$ functions.

Notice that the derivatives appearing in the matrix are unaffected by the possibility of multi-valued functions for $X^1$, $X^2$, $X^3$.

In the same manner as above, we can "eliminate" any triplet of variables $x_i$, $x_j$, $x_k$ ( $i \neq j \neq k$) from the three constraint equations (C.13).

In the general case where there are S-1 constraints and **r** has N components $x_1, x_2 \ldots x_N$ with N > S, the dimensionality of the intersection surface is N - (S-1) in $E^N$. The conclusions of the Theorem 3 proof outlined in Section 6.2 (b) are similarly not affected by the possible existence of an intersection surface having multiple pieces and the functions $X^n$ possibly having multiple values.

Footnote: Why is $a(x_1, x_2, x_3, ...x_N) = 0$ an (N-1)-dimensional surface in $E^N$ ? (C.16)

This fact probably seems obvious to the reader and is certainly obvious in the first three examples above. Here we attempt a simple "engineering" explanation.

We *assume* that a constraint equation is a "smooth" equation, meaning it is continuous and differentiable in all its arguments except perhaps at certain isolated locations which can be handled on an *ad hoc* basis either by fiat or by taking limits. Consider that, since $a(\mathbf{r})$ = constant (namely 0), da = 0 so

$$0 = da = \Sigma_{i=1}^{N} a_i dx_i \ , \qquad // \ a_i = a_i(\mathbf{r}) = a_i(x_1, x_2, x_3, ...x_N) \tag{C.17}$$

where $a_i$ is usual means $\partial a/\partial x_i$. Let $\mathbf{r}$ be some point for which $a(\mathbf{r}) = 0$. At this point we shall assume that there exists at least one $a_s(\mathbf{r})$ which is non-zero. If *all* $a_i(\mathbf{r}) = 0$, we show below that $\mathbf{r}$ is a point on a surface which has a null normal vector, which is not possible, so such an $\mathbf{r}$ value could not lie on $a(\mathbf{r}) = 0$. Then (C.17) may be written

$$dx_s = (1/a_s) \Sigma_{i \neq s} a_i dx_i \ . \tag{C.18}$$

Create a coordinate system whose origin is located at this point $\mathbf{r}$ for which $a(\mathbf{r}) = 0$, and whose axes are aligned with those of $E^N$. Now imagine an arbitrary tiny displacement of all the $dx_i$ other than $dx_s$. For this set of N-1 $dx_i$ there is only one possible $dx_s$ which causes the point $\mathbf{r} + d\mathbf{r}$ to satisfy the equation $a(\mathbf{r}+d\mathbf{r}) = a(\mathbf{r}) + da = 0 + 0 = 0$, where $d\mathbf{r} \equiv (dx_1, dx_2.....dx_N)$. That one possible $dx_s$ value is that given by (C.18). Imagine repeating this process for a continuum of values for the $dx_i$ other than $dx_s$, and in each case we obtain the unique solution $dx_s$ from (C.18). We can think of the $x_s$ axis as being "vertical" and all the other axes being "horizontal" inasmuch as they are all perpendicular to the $x_s$ axis. The vectors $d\mathbf{r}$ generated in this manner comprise a tiny patch of a "plane" of dimension N-1 (a hyperplane) in $E^N$ which contains the point $\mathbf{r}$. This is so because $\Sigma_{i=1}^{N} a_i dx_i = \mathbf{a} \bullet d\mathbf{r} = 0$ is the equation of a plane in $E^N$ passing through our origin at $\mathbf{r}$ with $\mathbf{a}$ being that plane's normal vector. More generally, $\hat{\mathbf{n}} \bullet \mathbf{r} = d$ is the equation of a hyperplane in $E^n$ having a unit normal $\hat{\mathbf{n}}$ and whose closest approach to the origin is distance d. In

Thus at a point $\mathbf{r}$ satisfying $a(\mathbf{r}) = 0$ we have constructed a tiny neighborhood of nearby points $\mathbf{r}$ which also satisfy $a(\mathbf{r}) = 0$. This neighborhood is a patch of a hyperplane in $E^N$ which is certainly a piece of "surface" in $E^N$ having dimension N-1. By repeating this process using a mesh of points $\mathbf{r}_i$ satisfying $a(\mathbf{r}_i) = 0$, we then map out a triangulated surface of dimension N-1 in $E^N$. In the limit the mesh size goes to 0, we arrive at a smooth surface of dimension N-1 in $E^N$ and that surface is $a(\mathbf{r}) = 0$. The tiny planar patch at any point on the surface is part of the "tangent plane" to the surface at that point.

The constraint $a(\mathbf{r}) = 0$ is assumed to be "locally smooth" in the immediate region of any point $\mathbf{r}$ on the operational constraint surface, so a small local planar region (an open set) can be constructed around any such point. This is the basic idea of a surface being a manifold M, and the set of vectors in the tangent plane of a point $\mathbf{r}$ on M comprise the "tangent space" of M at $\mathbf{r}$, usually denoted by $T_{\mathbf{r}}M$.

**Appendix D: Lagrange Multipliers in Classical Mechanics**

Outline of this Appendix showing the nearest equation number:


Constraints and virtual displacements (D.1)
Assumption that constraints do no virtual work (D.11)
Comment on internal forces (D.13)
An Application of Lagrange Multipliers (D.13)
Summary of the above Lagrange Multiplier Application (D.25)
Generalized coordinates, generalized forces, and Lagrange's Equations (D.27)
Footnote: Derivation of the result (D.31) used above (D.40)
Lagrange Equations with non-holonomic constraints (D.49)
The Case of C holonomic constraints treated as if they were non-holonomic (D.55)
Footnote: Carry out the $\delta S = 0$ functional variation of the action (D.56)


Our main focus in this document is the use of Lagrange Multipliers to find candidates $\mathbf{r}$ for which a scalar *function* f($\mathbf{r}$) is stationary, df = 0, subject to constraints. In this Appendix, however, we consider applications of Lagrange Multipliers which relate to classical mechanics for a system of N particles in the presence of constraints. In this case the function f($\mathbf{r}$) with $\mathbf{r}$ lying in $E^{\mathbf{N}}$ is replaced by a *functional* f(φ) where φ lies in a space of functions. The goal is to find a function φ which is a stationary point of the functional, and one writes this as $\delta f(\varphi) = 0$.

A certain amount of background information is required to put this application into context, which we now present.

Constraints and virtual displacements

Let $\mathbf{r_k}(t)$ be the position of the $k^{\mathbf{th}}$ particle of an N particle system in 3 dimensional space. At any fixed time t, if the particles are unencumbered with constraints, the variables ($\mathbf{r_1}$, $\mathbf{r_2}$..... $\mathbf{r_N}$) can be regarded as a single point in the space $E^{\mathbf{3N}}$ and the "configuration space" of the system indicated by ($\mathbf{r_1}$, $\mathbf{r_2}$..... $\mathbf{r_N}$) is all of this space $E^{\mathbf{3N}}$. All points in $E^{\mathbf{3N}}$ are "legal" points for the system.

