scraps

Derive Equations (6.4.6)

This is amazingly tedious to derive, it took me half of a day to get all the algebra right!

Start with

$$\varepsilon_1 < \varepsilon_2 < \varepsilon_3 . \qquad \text{and recall} \qquad u = U/M \quad \text{and } \varepsilon_1 \le u \le \varepsilon_3 \ . \tag{5.4.3}$$

We regard both N_1 and N_2 as functions of N_3 where, according to (5.3.1),

$$\begin{aligned} N_1 &= +(\varepsilon_2-\varepsilon_1)^{-1}[\,\varepsilon_2 M - U + (\varepsilon_3-\varepsilon_2)N_3] \;\; = N_1(N_3) \\ N_2 &= -(\varepsilon_2-\varepsilon_1)^{-1}[\,\varepsilon_1 M - U + (\varepsilon_3-\varepsilon_1)N_3] \;\; = N_2(N_3) \ . \qquad \textit{(5.3.3)} \end{aligned} \tag{5.4.4}$$

It seems clear that one must have

$$\begin{aligned} &0 \le N_1 \le M \\ &0 \le N_2 \le M \ . \end{aligned} \tag{5.4.5}$$

In my original doc, I claim that these 4 inequalities can be summarized in this way

$$N_{3min} \le N_3 \le N_{3max}$$

$$N_{3min} = \max\left(\frac{u-\varepsilon_2}{\varepsilon_3-\varepsilon_2} M,\ 0\right)$$

$$N_{3max} = \min\left(\frac{u-\varepsilon_1}{\varepsilon_3-\varepsilon_1} M,\ M\right) = \frac{u-\varepsilon_1}{\varepsilon_3-\varepsilon_1} M \ . \qquad \text{// since } u \le \varepsilon_3 \tag{5.4.6}$$

I want now to *verify* this claim.

Look at each of these four inequalities

$0 \le N_1$:

$$0 \le (\varepsilon_2-\varepsilon_1)^{-1}[\,\varepsilon_2 M - U + (\varepsilon_3-\varepsilon_2)N_3]$$

$$0 \le [\,\varepsilon_2 M - U + (\varepsilon_3-\varepsilon_2)N_3]$$

$$U - \varepsilon_2 M \le (\varepsilon_3-\varepsilon_2)N_3$$

$$N_3 \ge (U-\varepsilon_2 M)/(\varepsilon_3-\varepsilon_2) = M\,(u-\varepsilon_2)/(\varepsilon_3-\varepsilon_2) \qquad \text{one form of } N_{3min} \tag{1}$$

$0 \le N_2$:

$0 \leq -(\varepsilon_2 - \varepsilon_1)^{-1}[\varepsilon_1 M - U + (\varepsilon_3 - \varepsilon_1)N_3]$

$0 \geq (\varepsilon_2 - \varepsilon_1)^{-1}[\varepsilon_1 M - U + (\varepsilon_3 - \varepsilon_1)N_3]$

$0 \geq [\varepsilon_1 M - U + (\varepsilon_3 - \varepsilon_1)N_3]$

$(\varepsilon_3 - \varepsilon_1)N_3 \leq U - \varepsilon_1 M$

$N_3 \leq (U - \varepsilon_1 M) / (\varepsilon_3 - \varepsilon_1)$

$N_3 \leq M(u - \varepsilon_1) / (\varepsilon_3 - \varepsilon_1)$ one form of N_{3max} (2)

$N_1 \leq M$:

$(\varepsilon_2 - \varepsilon_1)^{-1}[\varepsilon_2 M - U + (\varepsilon_3 - \varepsilon_2)N_3] \leq M$

$[\varepsilon_2 M - U + (\varepsilon_3 - \varepsilon_2)N_3] \leq (\varepsilon_2 - \varepsilon_1)M$

$(\varepsilon_3 - \varepsilon_2)N_3 \leq (\varepsilon_2 - \varepsilon_1)M + U - \varepsilon_2 M = -\varepsilon_1 M + U = (u - \varepsilon_1)M$

$N_3 \leq (u - \varepsilon_1)M/(\varepsilon_3 - \varepsilon_2)$ one form for N_3 max (3)

$N_2 \leq M$:

$-(\varepsilon_2 - \varepsilon_1)^{-1}[\varepsilon_1 M - U + (\varepsilon_3 - \varepsilon_1)N_3] \leq M$

$(\varepsilon_2 - \varepsilon_1)^{-1}[\varepsilon_1 M - U + (\varepsilon_3 - \varepsilon_1)N_3] \geq -M$

$[\varepsilon_1 M - U + (\varepsilon_3 - \varepsilon_1)N_3] \geq -(\varepsilon_2 - \varepsilon_1)M$

$(\varepsilon_3 - \varepsilon_1)N_3 \geq -(\varepsilon_2 - \varepsilon_1)M - \varepsilon_1 M + U = (-\varepsilon_2)M + U = [u - \varepsilon_2]M$

$N_3 \geq [u - \varepsilon_2]M/(\varepsilon_3 - \varepsilon_1)$ one form for N_3min (4)

So my four inequalities result in :

