The curvature calculation PhL 10.14.16

I have this all cleaned up and installed into new Lagrange. The result is the same as my original result. Keeping this only for the record, no need to read what is below.

Here is how I start off:

5.3 More details of the solution

Start with f and its partial derivatives,

$$f = \ln\Omega = \Sigma_i [(1+\ln g_i)N_i - N_i \ln N_i] = \Sigma_i N_i [1 + \ln(g_i/N_i)] \qquad (5.2.3)$$

$$f_i = \ln(g_i/N_i) \qquad // f_i \equiv \partial f/\partial N_i \qquad (5.2.6)$$

The constraints $\Sigma_{i=1}^{m} N_i = M$ and $\Sigma_{i=1} N_i \varepsilon_i = U$ can be regarded as two equation in the two unknowns N_1 and N_2 which are easily solved to obtain,

$$\begin{aligned} N_1 &= +(\varepsilon_2 - \varepsilon_1)^{-1}[\varepsilon_2 M - U + \Sigma_{i=3}^{m} (\varepsilon_i - \varepsilon_2)N_i] &&= N_1(N_3, \ldots N_m) \\ N_2 &= -(\varepsilon_2 - \varepsilon_1)^{-1}[\varepsilon_1 M - U + \Sigma_{i=3}^{m} (\varepsilon_i - \varepsilon_1)N_i] &&= N_2(N_3, \ldots N_m) \ . \end{aligned} \qquad (5.3.1)$$

Notice that,

$$\begin{aligned} \frac{\partial N_1}{\partial N_i} &= +(\varepsilon_2 - \varepsilon_1)^{-1}[(\varepsilon_i - \varepsilon_2)] = \frac{\varepsilon_i - \varepsilon_2}{\varepsilon_2 - \varepsilon_1} && i = 3,4..m \\ \frac{\partial N_2}{\partial N_i} &= -(\varepsilon_2 - \varepsilon_1)^{-1}[(\varepsilon_i - \varepsilon_1)] = -\frac{\varepsilon_i - \varepsilon_1}{\varepsilon_2 - \varepsilon_1} \ . && i = 3,4..m \end{aligned} \qquad (5.3.2)$$

Now, using (5.3.1), think of f as function F of m-2 *independent* variables of $N_3,,,,N_m$ as follows,

$$F(N_3,N_4.....N_m) \equiv f(N_1(N_3, \ldots N_m), N_2(N_3, \ldots N_m), N_3, ...N_m) \ . \qquad (5.3.3)$$

F has the same value as f but has a different functional form.

What are the implications of this last definition? How do you rewrite earlier equations in terms of F?

$$\begin{aligned} F(N_3,N_4.....N_m) &= N_1(N_3, \ldots N_m)[1 + \ln(g_1) - \ln N_1(N_3, \ldots N_m)] \\ &+ N_2(N_3, \ldots N_m)[1 + \ln(g_2) - \ln N_1(N_3, \ldots N_m)] \\ &+ \Sigma_{j=3}^{m} N_j [1 + \ln(g_j) - \ln N_j] \end{aligned}$$

where now I expose functional dependency. Now derivatives are very complicated. Letting i = 3,4.... get

$$\frac{\partial F}{\partial N_i} = \frac{\partial N_1}{\partial N_i}\,[\,1 + \ln(g_1) - \ln N_1(N_3,\ldots N_m)\,] \;-\; N_1(N_3,\ldots N_m)\,[\,1/N_1(N_3,\ldots N_m)\,]\,\frac{\partial N_1}{\partial N_i}$$
$$+\,\frac{\partial N_2}{\partial N_i}\,[\,1 + \ln(g_2) - \ln N_2(N_3,\ldots N_m)\,] \;-\; N_2(N_3,\ldots N_m)\,[\,1/N_2(N_3,\ldots N_m)\,]\,\frac{\partial N_2}{\partial N_i}$$

$$+\,\Sigma_{j=3}^{m}\,\{\,\delta_{i,j}\,[\,1 + \ln(g_j) - \ln N_j\,] \;-\; N_j\,[\,1/N_j\,]\,\delta_{i,j}\,\}$$

