A slightly different view PhL 20.21.16

I was going to put this at the end of Section 5.5, but decided not to. I will explain why below. First here is the section I was going to add:

A slightly different view

In the above discussion, the functions N(v) and g(v) are densities in "speed space" v and that is the usual path one takes with the Maxwell distribution. One can alternatively define $\mathcal{N}(\mathbf{v})$ to be the density of particles in velocity space so that $N_i = \mathcal{N}(\mathbf{v})d^3v$. The degeneracy in this case is just $g_i = (d^3v/V)$ where V is the same tiny lattice volume in v-space we used before. Quantum theory tells us that $V = (\pi\hbar/mL)^3$ for our box of helium gas of edge L. Then

$$N_i = A\ g_i\ e^{-\beta\varepsilon_i} \Rightarrow \mathcal{N}(\mathbf{v})d^3v = A\ (1/V)\ d^3v \exp(-\beta m v^2/2) \Rightarrow$$

$$\mathcal{N}(\mathbf{v}) = (A/V) \exp(-\beta m v^2/2)$$

To eliminate the derived Lagrange multiplier A we write

$$M = \int_{\text{first octant}} \mathcal{N}(\mathbf{v})d^3v = \int \mathcal{N}(\mathbf{v})\, v^2 dv\, d\Omega = \int d\Omega \int \mathcal{N}(\mathbf{v})\, v^2 dv$$

$$= (1/8)\, 4\pi \int (A/V) \exp(-\beta m v^2/2)\, v^2 dv$$

$$= (\pi/2)\,(A/V)\ \{(1/2)\Gamma(3/2)\,(\beta m/2)^{-3/2}\} \qquad \text{// from ***}$$

$$= (\pi/2)\,(A/V)\,(1/2)(\sqrt{\pi}/2)\,(\beta m/2)^{-3/2}$$

$$= (\pi^{3/2}/8)\,(A/V)\,(\beta m/2)^{-3/2}.$$

Then

$$(A/V) = M\,(8\,\pi^{-3/2})\,(\beta m/2)^{3/2}$$

and so

$$\mathcal{N}(\mathbf{v}) = M\,(8\,\pi^{-3/2})\,(\beta m/2)^{3/2} \exp(-\beta m v^2/2)$$

Thus $\mathcal{N}(\mathbf{v})$ is really just a function of speed v, but we write it as $\mathcal{N}(\mathbf{v})$ as a reminder that it is the particle density in 3D velocity space. Recall now from *** that

$$N(v) = M\,(4\pi^{-1/2})\,(\beta m/2)^{3/2}\, v^2 \exp(-\beta m v^2/2)$$

Comparison shows that

$$N(v) = (1/8)\,(4\pi v^2)\,\mathcal{N}(\mathbf{v}) = (\pi v^2/2)\,\mathcal{N}(\mathbf{v}) \quad \text{so} \quad \mathcal{N}(\mathbf{v}) = (2/\pi v^2)N(v)$$

We can interpret this to say that N(v) is $\mathcal{N}(\mathbf{v})$ times the area of the octant shell of radius v in v-space. The result is in retrospect obvious if one writes $d^3v = v^2 dv d\Omega$ so

$$N(v)dv = \int_{\texttt{octant shell volume}} \mathcal{N}(\mathbf{v})\, v^2 dv d\Omega = (1/8)4\pi\, \mathcal{N}(\mathbf{v})\, v^2 dv$$

We are now in a better position to answer the following question:

How may helium atoms in the 1 cm^3 box have the specific "quantum numbers" (n_x,n_y,n_z) ? This set of integers is associated with a tiny volume $V = (\pi\hbar/mL)^3$ so the answer to the question is ,

$$N_{n_x n_y n_z} = \mathcal{N}(\mathbf{v}_{n_x n_y n_z})\, V = [\, M\,(8\,\pi^{-3/2})\,(\beta m/2)^{3/2} \exp(-\beta m\, v^2_{n_x n_y n_z}/2)] * (\pi\hbar/mL)^3$$

$$= (2/\pi v_{n_x n_y n_z}{}^2) N(v_{n_x n_y n_z}) * (\pi\hbar/mL)^3$$

where recall from ** that

$$v^2_{n_x n_y n_z} = (\pi\hbar/mL)^2\,(\, n_x^2 + n_y^2 + n_z^2)$$

This count $N_{n_x n_y n_z}$ is then analogous to the count N_i in a discrete-energy-level system where the energy levels are labeled by the single index i.

Now here is some Maple code (continues previous Maple code)

```
V := (h/(2*mHe*L))^3;
```

$$V := .1214240632\ 10^{-15}\ \frac{m^3}{s^3}$$

```
v := (nx,ny,nz) ->sqrt((h/(2*mHe*L))^2*(nx^2+ny^2+nz^2));
```

$$v := (nx, ny, nz) \rightarrow \sqrt{\frac{1}{4}\ \frac{h^2\ (nx^2 + ny^2 + nz^2)}{mHe^2\ L^2}}$$

```
N_:= (nx,ny,nz) -> (2/Pi)*(1/v(nx,ny,nz)^2)*N(v(nx,ny,nz))*V;
```

$$N_ := (nx, ny, nz) \rightarrow 2\ \frac{N(v(nx, ny, nz))\ V}{\pi\ v(nx, ny, nz)^2}$$

```
v(1e8,1e8,1e8);
```

$$857.6871058\ \frac{m}{s}$$

```
N_(1e8,1e8,1e8): simplify(%);
```

$$.1776883745\ 10^{-5}$$

When I wrote the above, I was assuming that perhaps N_(1,2,5) would be maybe 26 particles. But what I learned is that the quantum numbers n_i have to be on the order of 10^8 to get a speed v which is out in the middle of the Maxwell distribution. But then you find that there is only 10^{-6} of one particle in the state corresponding to $(n_x,n_y,n_z) = (10^8,10^8,10^8)$.

Basically here is what you have:

$$N_{n_x n_y n_z} = \mathcal{N}(\mathbf{v})V = M\ (8\ \pi^{-3/2})\ (\beta m/2)^{3/2}\ \exp(-\beta m v^2/2)V$$

The right side has a max value at v = 0 which I see is around 0.3×10^{-5} and for any larger v the result is just going to be smaller. So you never get a number like 26!

The cube V is so small that you have to take a million cubes just to get a single particle being in that region of v-space!

How many V cubes are in a sphere whose radius is the mean speed?

$$(4/3)\pi v^3/V \sim 4 * (1243)^3 * 10^{16} = 10^{26}$$

By showing g(v) >> N(v) I have already shown that there are many more states (think cubes) than there are particles in a shell near any v.

Think of a large sphere in velocity space where you can see all the tiny cubes. Most of the 10^{26} cubes are empty, but 10^{19} of the cubes are occupied and appear as black cubes in my picture. The unoccupied cubes are clear. So if you ask about the probability of there being a particle in a particular cube (n_x,n_y,n_z), that

probability is going to be 10^{-7} or some small number like that. That is why my number $N_{n_x n_y n_z}$ is coming out so small.

So I conclude that computing $N_{n_x n_y n_z}$ is just not a very interesting thing to do.