A **holonomic** constraint has the form

$$a(\mathbf{r_1}, \mathbf{r_2}..... \mathbf{r_N}, t) = 0. \qquad (D.1)$$

At a given time t, this equation represents a surface of dimension 3N-1 in $E^{\mathbf{N}}$. If there are s such holonomic constraints

$$a_{\mathbf{i}}(\mathbf{r_1}, \mathbf{r_2}..... \mathbf{r_N}, t) = 0 \qquad i = 1,2...s \qquad (D.2)$$

then at time t the system vector ($\mathbf{r_1}$, $\mathbf{r_2}$..... $\mathbf{r_N}$) is constrained to lie on a surface of dimension 3N-s in $E^{\mathbf{N}}$, as discussed in Appendix C, so each extra constraint lowers the operating surface dimension by 1. One can think of the term holonomic as meaning that the constraints are functions of "whole" coordinates like $\mathbf{r_2}$ as opposed to differential coordinates like $d\mathbf{r_2}$.

The constraints which appear in (1.7) are holonomic constraints of the form $a(x_1, x_2, \ldots x_n) = 0$.

A **non-holonomic** constraint is one that is not holonomic. An inequality constraint like $a(\mathbf{r}_1, \mathbf{r}_2 \ldots \mathbf{r}_N, t) < 0$ which might bound the particles to one side of a surface is an example of a nonholonomic constraint, but our interest lies more in nonholonomic constraints of the form

$$\Sigma_{k=1}^{N} \mathbf{b}_k \bullet d\mathbf{r}_k + b_t dt = 0 \qquad \text{(D.3)}$$

in which differentials rather than "whole" coordinates appear. The "coefficients" $\mathbf{b}_k$ and $b_t$ can in general be functions of $(\mathbf{r}_1, \mathbf{r}_2 \ldots \mathbf{r}_k, t)$. If it happens that such a constraint can be integrated by some hook or crook to give an equation of the form $a(\mathbf{r}_1, \mathbf{r}_2 \ldots \mathbf{r}_k, t) = 0$, then it is regarded as being a holonomic constraint, so a non-holonomic constraint must be non-integrable. If there are m such non-holonomic constraints imposed on a system, one could write them as

$$\begin{aligned} &\Sigma_{k=1}^{N} \mathbf{A}_{ik} \bullet d\mathbf{r}_k + A_{it} dt = 0 \qquad i = 1,2 \ldots m \\ &\text{or} \qquad \text{(D.4)} \\ &\Sigma_{k=1}^{N} \mathbf{A}_{ik} \bullet \dot{\mathbf{r}}_k + A_{it} = 0 \qquad i = 1,2 \ldots m \ . \end{aligned}$$

Here bolded $\mathbf{A}_{ik}$ represents 3 matrices each of which is m x N (rows x columns). Since these constraints are non-integrable, one cannot really described them in terms of surfaces in $E^{3N}$.

If one differentiates the holonomic constraints (D.2) one gets

$$\begin{aligned} &\Sigma_{k=1}^{N} [\nabla^{(k)} a_i] \bullet d\mathbf{r}_k + [\partial_t a_i] \, dt = 0 \qquad i = 1,2 \ldots s \\ &\text{or} \qquad \text{(D.5)} \\ &\Sigma_{k=1}^{N} [\nabla^{(k)} a_i] \bullet \dot{\mathbf{r}}_k + [\partial_t a_i] = 0 \qquad i = 1,2 \ldots s \end{aligned}$$

where $\nabla^{(k)} \equiv (\partial/\partial(\mathbf{r}_k)_1, \partial/\partial(\mathbf{r}_k)_2, \partial/\partial(\mathbf{r}_k)_3)$ is a gradient with respect to the components of $\mathbf{r}_k$. These equations have the same form as those in (D.4), but these are holonomic because they are integrable to give $a_i(\mathbf{r}_1, \mathbf{r}_2 \ldots \mathbf{r}_N, t) = 0$.

For convenience below, we shall now combine (D.4) and (D.5) into a single set of equations,

$$\begin{aligned} &\Sigma_{k=1}^{N} \mathbf{B}_{ik} \bullet d\mathbf{r}_k + B_{it} dt = 0 \qquad i = 1,2 \ldots C \qquad // \ C = s+m \\ &\text{or} \qquad \text{(D.6)} \\ &\Sigma_{k=1}^{N} \mathbf{B}_{ik} \bullet \dot{\mathbf{r}}_k + B_{it} = 0 \qquad i = 1,2 \ldots C \ , \end{aligned}$$

where

$$\begin{array}{llllll} \mathbf{B}_{ik} & = \mathbf{A}_{ik} & B_{it} = A_{it} & \text{for } i = 1,2 \ldots s & \text{non-holonomic} & \\ \mathbf{B}_{ik} & = \nabla^{(k)} a_i & B_{it} = \partial_t a_i & \text{for } i = s+1, s+2 \ldots C & \text{holonomic} \quad . & \text{(D.7)} \end{array}$$

In general, the functions $a_i$, $\mathbf{B}_{ik}$ and $B_{it}$ are all functions of $(\mathbf{r}_1, \mathbf{r}_2 \ldots \mathbf{r}_k, t)$.

In the above we have followed the notation of Ray and Shamanna (2006) and we continue our path in the manner they propose.

Suppose the set of N differentials $\{d\mathbf{r}_k\}$ satisfies all of equations (D.6) for some time interval dt. Such a set $\{d\mathbf{r}_k\}$ is then called a set of **allowed displacements**. Imagine that $\{d\mathbf{r'}_k\}$ is some other distinct allowed displacement set. Elements of the difference set defined as

$$\{\delta\mathbf{r}_k\} = \{d\mathbf{r}_k\} - \{d\mathbf{r'}_k\} \qquad k = 1,2...N \tag{D.8}$$

are then called **virtual displacements**. This is how Ray and Shamanna define the vague term virtual displacement which appears in many texts (about which they have much to say in their Section I A). It follows then that for any virtual displacement set $\{\delta\mathbf{r}_k\}$ equations (D.6) take this form

$$\Sigma_{k=1}^{N} \mathbf{B}_{ik} \bullet \delta\mathbf{r}_k = 0 \qquad i = 1,2....C \tag{D.9}$$

simply because the last term in (D.6) cancels out when one writes $\delta\mathbf{r}_k = d\mathbf{r}_k - d\mathbf{r'}_k$. Notice that the $\delta\mathbf{r}_k$ are not arbitrary differential displacements. They are differences of allowed displacements.

Meanwhile, Newton's Law for a system of N particles subject to constraints has the form

$$m\ddot{\mathbf{r}}_k = \mathbf{F}_k + \mathbf{R}_k \qquad k = 1,2...N \tag{D.10}$$

where $\mathbf{R}_k$ is the sum of all constraint forces applied to particle k and $\mathbf{F}_k$ is the sum of all *other* forces applied to particle k.