I can compact this down a bit

$$\frac{\partial F}{\partial N_i} = \frac{\partial N_1}{\partial N_i}\,[\,1 + \ln(g_1) - \ln N_1(N_3,\ldots N_m)\,] \;-\; \frac{\partial N_1}{\partial N_i}$$
$$+\,\frac{\partial N_2}{\partial N_i}\,[\,1 + \ln(g_2) - \ln N_2(N_3,\ldots N_m)\,] \;-\; \frac{\partial N_2}{\partial N_i}$$

$$+\,\{\,[\,1 + \ln(g_i) - \ln N_i\,] \;-\; 1\,\}$$

and then still more

$$\frac{\partial F}{\partial N_i} = \frac{\partial N_1}{\partial N_i}\,[\,0 + \ln(g_1) - \ln N_1(N_3,\ldots N_m)\,]$$
$$+\,\frac{\partial N_2}{\partial N_i}\,[\,0 + \ln(g_2) - \ln N_2(N_3,\ldots N_m)\,]$$

$$+\,[\,0 + \ln(g_i) - \ln N_i\,] \qquad\qquad i = 3,4....$$

Now what happens if I insert the two derivatives shown

$$\frac{\partial F}{\partial N_i} = \frac{\varepsilon_i - \varepsilon_2}{\varepsilon_2 - \varepsilon_1}\,[\,0 + \ln(g_1) - \ln N_1(N_3,\ldots N_m)\,]$$
$$-\;\frac{\varepsilon_i - \varepsilon_1}{\varepsilon_2 - \varepsilon_1}\,[\,0 + \ln(g_2) - \ln N_2(N_3,\ldots N_m)\,]$$

$$+\,[\,0 + \ln(g_i) - \ln N_i\,] \qquad\qquad i = 3,4....$$

which I rewrite as

$$\frac{\partial F}{\partial N_i} = \frac{\varepsilon_i - \varepsilon_2}{\varepsilon_2 - \varepsilon_1}\,[\,0 + \ln(g_1/N_1(N_3,\ldots N_m))\,] \;-\; \frac{\varepsilon_i - \varepsilon_1}{\varepsilon_2 - \varepsilon_1}\,[\,0 + \ln(g_2/N_2(N_3,\ldots N_m))\,] \;+\,[\,0 + \ln(g_i/N_i)\,]$$

Now recall this earlier result

$$\ln(N_i/g_i) = \lambda_1 + \lambda_2\varepsilon_i \qquad i = 1,2....m \qquad\qquad (5.2.9)$$

This is valid only at the solution N_i point, and it is valid for all values of i. So insert into the above

$$\frac{\partial F}{\partial N_i} = \frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\,[\,0 - \lambda_1 - \lambda_2\varepsilon_1\,] \;-\; \frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\,[\,0 - \lambda_1 - \lambda_2\varepsilon_2\,] \;+[\,0 - \lambda_1 - \lambda_2\varepsilon_i\,]$$

$$= \lambda_1\,[\,-\frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1} + \frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1} - 1\,] + \lambda_2\,[\,-\frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\,\varepsilon_1 + \frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\,\varepsilon_2 - \varepsilon_i\,] + [\,0\,\frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1} - 0\,\frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1} + 0\,]$$

Now add a factor $1 = \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2-\varepsilon_1}$ into the last term in each square bracket,

$$\frac{\partial F}{\partial N_i} = \lambda_1\,[\,-\frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1} + \frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1} - \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\,] + \lambda_2\,[\,-\frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\,\varepsilon_1 + \frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\,\varepsilon_2 - \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\,\varepsilon_i\,]$$
$$+ [\,0\,\frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1} - 0\,\frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1} + 0\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\,]$$