$N_3 \geq M(u - \varepsilon_2)/(\varepsilon_3 - \varepsilon_2)$ (1)
$N_3 \geq M(u - \varepsilon_2)/(\varepsilon_3 - \varepsilon_1)$ (4)

and

$N_3 \leq M(u - \varepsilon_1)/(\varepsilon_3 - \varepsilon_1)$ (2)
$N_3 \leq M(u - \varepsilon_1)/(\varepsilon_3 - \varepsilon_2)$ (3)

So it would seem that

$N_{3min} = M \max [(u-\varepsilon_2)/(\varepsilon_3-\varepsilon_2),\ (u-\varepsilon_2)/(\varepsilon_3-\varepsilon_1)]$
$N_{3max} = M \min [(u-\varepsilon_1)/(\varepsilon_3-\varepsilon_2),\ (u-\varepsilon_1)/(\varepsilon_3-\varepsilon_1)]$

Now in each line, the second denominator is larger than the first, so the second term is smaller than the first.

In the first line we therefore select the first term.

In the second line we therefore select the second term term.

Therefore

$N_{3min} = M\ (u-\varepsilon_2)/(\varepsilon_3-\varepsilon_2)$ but also need $N_{3min} > 0$
$N_{3max} = M\ (u-\varepsilon_1)/(\varepsilon_3-\varepsilon_1)$ but also need $N_{3max} < M$

So final result is

$N_{3min} = M\ \max[(u-\varepsilon_2)/(\varepsilon_3-\varepsilon_2), 0]$
$N_{3max} = M\ \min[(u-\varepsilon_1)/(\varepsilon_3-\varepsilon_1), 1]$.

But since $u \le \varepsilon_3$ the first fraction is always ≤ 1, so it always wins in the min, and then

$N_{3min} = M\ \max[(u-\varepsilon_2)/(\varepsilon_3-\varepsilon_2), 0]$
$N_{3max} = M\ (u-\varepsilon_1)/(\varepsilon_3-\varepsilon_1)$

This replicates my original Lagrange doc results (6.4.6)

Note: This result depends on the fact that we assumed $\varepsilon_1 < \varepsilon_2 < \varepsilon_3$.

What happens if you treat N_1 or N_2 as the independent variable?

In the above, I started with

$N_1 = +(\varepsilon_2-\varepsilon_1)^{-1}[\ \varepsilon_2 M - U + (\varepsilon_3-\varepsilon_2)N_3] = N_1(N_3)$
$N_2 = -\ (\varepsilon_2-\varepsilon_1)^{-1}[\ \varepsilon_1 M - U + (\varepsilon_3-\varepsilon_1)N_3] = N_2(N_3)$
$0 \le N_1 \le M$
$0 \le N_2 \le M$

where N_3 is the independent variable. Suppose more generally I want to have N_a and N_b be the two dependent variables. I then would start with

$N_a = +(\varepsilon_b-\varepsilon_a)^{-1}[\ \varepsilon_b M - U + \Sigma_{i\neq a,b}\ (\varepsilon_i-\varepsilon_b)N_i] = N_a(N_{i\neq a,b})$
$N_b = -\ (\varepsilon_b-\varepsilon_a)^{-1}[\ \varepsilon_a M - U + \Sigma_{i\neq a,b}\ (\varepsilon_i-\varepsilon_a)N_i] = N_b(N_{i\neq a,b})$ (5.3.18a)

But if there are only 3 items, I can call the last one "c" and these then read

$$N_a = +(\varepsilon_b - \varepsilon_a)^{-1}[\varepsilon_b M - U + (\varepsilon_c - \varepsilon_b)N_c] = N_a(N_c)$$
$$N_b = -(\varepsilon_b - \varepsilon_a)^{-1}[\varepsilon_a M - U + (\varepsilon_c - \varepsilon_a)N_c] = N_b(N_c) \quad (5.3.18a)$$
$$0 \le N_a \le M$$
$$0 \le N_b \le M$$

This then is the system I would have to analyze. In general we don't know the sign of any ε-ε differences, so I think this general case is not each. Let's write the equations for the three cases

abc = 123 :

$$N_1 = +(\varepsilon_2 - \varepsilon_1)^{-1}[\varepsilon_2 M - U + (\varepsilon_3 - \varepsilon_2)N_c] = N_1(N_3)$$
$$N_2 = -(\varepsilon_2 - \varepsilon_1)^{-1}[\varepsilon_1 M - U + (\varepsilon_3 - \varepsilon_1)N_c] = N_2(N_3) \quad (5.3.18a)$$
$$0 \le N_1 \le M$$
$$0 \le N_2 \le M \qquad \text{// this does replicate (5.3.3)}$$

abc = 132 :

$$N_1 = +(\varepsilon_3 - \varepsilon_1)^{-1}[\varepsilon_3 M - U + (\varepsilon_2 - \varepsilon_3)N_2] = N_1(N_2)$$
$$N_3 = -(\varepsilon_3 - \varepsilon_1)^{-1}[\varepsilon_1 M - U + (\varepsilon_2 - \varepsilon_1)N_2] = N_3(N_2) \quad (5.3.18a)$$
$$0 \le N_1 \le M$$
$$0 \le N_3 \le M$$