Assumption that constraints do no virtual work

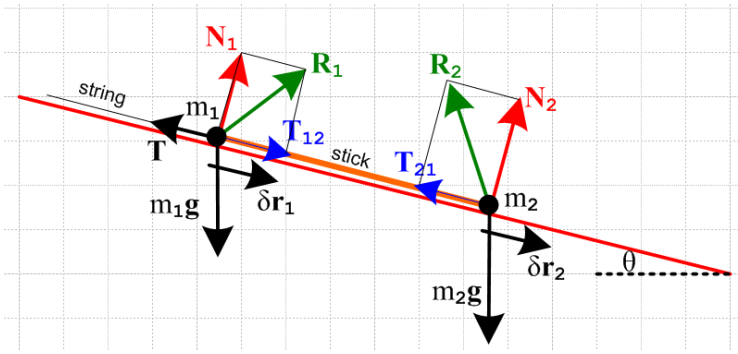
Next, *assume* that ( R&S refer to this as being an "ideal constraint" situation),

$$0 = \Sigma_{k=1}^{N} \mathbf{R}_k \bullet \delta\mathbf{r}_k \; . \tag{D.11}$$

In Goldstein this equation corresponds to setting the second term in (1-40) to zero. The main justification for making this assumption is that it is valid for particles which form a rigid body and for many other constraint situations. There seems to be no general justification of this equation for *all* constraint situations, and it is certainly not valid if friction is present. The equation can be interpreted as saying that the **total virtual work** done by all the constraint forces on all the particles of the system vanishes.

As a first simple example, consider two particles 1 and 2 independently sliding down a frictionless static ramp. In this case for particle 1 the constraint force $\mathbf{R}_1$ is the normal force of the ramp acting on the particle, and this is clearly perpendicular to $d\mathbf{r}_1$ for any allowed displacement set $\{d\mathbf{r}_k\}$, and thus $\mathbf{R}_1$ is also perpendicular to any $\delta\mathbf{r}_1$ which is part of any virtual displacement set $\{\delta\mathbf{r}_k\}$. In this case we have $\mathbf{R}_1 \bullet \delta\mathbf{r}_1 = 0$ and $\mathbf{R}_2 \bullet \delta\mathbf{r}_2 = 0$ so the terms in (D.11) are separately zero.

A second example is more interesting. Particles 1 and 2 of arbitrary masses slide down a one dimensional ramp but are glued to the two ends of a massless stick so they comprise a rigid body. The upper mass is attached to a string which provides a constant tension T. Here is a picture,




(D.12)

Particle 2 exerts a constraint force $\mathbf{T_{12}}$ on particle 1 through the stick, and Particle 1 exerts a constraint force $\mathbf{T_{21}}$ on particle 2. Since there is only one force of tension (or compression) in such a stick, $T_{12} = T_{21}$. All allowed displacement sets $\{d\mathbf{r_1}, d\mathbf{r_2}\}$ are of the form $\{d\mathbf{r}, d\mathbf{r}\}$, and therefore all *virtual* displacement sets are of the form $\{\delta\mathbf{r_1}, \delta\mathbf{r_2}\} = \{\delta\mathbf{r}, \delta\mathbf{r}\}$. In (D.12) we have drawn the two normal forces $\mathbf{N_i}$ and the two *total* constraint forces $\mathbf{R_i}$. It is no longer true that $\mathbf{R_1} \bullet \delta\mathbf{r_1} = 0$ and $\mathbf{R_2} \bullet \delta\mathbf{r_2} = 0$ as the picture shows. However, it is true that $\Sigma_{k=1}^{2}\, \mathbf{R_k} \bullet \delta\mathbf{r_k} = [\, \Sigma_{k=1}^{2}\, \mathbf{R_k}\,] \bullet \delta\mathbf{r} = 0$ because $\mathbf{R_1}$ and $\mathbf{R_2}$ have equal and opposite components along the ramp, so equation (D.11) is valid for this example.

Now suppose the ramp is translating in some direction not along the ramp. The displacements $d\mathbf{r_1} = d\mathbf{r_2}$ are then no longer along the ramp so that $\Sigma_{k=1}^{2}\, \mathbf{R_k} \bullet d\mathbf{r_k} \neq 0$. However, any virtual displacement set would still have $\delta\mathbf{r_1} = \delta\mathbf{r_2}$ along the ramp and then again $\Sigma_{k=1}^{2}\, \mathbf{R_k} \bullet \delta\mathbf{r_k} = 0$. Remember that a virtual displacement set is the difference of any two allowed displacement sets, and in this subtraction the effect of the ramp translation cancels out. So equation (D.11) is still valid.

Other examples are given in Ray and Shamanna Section III.

Reader Exercise: show for (D.12) that $T_{12} = T\, m_2 / (m_1+m_2)$ and that this makes sense if $m_1 >> m_2$ and vice versa.

We proceed then assuming our system of interest respects equation (D.11).

Comment on internal forces

In (D.10) we have classified the forces acting on particle k into two groups and we write $\mathbf{R_k} + \mathbf{F_k}$, where $\mathbf{R_k}$ are "constraint" forces and the $\mathbf{F_k}$ are any "other" forces. Goldstein refers to the $\mathbf{F_k}$ forces as "applied" forces in (1-39), while Ray and Shamanna call them "external" in Sec IV. Consider a system consisting of N masses all connected by a network of massless springs. The force acting on particle k due to all the springs to which it is attached would be best classified in the $\mathbf{F_k}$ "other" category. Were we to classify this force into the $\mathbf{R_k}$ group, we could not use the no-constraint-work property (D.11) since spring forces do work as the springs compress and expand. An example of such a system is the Solar System where the spring forces are represented by gravity. It would seem a misnomer to classify the internal forces in this case as "applied" or "external". However, if all the springs are slowly adjusted until they become infinitely stiff, in that limit it would be better to classify the internal spring forces into the $\mathbf{R_k}$ constraint forces group, since these springs do no work and (D.11) applies. The system is then a "rigid body" and in this case, with the internal forces classified as $\mathbf{R_k}$ constraint forces, one benefits from the many simplifications which arise for rigid body mechanics. The internal forces in our spring example fall into

the category where $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ and $\mathbf{F}_{ij}$ is in the direction $\mathbf{r}_i - \mathbf{r}_k$ . When this is not the case, the rigid body limit does not apply, but the internal forces can still be classified as "constraint" plus "other".