$$= \frac{\lambda_1}{\varepsilon_2-\varepsilon_1}\,[\,-(\varepsilon_i-\varepsilon_2) + (\varepsilon_i-\varepsilon_1) - (\varepsilon_2-\varepsilon_1)\,] + \frac{\lambda_2}{\varepsilon_2-\varepsilon_1}\,[\,-(\varepsilon_i-\varepsilon_2)\varepsilon_1 + (\varepsilon_i-\varepsilon_1)\varepsilon_2 - (\varepsilon_2-\varepsilon_1)\varepsilon_i\,]$$
$$+ \frac{2}{\varepsilon_2-\varepsilon_1}\,[\,(\varepsilon_i-\varepsilon_2) - (\varepsilon_i-\varepsilon_1) + (\varepsilon_2-\varepsilon_1)\,]\,0$$

$$= \frac{\lambda_1}{\varepsilon_2-\varepsilon_1}\,[\,-\varepsilon_i+\varepsilon_2 + \varepsilon_i-\varepsilon_1 - \varepsilon_2+\varepsilon_1)\,] + \frac{\lambda_2}{\varepsilon_2-\varepsilon_1}\,[\,-\varepsilon_i\varepsilon_1 + \varepsilon_2\varepsilon_1 + \varepsilon_i\varepsilon_2 - \varepsilon_1\varepsilon_2 - \varepsilon_2\varepsilon_i + \varepsilon_1\varepsilon_i\,]$$
$$+ \frac{2}{\varepsilon_2-\varepsilon_1}\,[\,\varepsilon_i-\varepsilon_2 - \varepsilon_i+\varepsilon_1 + \varepsilon_2-\varepsilon_1\,]\,0$$

$$= \frac{\lambda_1}{\varepsilon_2-\varepsilon_1}\,[\,0\,] + \frac{\lambda_2}{\varepsilon_2-\varepsilon_1}\,[\;\;]$$
$$+ \frac{2}{\varepsilon_2-\varepsilon_1}\,[\,0\,]\,0$$

$$= 0$$

Wow. This finally confirms what I want which is that the partial vanishes at the stationary point solution **N**. Now we can go on to attempt the second derivative starting with this result valid for i = 3,4....

$$\frac{\partial F}{\partial N_i} = \frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\,[\,2 + \ln(g_1) - \ln N_1(N_3,\ldots N_m)\,]$$
$$- \frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\,[\,2 + \ln(g_2) - \ln N_2(N_3,\ldots N_m)\,]$$
$$+ [\,2 + \ln(g_i) - \ln N_i\,] \qquad\qquad i = 3,4....$$

Then :

$$\frac{\partial^2 F}{\partial^2 N_i} = \frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\left[-1/N_1 * \frac{\partial N_1}{\partial N_i}\right] - \frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\left[-1/N_2 * \frac{\partial N_2}{\partial N_i}\right] + [-1/N_i]$$

Now insert (5.3.2) to get

$$\frac{\partial^2 F}{\partial^2 N_i} = \frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\left[-1/N_1 * \frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\right] - \frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\left[+1/N_2 * \frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\right] + [-1/N_i]$$

$$= -\left(\frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\right)^2 /N_1 - \left(\frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\right)^2 /N_2 - 1/N_i$$

$$- \left[\left(\frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\right)^2 /N_1 + \left(\frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\right)^2 /N_2 + 1/N_i \right]$$

and this is exactly the result I got in the original Lagrange doc!!! But this time I have shown an actual derivation, not just some words.

Try to compact down the algebra shown abovwe

$$F(N_3,N_4.....N_m) = N_1[1 + \ln(g_1) - \ln N_1] + N_2 [1 + \ln(g_2) - \ln N_2] + \Sigma_{j=3}^{m} N_j [1 + \ln(g_j) - \ln N_j]$$

where now I expose functional dependency. Now derivatives are very complicated. Letting i = 3,4.... get

$$\frac{\partial F}{\partial N_i} = \frac{\partial N_1}{\partial N_i}[1 + \ln(g_1) - \ln N_1] + N_1 [1/N_1] \frac{\partial N_1}{\partial N_i}$$
$$+ \frac{\partial N_2}{\partial N_i}[1 + \ln(g_2) - \ln N_2] + N_2 [1/N_2] \frac{\partial N_2}{\partial N_i}$$
$$+ \Sigma_{j=3}^{m} \{ \delta_{i,j} [1 + \ln(g_j) - \ln N_j] + N_j [1/N_j] \delta_{i,j} \}$$

$$= \frac{\partial N_1}{\partial N_i}[2 + \ln(g_1) - \ln N_1] + \frac{\partial N_2}{\partial N_i}[2 + \ln(g_2) - \ln N_2] + [2 + \ln(g_i) - \ln N_i]$$

$$= \frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}[2 - \ln(N_1/g_1)] - \frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}[2 - \ln(N_2/g)] + [2 - \ln(N_i/g_i)] \qquad // (5.3.2)$$

As a check on the algebra, we evaluate the above expression at the stationary point N where we know that

$$\ln(N_i/g_i) = \lambda_1 + \lambda_2 \varepsilon_i \qquad (6.2.9)$$

Then at the stationary point we find (adding $1 = \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2-\varepsilon_1}$ to the third term)

$$\frac{\partial F}{\partial N_i} = \frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\,[\,2-\lambda_1-\lambda_2\varepsilon_1] - \frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\,[\,2-\lambda_1-\lambda_2\varepsilon_2] + \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\,[\,2-\lambda_1-\lambda_2\varepsilon_i]$$

$$= \frac{1}{\varepsilon_2-\varepsilon_1}\{(\varepsilon_i-\varepsilon_2)[\,2-\lambda_1-\lambda_2\varepsilon_1] - (\varepsilon_i-\varepsilon_1)[\,2-\lambda_1-\lambda_2\varepsilon_2] + (\varepsilon_2-\varepsilon_1)[\,2-\lambda_1-\lambda_2\varepsilon_i]\}$$

Inside the Curly Bracket all terms cancel, as Maple verifies

```
CB := (ei-e2)*(2-L1-L2*e1)-(ei-e1)*(2-L1-L2*e2)+(e2-e1)*(2-L1-L2*ei);
```

$$CB := (ei-e2)(2-L1-L2\,e1)-(ei-e1)(2-L1-L2\,e2)+(e2-e1)(2-L1-L2\,ei)$$

```
simplify(CB);
```

$$0$$

with the final result

$$\frac{\partial F}{\partial N_i} = 0 \qquad\qquad i = 3,4...m$$

Now compact down the second derivative algebra

$$\frac{\partial F}{\partial N_i} = \frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\,[\,0+\ln(g_1)-\ln N_1(N_3,\ldots N_m)]$$

$$-\frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\,[\,0+\ln(g_2)-\ln N_2(N_3,\ldots N_m)]$$

$$+[\,0+\ln(g_i)-\ln N_i] \qquad\qquad i = 3,4....$$

Then :

$$\frac{\partial^2 F}{\partial^2 N_i} = \frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\,[\,-1/N_1 * \frac{\partial N_1}{\partial N_i}\,] - \frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\,[\,-1/N_2 * \frac{\partial N_2}{\partial N_i}\,] + [\,-1/N_i\,]$$

Now insert (5.3.2) to get

$$\frac{\partial^2 F}{\partial^2 N_i} = \frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\,[\,-1/N_1 * \frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\,] - \frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\,[\,+1/N_2 * \frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\,] + [\,-1/N_i\,]$$

$$= -\left(\frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\right)^2/N_1 - \left(\frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\right)^2/N_2 - 1/N_i$$

$$-\left[\left(\frac{\varepsilon_i-\varepsilon_2}{\varepsilon_2-\varepsilon_1}\right)^2/N_1 + \left(\frac{\varepsilon_i-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\right)^2/N_2 + 1/N_i\,\right]$$

I can compact this down a bit

$$\frac{\partial F}{\partial N_i} = \frac{\partial N_1}{\partial N_i}\,[\,1 + \ln(g_1) - \ln N_1(N_3, \ldots N_m)] + \frac{\partial N_1}{\partial N_i}$$
$$+ \frac{\partial N_2}{\partial N_i}\,[\,1 + \ln(g_2) - \ln N_2(N_3, \ldots N_m)] + \frac{\partial N_2}{\partial N_i}$$