An Application of Lagrange Multipliers

Recall the C = m+s constraint equations (D.9),

$$\Sigma_{k=1}^{N} \mathbf{B}_{ik} \bullet \delta\mathbf{r}_k = 0 \qquad i = 1,2...C . \qquad // \; C \equiv \text{ m+s = total number of constraints} \qquad (D.9)$$

Multiply each of these C equations by a Lagrange Multiplier $\lambda_i$ and then add the equations to get

$$0 = \Sigma_{i=1}^{C} \lambda_i \; [\Sigma_{k=1}^{N} \mathbf{B}_{ik} \bullet \delta\mathbf{r}_k] = 0$$

$$= \Sigma_{k=1}^{N} [\; \Sigma_{i=1}^{C} \lambda_i \; \mathbf{B}_{ik} \;] \bullet \; \delta\mathbf{r}_k \; . \qquad (D.13)$$

Now subtract equation (D.13) from (D.11) to get

$$0 = \Sigma_{k=1}^{N} [\; \mathbf{R}_k - \Sigma_{i=1}^{C} \lambda_i \; \mathbf{B}_{ik}] \bullet \delta\mathbf{r}_k \; . \qquad (D.14)$$

We know that the N vectors in the set $\{\delta\mathbf{r}_k\}$ are not linearly independent because they must satisfy the equations (D.9), so we cannot claim from (D.14) that $[\; \mathbf{R}_k - \Sigma_{i=1}^{C} \lambda_i \; \mathbf{B}_{ik}] = 0$ for each k. But now write out equation (D.14) in full detail, showing all vector components,

$$0 = \Sigma_{k=1}^{N} \; \Sigma_{j=1}^{3} [\; (\mathbf{R}_k)_j - \Sigma_{i=1}^{C} \lambda_i \; (\mathbf{B}_{ik})_j] \, (\delta\mathbf{r}_k)_j \; . \qquad (D.15)$$

Next, relabel the 3N terms in this sum using a single index n defined in this way

$$\begin{array}{l} (k,j) = (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), \ldots\ldots\ldots (N,1), \; (N,2), \; (N,3) \\ \quad n \; = \; 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6 \; , \qquad\qquad 3N\text{-}2, \; 3N\text{-}1, \; 3N \end{array}$$

so (D.16)

$$n = 3(k-1)+j \qquad k = 1 + \text{Int}[(n-1)/3] \qquad j = 1 + \text{Rem}[(n-1)/3] \; .$$

Then define (here the n value is implied by the k,j labels as per above)

$$x_n \equiv (\mathbf{r}_k)_j \qquad R_n \equiv (\mathbf{R}_k)_j \qquad B_{in} \equiv (\mathbf{B}_{ik})_j \quad . \qquad (D.17)$$

Equation (D.15) can then be expressed as

$$0 = \Sigma_{n=1}^{3N} [\; R_n - \Sigma_{i=1}^{C} \lambda_i \, B_{in}] \, \delta x_n \qquad (D.18)$$

and (D.9) becomes $0 = \Sigma_{k=1}^{N} \; \Sigma_{j=1}^{3} \; (\mathbf{B}_{ik})_j \, (\delta\mathbf{r}_k)_j$ or

$$0 = \; \Sigma_{n=1}^{3N} B_{in} \; \delta x_n \qquad i = 1,2...C \; . \qquad (D.19)$$

Once again, we cannot set the square brackets in (D.18) to 0 because the 3N coordinates $\delta x_n$ are not linearly independent due to the C equations (D.19).

At any given time t, due to these equations (D.19), there must exist at least one subset of the 3N variables $\delta x_n$ which are independent and the remaining C $\delta x_n$ are dependent. Assume that the dependent variables have indices $n = n_1, n_2,...n_C$. Consider then the following set of C equations involving the dependent-term square brackets in (D.18),

$$[\ R_n - \Sigma_{i=1}^{C} \lambda_i B_{in}] = 0 \qquad n = n_1, n_2,...n_C \ . \tag{D.20}$$

This represents C linear equations in the C parameters $\lambda_i$ so, when all is said and done, there must exist a set of values $\{\lambda_i\}$ which make these C equations be true. We assume these values will be $\lambda_i(t)$ and this is justified below when the full set of equations is considered.

Now the terms in (D.18) indicated by $n = n_1, n_2,...n_C$ all vanish due to (D.20), so (D.18) can be written,

$$0 = \Sigma_{n \neq n_1, n_2, .... n_C} [\ R_n - \Sigma_{i=1}^{C} \lambda_i B_{in}] \delta x_n \ . \tag{D.21}$$

But all the $\delta x_n$ appearing in (D.21) are independent, so the square brackets here must also vanish. We then end up with *all* the square brackets in (D.18) being 0 (assuming the correct solution values of $\lambda_i$),

$$[\ R_n - \Sigma_{i=1}^{C} \lambda_i B_{in}] = 0 \qquad n = 1,2....3N$$

or

$$R_n = \Sigma_{i=1}^{C} \lambda_i \ B_{in} \qquad n = 1,2....3N$$

or

$$(\mathbf{R}_k)_j = \Sigma_{i=1}^{C} \lambda_i \ (\mathbf{B}_{ik})_j \qquad k = 1,2..N \qquad j = 1,2,3$$

or

$$\mathbf{R}_k = \Sigma_{i=1}^{C} \lambda_i \ \mathbf{B}_{ik} \qquad k = 1,2...N \ . \tag{D.22}$$

Thus the constraint forces are certain linear combinations of the $\mathbf{B}_{ik}$ constraint functions shown in (D.7).

Meanwhile, Newton's Law (D.10) says that $m_k\mathbf{a}_k = \mathbf{F}_k + \mathbf{R}_k$ where the $\mathbf{F}_k$ are the non-constraint forces for the problem. Inserting (D.22) gives

$$m_k\ddot{\mathbf{r}}_k = \mathbf{F}_k + \Sigma_{i=1}^{C} \lambda_i \mathbf{B}_{ik} \ . \qquad \mathbf{B}_{ik} = \begin{cases} \mathbf{A}_{ik} & i = 1 \text{ to m (non-holonomic)} \\ [\nabla^{(k)} a_i] & i = m+1 \text{ to C (holonomic)} \end{cases} \tag{D.23}$$

The problem is then summarized in the following set of equations from (D.23) and (D.6),

$$m_k\dot{\mathbf{v}}_k(t) = \mathbf{F}_k(\mathbf{r}_1, \mathbf{r}_2..... \mathbf{r}_N, t) + \Sigma_{i=1}^{C} \lambda_i(t)\mathbf{B}_{ik}(\mathbf{r}_1, \mathbf{r}_2..... \mathbf{r}_N, t) \qquad k = 1..N \qquad 3N \text{ equations}$$

$$\dot{\mathbf{r}}_k(t) = \mathbf{v}_k(t) \qquad k = 1..N \qquad 3N \text{ equations}$$

$$\Sigma_{k=1}^{N} \mathbf{B}_{ik}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) \bullet \dot{\mathbf{r}}_k(t) + B_{it}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) = 0 \qquad i = 1,2....C \ . \quad C \text{ equations}$$

(D.24)

This can be regarded as a set of 6N+C scalar first-order ODEs in the single variable t for the 6N+C unknown scalar functions $\mathbf{r}_k(t)$, $\mathbf{v}_k(t)$ and $\lambda_i(t)$. Although the first 3N equations happen to be linear in the $\lambda_i(t)$, the equations are in general non-linear in the $\mathbf{r}_k(t)$ so this is then a *non-linear* system of ODE's. Nevertheless, a well known existence theorem (e.g., Ince Section 3.3) states that, provided the $\mathbf{F}_k$, $\mathbf{B}_{ik}$ and $B_{it}$ functions are continuous and differentiable in all their arguments, a unique solution exists for any set of initial values $\mathbf{r}_k(0)$, $\mathbf{v}_k(0)$ and $\lambda_i(0)$. As (D.22) shows, the $\lambda_i(0)$ are related to the constraint forces at time 0.

The uniqueness of the solution agrees with our intuition that a physical system will evolve in a unique manner. Finding an analytic form for the solution might not be possible, but a numeric form can always be obtained.

Summary of the above Lagrange Multiplier Application

We have used a set of Lagrange Multipliers as an aid to solving a mechanics problem involving N particles with C constraints. As already noted, the above presentation is based on Section IV of Ray and Shamanna. The steps were as follows, considered more generically:

1. One has a sum $\Sigma_{n=1}^{N} R_n \delta x_n = 0$. (D.25)

2. One also has a set of sums $\Sigma_{n=1}^{N} B_{in} \delta x_n = 0$ for i = 1,2...C with C < N (constraints in the above case). This implies that one can find C of the $\delta x_n$ which are linearly dependent on the other $\delta x_n$.

3. If $\lambda_i$ are C arbitrary factors, certainly $\Sigma_{i=1}^{C} \lambda_i (\Sigma_{n=1}^{N} B_{in} \delta x_n) = 0$ based on 2. These $\lambda_i$ are "Undetermined Lagrange Multipliers".

4. Subtract sum 3 from sum 1 to get the new sum $\Sigma_{n=1}^{N} [ R_n - \Sigma_{i=1}^{C} \lambda_i B_{in}] \delta x_n = 0$.

5. The C $\lambda_i$ are selected so that [..] = 0 in the C dependent $\delta x_n$ terms of this sum. The remaining sum has only independent $\delta x_n$ terms so [..] = 0 for all those terms as well. Then *all* [...] = 0 in the sum of step 4.

6. The conclusion is that $[ R_n - \Sigma_i \lambda_i B_{in}] = 0$ for all n, assuming the solution $\lambda_i$ values of step 5.

In contrast with our main topic of finding stationary points df = 0 of a function f(**r**), in the Lagrange Multiplier application just presented there is no explicit function f(**r**) we are making stationary. On the other hand, one can regard (D.11) that $\Sigma_k \mathbf{R}_k \bullet \delta \mathbf{r}_k = 0$ as the statement $\delta W = 0$ where $\delta W = \Sigma_k \mathbf{R}_k \bullet \delta \mathbf{r}_k$ is the total differential "virtual work" done by the constraint forces during a differential virtual displacement $\{\delta \mathbf{r}_k\}$. Using Newton's law (D.10) this equation can then be written

$$0 = \delta W = \Sigma_k [m_k \mathbf{a}_k - \mathbf{F}_k] \bullet \delta \mathbf{r}_k \qquad \text{// D'Alembert's Principle} \qquad \text{(D.26)}$$

in which form it is known as **D'Alembert's Principle** (1743, Goldstein (1-42)). The analog of f(**r**) is then an action integral W of the differential virtual work which integral one renders stationary to get δW = 0. The notion of "the action" is mentioned at the end of this appendix.

We now present another application of the Lagrange Multiplier technique outlined in (D.25). As before, a certain amount of background is needed before the application can be demonstrated.

Generalized coordinates, generalized forces, and Lagrange's Equations

Usually the Cartesian components of the particle positions $\mathbf{r}_k$ are not the most convenient variables to use in solving a problem. Angles are often more useful. With C holonomic constraints one can replace the 3N variables $\{x_n\}$ in (D.17) with a set of 3N-C *independent* variables $\{q_n\}$ where

$$x_n = x_n(q_1, q_2, \ldots q_{3N-C}, t) \qquad n = 1,2..3N \qquad \text{(D.27)}$$

and one can then rework the above presentation in these new **generalized coordinates** $q_j$ which then won't always have the dimensions of length. For example, one then has, for virtual displacements,

$$\delta x_n = \Sigma_{j=1}^{3N-C} (\partial x_n / \partial q_j)\, \delta q_j \qquad n = 1,2..3N$$

or

$$\delta \mathbf{r}_k = \Sigma_{j=1}^{3N-C} (\partial \mathbf{r}_k / \partial q_j)\, \delta q_j \qquad k = 1,2..N \qquad \text{// as in Goldstein (1-44).} \qquad \text{(D.28)}$$

The usual presentation of Lagrangian dynamics with holonomic constraints *avoids* dealing with the forces of constraint and also avoids the need for using a set of Lagrange Multipliers. That presentation appears in Section V of the Ray and Shamanna and Goldstein p 17 (and many other places) and proceeds like this:

1. Start with $\Sigma_k \mathbf{R}_k \bullet \delta \mathbf{r}_k = 0$ in (D.11) which we write in the form of D'Alembert's Principle (D.26) mentioned above,

$$0 = \Sigma_{k=1}^{N} [m_k \mathbf{a}_k - \mathbf{F}_k] \bullet \delta \mathbf{r}_k \qquad \textit{(D.26)}$$

where $\mathbf{F}_k$ is the total non-constraint force acting on particle k.

2. Install (D.28) into D'Alembert's Principle to get

$$\begin{aligned} 0 &= \Sigma_{k=1}^{N} [m_k \mathbf{a}_k - \mathbf{F}_k] \bullet \Sigma_{j=1}^{3N-C} \left(\frac{\partial \mathbf{r}_k}{\partial q_j}\right) \delta q_j \\ &= \Sigma_{j=1}^{3N-C} \left\{ \Sigma_{k=1}^{N} m_k \mathbf{a}_k \bullet \frac{\partial \mathbf{r}_k}{\partial q_j} - \Sigma_{k=1}^{N} \mathbf{F}_k \bullet \left(\frac{\partial \mathbf{r}_k}{\partial q_j}\right) \right\} \delta q_j \\ &= \Sigma_{j=1}^{3N-C} \left\{ \Sigma_{k=1}^{N} m_k \mathbf{a}_k \bullet \frac{\partial \mathbf{r}_k}{\partial q_j} - Q_j \right\} \delta q_j \end{aligned} \qquad \text{(D.29)}$$

where

$$Q_j \equiv \Sigma_{k=1}^{N} \mathbf{F}_k \bullet \left(\frac{\partial \mathbf{r}_k}{\partial q_j}\right) \ . \qquad \text{(D.30)}$$

The $Q_j$ defined above are the **generalized forces** associated with the "generalized coordinates" $q_j$ .

3. Make use of the following non-obvious result (derived in footnote below starting at (D.40)),

$$\Sigma_{k=1}^{N} m_k \mathbf{a}_k \bullet \frac{\partial \mathbf{r}_k}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \qquad \text{(D.31)}$$

where T is the total kinetic energy of the N particles,

$$T \equiv (1/2)\Sigma_{k=1}^{N} m_k (\mathbf{v}_k \bullet \mathbf{v}_k) \ . \qquad \text{(D.32)}$$

Then (D.29) reads,

$$0 = \Sigma_{j=1}^{3N-C} \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j \right\} \delta q_j \ . \qquad \text{(D.33)}$$

Since the $\delta q_j$ are independent variables, {..} = 0 and we end up with one form of "Lagrange's Equations"

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \qquad j = 1,2....3N\text{-}C \ . \qquad \text{// as in Goldstein (1-50)} \qquad \text{(D.34)}$$

If the applied forces can be derived from a potential $\mathcal{V}(\mathbf{r}_1, \mathbf{r}_2, ..... \mathbf{r}_N)$ such that

$$\mathbf{F}_k = -\nabla^{(k)} \mathcal{V}(\mathbf{r}_1, \mathbf{r}_2, ..... \mathbf{r}_N) \qquad k = 1,2..N \qquad \text{// no t or } \dot{\mathbf{r}}_i \text{ dependence in } \mathcal{V} \qquad \text{(D.35)}$$

then from (D.30),

$$Q_j \equiv \Sigma_{k=1}^{N} \mathbf{F}_k \bullet \left(\frac{\partial \mathbf{r}_k}{\partial q_j}\right) = -\Sigma_{k=1}^{N} \left(\frac{\partial \mathcal{V}}{\partial \mathbf{r}_k}\right) \bullet \left(\frac{\partial \mathbf{r}_k}{\partial q_j}\right) = -\frac{\partial V}{\partial q_j} \qquad j = 1,2....3N\text{-}C \qquad \text{(D.36)}$$

where $V(q_i) \equiv \mathcal{V}(\mathbf{r}_k(q_i))$ is a function of the generalized coordinates $q_i$. Since V is not a function of the $\dot{q}_i$ we know that $\frac{\partial V}{\partial \dot{q}_j} = 0$. Using this fact, and (D.36) that $Q_j = -\frac{\partial V}{\partial q_j}$, (D.34) can be written

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0 \ . \qquad j = 1,2....3N\text{-}C \qquad \text{(D.37)}$$

The last step is to define the classical **Lagrangian**

$$L \equiv T - V \qquad \text{(D.38)}$$

to obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad j = 1,2....3N\text{-}C \quad . \qquad \text{// as in Goldstein (1-53)} \qquad (D.39)$$

which is the more conventional form of "Lagrange's Equations" with C holonomic constraints. The equations of (D.39) are usually called the **Euler-Lagrange Equations**.

One can define **generalized (canonical, conjugate) momenta** $p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$ so the Lagrange equations (D.39) are just $\dot{p}_j = \frac{\partial L}{\partial q_j}$ or $\dot{\mathbf{p}} = \nabla_{(q)} L$ . If $V = V(q_i)$, then $p_j = \frac{\partial T}{\partial \dot{q}_j}$ and then if $T = \frac{1}{2}\Sigma_k m_k \dot{q}_k^2$ one has the familiar result $p_j = m_j \dot{q}_j$ . When $T = T(\dot{q}_i)$ one finds $Q_j \equiv -\frac{\partial V}{\partial q_j} = \frac{\partial L}{\partial q_j} = \dot{p}_j$ so $\dot{\mathbf{p}} = \mathbf{Q}$, which is the **generalized Newton's Law** in terms of generalized coordinates and generalized forces.

Footnote: Derivation of the result (D.31) used above

Define the 3N-C independent generalized coordinates $q_i$ as in (D.27) according to

$$\mathbf{r}_k = \mathbf{r}_k(q_1, q_2....q_{3N-C}, t) \equiv \mathbf{r}_k(q_i, t) \; . \qquad (D.40)$$

Compute the total time derivative of $\mathbf{r}_k$ ,

$$\mathbf{v}_k = \dot{\mathbf{r}}_k = \frac{d\mathbf{r}_k}{dt} = \Sigma_j \frac{\partial \mathbf{r}_k}{\partial q_j}\frac{\partial q_j}{\partial t} + \frac{\partial \mathbf{r}_k}{\partial t} = \Sigma_j \frac{\partial \mathbf{r}_k}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_k}{\partial t} \quad . \qquad // = \mathbf{v}_k(q_i, \dot{q}_i, t) \qquad (D.41)$$

Then based on this result compute these two partial derivatives of $\mathbf{v}_k$ ,

$$\frac{\partial \mathbf{v}_k}{\partial q_i} = \Sigma_j \frac{\partial^2 \mathbf{r}_k}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 \mathbf{r}_k}{\partial q_i \partial t} \qquad (D.42)$$

$$\frac{\partial \mathbf{v}_k}{\partial \dot{q}_i} = \frac{\partial \mathbf{r}_k}{\partial q_i} \; . \qquad (D.43)$$

Next, the total kinetic energy of all k particles is given by (D.32),

$$T \equiv (1/2)\,\Sigma_k \; m_k \,(\mathbf{v}_k \bullet \mathbf{v}_k) \qquad \textit{(D.32)}$$

so that

$$\frac{\partial T}{\partial \dot{q}_i} = \Sigma_k\, m_k\, \mathbf{v}_k \bullet \frac{\partial \mathbf{v}_k}{\partial \dot{q}_i} = \Sigma_k\, m_k\, \mathbf{v}_k \bullet \frac{\partial \mathbf{r}_k}{\partial q_i} \quad . \qquad \text{// using (D.43)} \tag{D.44}$$

Apply d/dt to the above, with $\mathbf{a}_k = \ddot{\mathbf{r}}_k$ :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \Sigma_k\, m_k\, \mathbf{a}_k \bullet \frac{\partial \mathbf{r}_k}{\partial q_i} + \Sigma_k\, m_k\, \mathbf{v}_k \bullet \frac{d}{dt}\left(\frac{\partial \mathbf{r}_k}{\partial q_i}\right) \quad . \tag{D.45}$$

Compute the total time derivative appearing in the second term,

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_k}{\partial q_i}\right) = \Sigma_j \frac{\partial^2 \mathbf{r}_k}{\partial q_j \partial q_i}\dot{q}_j + \frac{\partial^2 \mathbf{r}_k}{\partial t\, \partial q_i} \quad . \tag{D.46}$$

Meanwhile,

$$\frac{\partial T}{\partial q_i} = \Sigma_k\, m_k\, \mathbf{v}_k \bullet \frac{\partial \mathbf{v}_k}{\partial q_i}$$

$$= \Sigma_k\, m_k\, \mathbf{v}_k \bullet \left[\, \Sigma_j \frac{\partial^2 \mathbf{r}_k}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 \mathbf{r}_k}{\partial q_i \partial t}\,\right] \qquad \text{// (D.42)}$$

$$= \Sigma_k\, m_k\, \mathbf{v}_k \bullet \frac{d}{dt}\left(\frac{\partial \mathbf{r}_k}{\partial q_i}\right) \quad . \qquad \text{// (D.46)} \tag{D.47}$$

Subtract (D.47) from (D.45) to get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \Sigma_k\, m_k\, \mathbf{a}_k \bullet \frac{\partial \mathbf{r}_k}{\partial q_i} \tag{D.48}$$

which is the result quoted above in (D.31).

Lagrange Equations with non-holonomic constraints

This is our final Lagrange Multiplier example, and it exactly follows the generic prescription of (D.25). The general topic is well described in Goldstein Chapter 2.

As the starting point we invoke **Hamilton's Principle** which is this :

$$\delta S = 0 \qquad \text{where} \qquad S = \int_{t1}^{t2} dt\, L(q_n(t), \dot{q}_n(t), t) \qquad n = 1,2....M \tag{D.49}$$

where there are M independent generalized coordinates $q_n$. The notation is a shorthand to imply that L is a function of *all* the $q_n(t)$ and *all* the $\dot{q}_n(t)$ functions. One might better write $L(\{q_n(t)\}, \{\dot{q}_n(t)\}, t)$ where the {..} indicate sets.

L = T - V is the Lagrangian (D.38) with $T(\dot{q}_i)$ and $V(q_i)$ being the kinetic and potential energies associated with the M generalized coordinates. We shall assume that M = N-D where there were D holonomic constraints on N initial problem coordinates. As shown in (D.60) below ( or see Goldstein p 41 (2-20') based on p 37 (2-15) ) the equation $\delta S = 0$ results in the following sum being 0,

1. $\Sigma_{n=1}^{M} \left[ \int_{t1}^{t2} dt \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \frac{\partial L}{\partial q_n} \right\} \right] \delta q_n(t) = 0 \ .$ (D.50)

The number 1. on the left refers to the first step outlined in (D.25). In this equation the $\delta q_n(t)$ are M independent functions of t ("path variations"), restricted only by $\delta q_n(t_1) = \delta q_n(t_2) = 0$. In order for (D.50) to be true for M arbitrary independent *functions* $\delta q_n(t)$, one must have {} = 0,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \frac{\partial L}{\partial q_n} = 0 \qquad n = 1,2....M \qquad (D.51)$$

which are the Euler-Lagrange Equations (D.39). Thus it is that the Lagrange Equations can be derived from Hamilton's Principle in the case of only holonomic constraints.

But suppose that in addition to the D holonomic constraints there are C non-holonomic constraints, so only M-C of the $\delta q_n$ are independent. In this case the above conclusion (D.51) is incorrect.

2. The C non-holonomic constraints result in C sums $\Sigma_n A_{ni} \delta q_n = 0$ i = 1,2...C analogous to (D.9) above.

3. If $\lambda_i$ are C arbitrary factors, certainly $\Sigma_i \lambda_i (\Sigma_n A_{in} \delta q_n) = 0$ based on 2. These $\lambda_i$ are "Undetermined Lagrange Multipliers". Then $\int_{t1}^{t2} dt \ ( \Sigma_i \lambda_i \Sigma_n A_{in} \delta q_n ) = 0$ as well.

4. Subtract sum 3 from sum 1 to get the new sum

$$\Sigma_{n=1}^{M} \left[ \int_{t1}^{t2} dt \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \frac{\partial L}{\partial q_n} - \Sigma_{i=1}^{C} \lambda_i A_{in} \right\} \right] \delta q_n(t) = 0 \ . \quad \text{// as in Goldstein (2-26)} \qquad (D.52)$$

5. The C $\lambda_i$ are selected so that {..} = 0 in the C dependent $\delta q_n(t)$ terms of this sum. The remaining sum has only independent $\delta q_n(t)$ terms so {..} = 0 for all those terms as well. Then *all* {...} = 0 in the sum of 4.

6. The conclusion is that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \frac{\partial L}{\partial q_n} = \Sigma_{i=1}^{C} \lambda_i A_{in} \ . \qquad n = 1,2....M \quad \text{// as in Goldstein (2-30)} \qquad (D.53)$$

This then is the corrected form of the Lagrange Equations in the presence of C non-holonomic constraints. Recalling from (D.7) that $\mathbf{B_{ik}} = \mathbf{A_{ik}}$ for non-holonomic constraints, the second line of (D.22) with $B_{in} = A_{in}$ reads $R_n = \Sigma_{i=1}^{C} \lambda_i A_{in}$ . Thus one can interpret the sum on the right of (D.53) as being the force of constraint $R_n$ arising from the C non-holonomic constraints. The forces of the D holonomic constraints are already incorporated into the left side of (D.53) in the reduction of independent coordinates $q_n$ from N to M = N-D.

The integral S shown in (D.49) is called "the action" and Hamilton's Principle is a classical mechanics incarnation of the **Principle of Least Action**. In relativistic quantum field theory the action is a spacetime

integral of the Lagrangian density $\mathcal{L}(\varphi_n(x^\mu), \partial_\mu\varphi_n(x^\mu))$ where the coordinates $q_n$ of L are replaced by fields $\varphi_n(x^\mu)$, and t is replaced by $x^\mu$, a point in spacetime. In this case the Lagrange Equations (D.39) take the form (the second line is for comparison),

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi_n)}\right) - \frac{\partial \mathcal{L}}{\partial \varphi_n} = 0 \quad \text{where} \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad \text{// Bjorken and Drell (11.30)} \qquad \text{(D.54)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \frac{\partial L}{\partial q_n} = 0 \qquad t \rightarrow x^\mu \qquad q_n(t) \rightarrow \varphi_n(x^\mu) \qquad \dot{q}_n(t) \rightarrow \partial_\mu\varphi_n(x^\mu) \qquad \textit{(D.39)}$$

<u>The Case of C holonomic constraints treated as if they were non-holonomic</u>

Suppose there are N generalized coordinates $q_i$ with C holonomic constraints and 0 non-holonomic constraints. As noted in (D.6) and (D.7), the holonomic constraints can be written in differential form so they look just like non-holonomic constraints. For generalized coordinates $q_n$ these two equations would appear as

$$\Sigma_{n=1}^{N} B_{in} dq_n + B_{it} dt = 0 \qquad i = 1,2....C \qquad \text{// } C = s+m$$

or (D.6)'

$$\Sigma_{n=1}^{N} B_{in} \dot{q}_n + B_{it} = 0 \qquad i = 1,2....C$$

where

$$\begin{array}{llllll} B_{in} = A_{in} & B_{it} = \mathbf{A}_{it} & \text{for } i = 1,2...s & \text{non-holonomic} & & \\ B_{in} = \partial a_i/\partial q_n & B_{it} = \partial_t a_i & \text{for } i = s+1, s+2....C & \text{holonomic} & . & \text{(D.7)'} \end{array}$$

Setting s = 0 we can carry out the procedure of the previous section with $\mathbf{A}_{ik}$ replaced by $\partial a_i/\partial q_k$ to obtain this result of rendering the action S stationary,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \frac{\partial L}{\partial q_n} = \Sigma_{i=1}^{C} \lambda_i \frac{\partial a_i}{\partial q_n} \qquad n = 1,2....N \qquad \text{(D.55)}$$

where the Lagrange multipliers $\lambda_i$ are chosen to make this equation be valid for a set of C dependent $q_n$ coordinates, as in Step 5 above. Only N-C of the coordinates $q_n$ are independent but the equations are written for all N coordinates. Here the functions $a_i(q_1, q_2, .....q_n, t) = 0$ are the C holonomic constraint equations.

Footnote: Carry out the $\delta S = 0$ functional variation of the action .

The variation in each function $q_n(t)$ is parameterized in terms of an arbitrary independent function $\eta_n(t)$ (but which vanishes at both time integration endpoints) and a scalar parameter $\alpha$ :

$$q_n(t,\alpha) = q_n(t,0) + \alpha\,\eta_n(t) \qquad \delta q_n(t) = \alpha\,\eta_n(t) \qquad \frac{\partial q_n}{\partial \alpha} = \eta_n(t) \qquad \eta_n(t_1) = \eta_n(t_2) = 0$$

$$\dot{q}_n(t,\alpha) = \dot{q}_n(t,0) + \alpha\,\dot{\eta}_n(t) \qquad \delta \dot{q}_n(t) = \alpha\,\dot{\eta}_n(t \qquad \frac{\partial \dot{q}_n}{\partial \alpha} = \dot{\eta}_n(t) \quad . \tag{D.56}$$

Then

$$S(\alpha) \equiv \int_{t1}^{t2} dt\, L(q_n(t,\alpha), \dot{q}_n(t,\alpha), t) \qquad \text{// action} \tag{D.49}$$

$$\frac{\partial S}{\partial \alpha} = \int_{t1}^{t2} dt\ \Sigma_n \left( \frac{\partial L}{\partial q_n} \frac{\partial q_n}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_n} \frac{\partial \dot{q}_n}{\partial \alpha} \right) = \int_{t1}^{t2} dt\ \Sigma_n \left( \frac{\partial L}{\partial q_n}\ \eta_n + \frac{\partial L}{\partial \dot{q}_n} \frac{d\eta_n}{dt} \right) \quad . \tag{D.57}$$

But for the second term do a parts integration,

$$\int_{t1}^{t2} dt \left( \frac{\partial L}{\partial \dot{q}_n} \frac{d\eta_n}{dt} \right) = \left[ \frac{\partial L}{\partial \dot{q}_n} \eta_n(t) \right]^{t2}_{t1} - \int_{t1}^{t2} dt \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} \right) \eta_n(t) \,. \tag{D.58}$$

The "parts" terms vanish since $\eta_n$ vanishes at both endpoints, so (D.57) becomes,

$$\frac{\partial S}{\partial \alpha} = \int_{t1}^{t2} dt\ \Sigma_n \left( \frac{\partial L}{\partial q_n} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} \right) \eta_n(t) \tag{D.59}$$

so

$$\delta S \equiv \frac{\partial S}{\partial \alpha} d\alpha = \left\{ \int_{t1}^{t2} dt\ \Sigma_n \left( \frac{\partial L}{\partial q_n} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} \right) \eta_n(t) \right\} d\alpha . (\alpha/\alpha)$$

$$= \left\{ \int_{t1}^{t2} dt\ \Sigma_n \left( \frac{\partial L}{\partial q_n} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} \right) \delta q_n(t) \right\} d\alpha/\alpha \quad . \tag{D.60}$$

Then,

$$\delta S = 0 \quad \Leftrightarrow \quad \int_{t1}^{t2} dt\ \Sigma_n \left[ \left( \frac{\partial L}{\partial q_n} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} \right) \eta_n(t) \right] = 0 \,.$$

or (D.61)

$$\delta S = 0 \quad \Leftrightarrow \quad \int_{t1}^{t2} dt\ \Sigma_n \left[ \left( \frac{\partial L}{\partial q_n} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} \right) \delta q_n(t) \right] = 0 \,.$$

But the functions $\delta q_n(t,\alpha) = \alpha\eta_n(t)$ are arbitrary and independent functions of t, so the integral can only vanish if the function in the parentheses vanishes for each term in the sum on n, so

$$\delta S = 0 \qquad \Leftrightarrow \qquad \left( \frac{\partial L}{\partial q_n} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} \right) = 0, \quad n = 1,2...m \tag{D.62}$$

which are the Euler-Lagrange Equations (D.39) and (D.51). For more detail see Goldstein Chapter 2.

**References**

The method of Lagrange multipliers often makes cameo appearances in textbooks on the calculus of multiple variables. There are also several pages and pdf's on the web, search on "Lagrange Multipliers". Whole books on the use the Lagrange multipliers for specialized applications may be found at the usual book vendor websites. The links below were last verified Oct 4, 2016.


J.D. Bjorken and S.D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

R.C. Buck with E.F. Buck, *Advanced Calculus*, 2nd Ed. (McGraw-Hill, New York, 1965). The discussion of extremum problems with constraints occupies pp 359-363. In the 3rd Edition (McGraw-Hill, New York, 1978), reprinted by (Waveland Press, Long Grove IL, 2003), this discussion has moved to pages 536-540. This excellent and now classic book was first published in 1956.

H. Goldstein, *Classical Mechanics* (Addison-Wesley, Boston, 1950), another classic. There is a 3rd edition 2001 by Goldstein, Safco and Poole, but our page and equation references are to the 1950 edition.

E.L. Ince, *Ordinary Differential Equations* (Longmans, Green & Co. (Ltd.), London, 1927). This classic text became a Dover paperback in 1956 and is available online for $4.

S. Jensen, *An Introduction to Lagrange Multipliers,*
http://www.slimy.com/~steuard/teaching/tutorials/Lagrange.html ( includes video lectures)

P. Lucht, *Tensor Analysis and Curvilinear Coordinates* (2016), http://user.xmission.com/~rimrock . If not there, search on "Phil Lucht Documents".

P.M. Morse and H. Feshbach, *Methods of Theoretical Physics* ( McGraw-Hill, New York, 1953).

S. Ray and J. Shamanna, "On Virtual Displacement and Virtual Work in Lagrangian Dynamics", European Journal of Physics, 27 (2006) 311-329. Also https://arxiv.org/abs/physics/0510204v2 .

G.E. Shilov, *Linear Algebra* (Dover Publications, 1977).

M.R. Spiegel, *Mathematical Handbook of Formulas and Tables* ( Schaum's Outline Series, McGraw-Hill, New York, 1968). This is now at 4th Ed (2012) with two coauthors and new reference numbers.

W. F. Trench, *The Method of Lagrange Multipliers* (2013), http://digitalcommons.trinity.edu/mono/7/ , Supplement 2 (31p). This paper also has a "Theorem 1" which is related to but different from ours.

Wiki on Lagrange Multipliers: https://en.wikipedia.org/wiki/Lagrange_multiplier

M.W. Zemansky, *Heat and Thermodynamics* 5th Ed (McGraw-Hill, New York, 1968). Our page references are to this edition, but there is a 7th Ed. (1997) with coauthor R.H Dittman and an 8th Ed (2011) in the Special Indian Edition series.